Kassim et al. *Advances in Difference Equations* ( 2021) 2021:375
https://doi.org/10.1186/s13662-021-03530-6

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH **Open Access**

# A qualitative study on generalized Caputo fractional integro-differential equations

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**Abstract**

The aim of this article is to discuss the uniqueness and Ulam–Hyers stability of solutions for a nonlinear fractional integro-differential equation involving a generalized Caputo fractional operator. The used fractional operator is generated by iterating a local integral of the form $(I_a^\rho f)(t) = \int_a^t f(s)s^{\rho-1}\,ds$. Our reported results are obtained in the Banach space of absolutely continuous functions that rely on Babenko's technique and Banach's fixed point theorem. Besides, our main findings are illustrated by some examples.

**MSC:** 34E10; 34A08; 26A33; 34A12

**Keywords:** Fractional differential equation; Generalized fractional derivative; Fixed point approach; Babenko's technique

## 1 Introduction

Fractional calculus has gotten much consideration from analysts and engineers, as well as it provided important tools for various areas of applied mathematics, physics, and engineering. Fractional differential equations (FDEs) are used to study plentiful phenomena such as fluid mechanics, plasma physics, optical fibers, biology, flow in nonlinear electric circuits, nonlinear oscillations of the earthquake, mechanics, aerodynamics, regular variations in thermodynamics, etc. Actually, the transform from theoretical to the application aspect of fractional calculus was strongly apparent in the works of Bagley and Torvik in [1–3]. In this regard, the researchers studied many models and used fractional-order derivatives to describe the solution of them. For instance, studying the qualitative properties of solutions of various kinds of FDEs.

Various problems may be modeled by fractional integro-differential equations (FIDEs) such as those representing applications in science and engineering. Up to a recent time, numerous analysts and researchers have discussed the FDEs and FIDEs and got many interesting outcomes utilizing a wide range of fixed point techniques, for instance, Zhang et al. [4], Ahmad et al. [5], Benchohra et al. [6, 7], Ravichandran et al. [8], Trujillo et al. [9], and the following recent papers series [10–18].

Some similar techniques have been applied to get interesting results about some different types of FDEs, see [19–23]. The existence and stability of solutions were studied for many of FDEs with some generalized fractional operators in [24–32]. Also here we refer to some recent works that have dealt with Hadamard fractional derivative [33–39]. For instance, Li [38] investigated the uniqueness of solutions of integral equations with Hadamard-type, that is,

$$a_n I_a^{\alpha_n} \varpi(\varkappa) + \cdots + a_1 I_a^{\alpha_1} \varpi(\varkappa) + \varpi(\varkappa) = \mathbb{G}\big(\varkappa, \varpi(\varkappa)\big).$$

It is realized that the standard definitions for fractional derivatives (Caputo, Riemann–Liouville, etc.), which are introduced in the classical monographs, do not fulfill the index law. A few analysts proposed that a differential operator cannot be known as a derivative or fractional derivative if it does not fulfill the index law, see [40]. However, there are some special cases that have been studied on smooth function spaces that make these operators subject to some laws. In this regard, the considered fractional (so-called Katugampola [41, 42]) operator generalizes both the Riemann–Liouville and Hadamard fractional operators in one form, and it is also most regarding the Erdélyi–Kober fractional operator, especially, when $\rho = 1$, we get a Caputo fractional derivative, and doing $\rho \downarrow 0$, we get a Caputo–Hadamard fractional derivative. Consequently, the current results are a generalization of the works of Li [38, 39] and inspired by [43, 44]. Motivated by the aforesaid discussion, in this research paper, we concentrate on the uniqueness and Ulam–Hyers stability results for the nonlinear FIDEs of the form

$$\begin{cases} {}^{C}D_a^{\rho,\alpha_n} \varpi(\varkappa) + a_{n-1} {}^{C}D_a^{\rho,\alpha_{n-1}} \varpi(\varkappa) + \cdots + a_0 {}^{C}D_a^{\rho,\alpha_0} \varpi(\varkappa) \\ \quad = \int_a^{\varkappa} \mathbb{F}(\tau, \varpi'(\tau))\, d\tau, \\ \varpi(a) = 0, \end{cases} \tag{1.1}$$

where $0 < \alpha_i < 1$, $i = 0, 1, \ldots, n$, $n \in \mathbb{N}$, ${}^{C}D_a^{\rho,\sigma}$ is the generalized Caputo fractional derivative of order $\sigma (> 0) \in \{\alpha_i; i = 0, 1, \ldots, n\}$ generated by local integrals of the form $(I_a^{\rho} f)(t) = \int_a^t f(s) s^{\rho-1}\, ds$, and $\mathbb{F} : [a, b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

There is an absence of various analytical strategies to obtain the qualitative properties of solutions to such problems under generalized fractional operators. To fill this vacuum, we are keen to obtain the existence of a unique solution and the Ulam–Hyers stability of a solution to (1.1). Compared with preceding investigations of such problems, (1.1) is more general because it has a generalized fractional operators. Moreover, the current results are obtained in the Banach space of absolutely continuous functions along with Banach's fixed point technique and Babenko's method [45].

The article is organized as follows: In Sect. 2 we present some necessary tools about the essential properties of generalized fractional operators and the abstract function spaces, in which we aim to employ our analysis techniques. Sections 3 and 4 are devoted to our main analysis results and their illustrated examples. Finally, Sect. 5 contains our short conclusion.

## 2 Preliminaries

In this section, we briefly recall some definitions, lemmas, properties, notations, and well-known estimations that we will use later.

Let $-\infty < a < b < \infty$. Let $AC[a, b]$ denote the space of absolutely continuous functions on $[a, b]$ [46]. We denote by $L^p(a, b)$, $p \geq 1$, the spaces of Lebesgue integrable functions on $(a, b)$ [46].

**Definition 2.1** ([46, (1,9,27), (1,9,28)]) The so-called multivariate Mittag-Leffler function $E_{(a_1,\dots,a_n),b}(z_1,\dots,z_n)$ of $n$ complex variables $z_1,\dots,z_n \in \mathbb{C}$ with complex parameters $a_1,\dots,a_n,b \in \mathbb{C}$ is defined by

$$E_{(a_1,\dots,a_n),b}(z_1,\dots,z_n) = \sum_{\kappa=0}^{\infty} \sum_{\kappa_1+\cdots+\kappa_n=\kappa} \binom{\kappa}{\kappa_1,\kappa_2,\dots,\kappa_k} \frac{\prod_{j=1}^{n} z_j^{\kappa_j}}{\Gamma(b+\sum_{j=1}^{n} a_j\kappa_j)}, \tag{2.1}$$

in terms of multinomial coefficients

$$\binom{\kappa}{\kappa_1,\kappa_2,\dots,\kappa_k} = \frac{\kappa!}{\kappa_1!\kappa_2!\dots\kappa_k!}, \quad \kappa,\kappa_1,\dots,\kappa_k \in \mathbb{N}_0.$$

**Theorem 2.2** ([47] (Multinomial theorem)) *For a positive integer $n$ and a nonnegative integer $k$,*

$$(z_1+z_2+\cdots+z_k)^n = \sum_{\kappa_1+\kappa_2+\cdots+\kappa_k=n} \binom{n}{\kappa_1,\kappa_2,\dots,\kappa_k} \prod_{j=1}^{k} z_j^{\kappa_j}.$$

**Definition 2.3** ([48, (6,3,1)](Babenko's method)) Given the FDE

$$\big(1+\lambda I_0^{\alpha}\big)\vartheta(\varkappa) = f(\varkappa), \tag{2.2}$$

where $\alpha > 0$, $\lambda$ is a constant, and $I_0^{\alpha}\vartheta(\varkappa) = \frac{1}{\Gamma(\alpha)}\int_0^{\varkappa}(\varkappa-\tau)^{\alpha-1}\vartheta(\tau)\,d\tau$. The solution of (2.2) is

$$\vartheta(\varkappa) = \big(1+\lambda I^{\alpha}\big)^{-1} f(\varkappa), \tag{2.3}$$

where $(1+\lambda I^{\alpha})^{-1}$ denotes the left inverse operator to the operation $(1+\lambda D^{-\alpha})$. Using the binomial expansion of $(1+\lambda I^{\alpha})^{-1}$, solution (2.3) can be expressed by

$$\vartheta(\varkappa) = \sum_{n=0}^{\infty}(-1)^n\lambda^n I^{\alpha n} f(\varkappa). \tag{2.4}$$

**Definition 2.4** ([49] (Dirichlet formula)) Let $f$ be a continuous function on $[a,b]$ and $\alpha,\beta > 0$. Then

$$\begin{aligned} &\int_a^{\varkappa}(\varkappa-\tau)^{\alpha-1}\,d\tau \int_a^{\tau}(\tau-s)^{\beta-1} f(\tau,s)\,ds \\ &\quad = \int_a^{\varkappa} ds \int_s^{\varkappa}(\varkappa-\tau)^{\alpha-1}(\tau-s)^{\beta-1} f(\tau,s)\,d\tau. \end{aligned} \tag{2.5}$$

Define the Banach space (see [46, (1,9,27), (1,9,28)])

$$AC_0[a,b] = \left\{\varpi : \varpi(\tau) \in AC[a,b] \text{ with } \varpi(a) = 0 \text{ and } \|\varpi\|_0 = \int_a^b \left|\varpi'(\tau)\right| d\tau < \infty\right\}.$$

Next, we introduce some definitions, notation, and properties of the generalized fractional integral and derivative.

**Definition 2.5** ([41]) The generalized fractional integral and derivative are defined, respectively, by

$$ {}^{\rho}I_a^{\alpha}\vartheta(\varkappa) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^{\varkappa}\left(\varkappa^{\rho}-\tau^{\rho}\right)^{\alpha-1}\tau^{\rho-1}\vartheta(\tau)\,d\tau, \quad \alpha>0, \rho>0, $$

and

$$ \begin{aligned} {}^{C}D_a^{\rho,\alpha}\vartheta(\varkappa) &= \left({}^{\rho}I_a^{n-\alpha}\delta_{\rho}^{n}\vartheta\right)(\varkappa) \\ &= \frac{\rho^{1-n+\alpha}}{\Gamma(n-\alpha)}\int_a^{\varkappa}\left(\varkappa^{\rho}-\tau^{\rho}\right)^{n-\alpha-1}\tau^{\rho-1}\delta_{\rho}^{n}\vartheta(\tau)\,d\tau, \quad \alpha>0, \rho>0, \end{aligned} $$

where

$$ n=-[-\alpha], \qquad \delta_{\rho}^{n} = \left(\varkappa^{1-\rho}\frac{d}{d\varkappa}\right)^{n}. $$

**Definition 2.6** ([50]) The incomplete gamma function is defined by

$$ \gamma(\alpha,\tau) = \int_0^{\tau} s^{\alpha-1}e^{-s}\,ds = \tau^{\alpha}\Gamma(\alpha)e^{-\tau}\sum_{i=0}^{\infty}\frac{\tau^{i}}{\Gamma(\alpha+i+1)}, \quad \alpha>0, \tau\geq 0. $$

**Property 2.7** ([44]) If $\alpha\geq 0$, $\rho>0$, and $\beta>0$, then

$$ \begin{aligned} {}^{\rho}I_a^{\alpha}\left(\frac{\varkappa^{\rho}-a^{\rho}}{\rho}\right)^{\beta} &= \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}\left(\frac{\varkappa^{\rho}-a^{\rho}}{\rho}\right)^{\beta+\alpha}, \quad \varkappa>a, \\ {}^{C}D_a^{\rho,\alpha}\left(\frac{\varkappa^{\rho}-a^{\rho}}{\rho}\right)^{\beta} &= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}\left(\frac{\varkappa^{\rho}-a^{\rho}}{\rho}\right)^{\beta-\alpha}, \quad \varkappa>a. \end{aligned} $$

The generalized fractional operators in Definition 2.5 fulfill the following properties.

**Property 2.8** ([51]) Let $\alpha$, $\rho$, and $\beta>0$. If $\vartheta\in AC_0[a,b]$, then

$$ \begin{aligned} &{}^{C}D_a^{\rho,\alpha}\,{}^{\rho}I_a^{\alpha}\vartheta(\tau) = \vartheta(\tau), \quad \tau>a, \\ &{}^{\rho}I_a^{\alpha}\,{}^{\rho}I_a^{\beta}\vartheta(\tau) = {}^{\rho}I_a^{\alpha+\beta}\vartheta(\tau), \quad \tau>a, \\ &{}^{C}D_a^{\rho,\beta}\,{}^{\rho}I_a^{\alpha}\vartheta(\tau) = {}^{\rho}I_a^{\alpha-\beta}\vartheta(\tau), \quad \alpha>\beta, \tau>a, \end{aligned} $$

and

$$ {}^{\rho}I_a^{\alpha}\vartheta(a) = 0. $$

**Lemma 2.9** *Let* $\alpha,\beta\in[0,1]$ *and* $\rho>0$. *If* $\vartheta\in AC_0[a,b]$, *then*

$$ \begin{aligned} &{}^{\rho}I_a^{\alpha}\,{}^{C}D_a^{\rho,\alpha}\vartheta(\tau) = \vartheta(\tau), \quad \tau>a, \\ &{}^{\rho}I_a^{\alpha}\,{}^{C}D_a^{\rho,\beta}\vartheta(\tau) = {}^{\rho}I_a^{\alpha-\beta}\vartheta(\tau), \quad \alpha>\beta, \tau>a. \end{aligned} $$

*Proof* Let $\vartheta \in AC_0[a,b]$. From Definition 2.5 and Property 2.8, we have

$$^{\rho}I_a^{\alpha\,C}D_a^{\rho,\alpha}\vartheta(\tau) = {}^{\rho}I_a^{\alpha\,\rho}I_a^{1-\alpha}\delta_\rho^1\vartheta(\tau) = {}^{\rho}I_a^1\delta_\rho^1\vartheta(\tau) = \vartheta(\tau) - \vartheta(a) = \vartheta(\tau), \quad \tau > a,$$

and

$$\begin{aligned} ^{\rho}I_a^{\alpha\,C}D_a^{\rho,\beta}\vartheta(\tau) &= {}^{\rho}I_a^{\alpha\,\rho}I_a^{1-\beta}\delta_\rho^1\vartheta(\tau) = {}^{\rho}I_a^{\alpha-\beta\,\rho}I_a^1\delta_\rho^1\vartheta(\tau) \\ &= {}^{\rho}I_a^{\alpha-\beta}\big(\vartheta(\tau)-\vartheta(a)\big) = {}^{\rho}I_a^{\alpha-\beta}\vartheta(\tau), \quad \alpha > \beta, \tau > a. \end{aligned}$$

□

**Lemma 2.10** *Let $\alpha, \rho > 0$. Then ${}^{\rho}I_a^{\alpha}$ is bounded from $AC_0[a,b]$ into $AC_0[a,b]$ and*

$$\left\| {}^{\rho}I_a^{\alpha}\vartheta \right\|_0 \le \frac{1}{\Gamma(\alpha+1)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha}\|\vartheta\|_0.$$

*Proof* Let $\vartheta \in AC_0[a,b]$. Then

$$\vartheta(\tau) = \int_a^\tau \vartheta'(s)\,ds = \int_a^\tau \theta(s)\,ds, \qquad \theta(\tau) = \vartheta'(\tau), \quad \text{and} \quad \vartheta(a) = 0.$$

From Definition 2.5, we obtain

$$\begin{aligned} ^{\rho}I_a^{\alpha}\vartheta(\varkappa) &= {}^{\rho}I_a^{\alpha}\left(\int_a^\tau \theta(s)\,ds\right)(\varkappa) \\ &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^\varkappa \left(\varkappa^\rho - \tau^\rho\right)^{\alpha-1}\tau^{\rho-1}\int_a^\tau \theta(s)\,ds\,d\tau. \end{aligned}$$

Using Dirichlet's formula (2.5), we obtain

$$\begin{aligned} ^{\rho}I_a^{\alpha}\vartheta(\varkappa) &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^\varkappa \theta(s)\int_s^\varkappa \left(\varkappa^\rho - \tau^\rho\right)^{\alpha-1}\tau^{\rho-1}\,d\tau\,ds \\ &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^\varkappa \theta(s)\left[-\frac{(\varkappa^\rho - \tau^\rho)^\alpha}{\alpha\rho}\right]_{\tau=s}^{\varkappa} ds \\ &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^\varkappa \theta(s)\left[\frac{(\varkappa^\rho - s^\rho)^\alpha}{\alpha\rho}\right] ds \\ &\le \frac{1}{\Gamma(\alpha+1)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha}\int_a^\varkappa \left|\theta(s)\right| ds \\ &= \frac{1}{\Gamma(\alpha+1)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha}\int_a^\varkappa \left|\vartheta'(s)\right| ds \\ &= \frac{1}{\Gamma(\alpha+1)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha}\|\vartheta\|_0. \end{aligned}$$

□

Next, we prove the following useful lemma.

**Lemma 2.11** *If $\alpha \ge 0$, $\rho > 0$, then*

$$^{\rho}I_a^{\alpha}e^{\varkappa^\rho} = e^{a^\rho}\left(\frac{\varkappa^\rho - a^\rho}{\rho}\right)^{\alpha}\sum_{i=0}^{\infty}\frac{(\varkappa^\rho - a^\rho)^i}{\Gamma(\alpha+i+1)}.$$

*Proof* By Definition 2.5, we have

$$ {}^{\rho}I_a^{\alpha} e^{\varkappa^\rho} = \frac{1}{\Gamma(\alpha)} \int_a^{\varkappa} \left( \frac{\varkappa^\rho - \tau^\rho}{\rho} \right)^{\alpha-1} \tau^{\rho-1} e^{\tau^\rho} \, d\tau. $$

Making the change of the variable

$$ s = \frac{\varkappa^\rho - \tau^\rho}{\rho} \quad \Rightarrow \quad \tau^\rho = \varkappa^\rho - \rho s. $$

Therefore,

$$ {}^{\rho}I_a^{\alpha} e^{\varkappa^\rho} = \frac{1}{\Gamma(\alpha)} \int_0^{(\varkappa^\rho - a^\rho)/\rho} s^{\alpha-1} e^{\varkappa^\rho - \rho s} \, ds = \frac{e^{\varkappa^\rho}}{\Gamma(\alpha)} \int_0^{(\varkappa^\rho - a^\rho)/\rho} s^{\alpha-1} e^{-\rho s} \, ds. $$

Let $r = \rho s$. Then

$$ {}^{\rho}I_a^{\alpha} e^{\varkappa^\rho} = \frac{e^{\varkappa^\rho}}{\rho^\alpha \Gamma(\alpha)} \int_0^{(\varkappa^\rho - a^\rho)} r^{\alpha-1} e^{-r} \, dr. $$

Using Definition 2.6, we get

$$ {}^{\rho}I_a^{\alpha} e^{\varkappa^\rho} = \gamma\left(\alpha, \varkappa^\rho - a^\rho\right) \frac{e^{\varkappa^\rho}}{\rho^\alpha \Gamma(\alpha)} = e^{a^\rho} \left( \frac{\varkappa^\rho - a^\rho}{\rho} \right)^{\alpha} \sum_{i=0}^{\infty} \frac{(\varkappa^\rho - a^\rho)^i}{\Gamma(\alpha + i + 1)}. \qquad \square $$

## 3 Main results

**Theorem 3.1** *Suppose* $a_i \in \mathbb{C}$ $(i = 0, 1, \ldots, n-1)$ *with* $0 < \alpha_0 < \alpha_1 < \cdots < \alpha_n < 1$. *If* $g \in AC_0[a, b]$, *then the linear problem*

$$ \begin{cases} {}^{C}D_a^{\rho,\alpha_n} \varpi(\varkappa) + a_{n-1} {}^{C}D_a^{\rho,\alpha_{n-1}} \varpi(\varkappa) + \cdots + a_0 {}^{C}D_a^{\rho,\alpha_0} \varpi(\varkappa) = g(\varkappa), \\ \varpi(a) = 0, \end{cases} \tag{3.1} $$

*has a solution*

$$ \begin{aligned} \varpi(\varkappa) = {} & \sum_{J=0}^{\infty} (-1)^J \sum_{J_1 + \cdots + J_n = J} \binom{J}{J_1, J_2, \ldots, J_n} \\ & \times a_{n-1}^{J_1} \cdots a_0^{J_n} {}^{\rho}I_a^{J_1(\alpha_n - \alpha_{n-1}) + \cdots + J_n(\alpha_n - \alpha_0) + \alpha_n} g(\varkappa). \end{aligned} \tag{3.2} $$

*Proof* Applying ${}^{\rho}I_a^{\alpha_n}$ to both sides of (3.1), we find that

$$ {}^{\rho}I_a^{\alpha_n} {}^{C}D_a^{\rho,\alpha_n} \varpi(\varkappa) + a_{n-1} {}^{\rho}I_a^{\alpha_n} {}^{C}D_a^{\rho,\alpha_{n-1}} \varpi(\varkappa) + \cdots + a_0 {}^{\rho}I_a^{\alpha_n} {}^{C}D_a^{\rho,\alpha_0} \varpi(\varkappa) = {}^{\rho}I_a^{\alpha_n} g(\varkappa). $$

Using Lemma 2.9, we obtain

$$ \varpi(\varkappa) + a_{n-1} {}^{\rho}I_a^{\alpha_n - \alpha_{n-1}} \varpi(\varkappa) + \cdots + a_0 {}^{\rho}I_a^{\alpha_n - \alpha_0} \varpi(\varkappa) = {}^{\rho}I_a^{\alpha_n} g(\varkappa). $$

By noting that $\varpi(a) = 0$ and $0 < \alpha_0 < \alpha_1 < \cdots < \alpha_n < 1$, then

$$ \left(1 + a_{n-1} {}^{\rho}I_a^{\alpha_n - \alpha_{n-1}} + \cdots + a_0 {}^{\rho}I_a^{\alpha_n - \alpha_0}\right) \varpi(\varkappa) = {}^{\rho}I_a^{\alpha_n} g(\varkappa). $$

Thanks to Babenko's method, we have

$$\varpi(\varkappa) = \left(1 + a_{n-1}^{\rho} I_a^{\alpha_n-\alpha_{n-1}} + \cdots + a_0^{\rho} I_a^{\alpha_n-\alpha_0}\right)^{-1} {}^{\rho}I_a^{\alpha_n} g(\varkappa).$$

Multinomial theorem and Property 2.8 give

$$\begin{aligned}
\varpi(\varkappa) &= \sum_{J=0}^{\infty}(-1)^J \left(a_{n-1}{}^{\rho}I_a^{\alpha_n-\alpha_{n-1}} + \cdots + a_0{}^{\rho}I_a^{\alpha_n-\alpha_0}\right)^{J} {}^{\rho}I_a^{\alpha_n} g(\varkappa) \\
&= \sum_{J=0}^{\infty}(-1)^J \sum_{J_1+\cdots+J_n=J} \binom{J}{J_1, J_2, \ldots, J_n} \\
&\quad \times \left(a_{n-1}{}^{\rho}I_a^{\alpha_n-\alpha_{n-1}}\right)^{J_1} \cdots \left(a_0{}^{\rho}I_a^{\alpha_n-\alpha_0}\right)^{J_n} {}^{\rho}I_a^{\alpha_n} g(\varkappa) \\
&= \sum_{J=0}^{\infty}(-1)^J \sum_{J_1+\cdots+J_n=J} \binom{J}{J_1, J_2, \ldots, J_n} \\
&\quad \times a_{n-1}^{J_1} \cdots a_0^{J_n} {}^{\rho}I_a^{J_1(\alpha_n-\alpha_{n-1})+\cdots+J_n(\alpha_n-\alpha_0)+\alpha_n} g(\varkappa).
\end{aligned}$$

By taking the limit as $\varkappa \to a$, we obtain $\varpi(a) = 0$. It remains to show that the series converges in the space $AC_0[a,b]$ and is absolutely continuous on $[a,b]$. By Lemma 2.10

$$\left\| {}^{\rho}I_a^{J_1(\alpha_n-\alpha_{n-1})+\cdots+J_n(\alpha_n-\alpha_0)+\alpha_n} g \right\|_0 \le \kappa \|g\|_0,$$

where

$$\kappa = \frac{\left(\frac{b^\rho - a^\rho}{\rho}\right)^{J_1(\alpha_n-\alpha_{n-1})+\cdots+J_n(\alpha_n-\alpha_0)+\alpha_n}}{\Gamma(J_1(\alpha_n-\alpha_{n-1})+\cdots+J_n(\alpha_n-\alpha_0)+\alpha_n+1)}. \tag{3.3}$$

Then

$$\begin{aligned}
\|\varpi\|_0 &\le \kappa \sum_{J=0}^{\infty} \sum_{J_1+\cdots+J_n=J} \binom{J}{J_1, J_2, \ldots, J_n} \left|a_{n-1}^{J_1}\right| \cdots \left|a_0^{J_n}\right| \\
&\quad \times \frac{\left(\frac{b^\rho - a^\rho}{\rho}\right)^{J_1(\alpha_n-\alpha_{n-1})+\cdots+J_n(\alpha_n-\alpha_0)+\alpha_n}}{\Gamma(J_1(\alpha_n-\alpha_{n-1})+\cdots+J_n(\alpha_n-\alpha_0)+\alpha_n+1)} \|g\|_0 \\
&= \kappa \sum_{J=0}^{\infty} \sum_{J_1+\cdots+J_n=J} \binom{J}{J_1, J_2, \ldots, J_n} \\
&\quad \times \frac{\left(|a_{n-1}|\left(\frac{b^\rho - a^\rho}{\rho}\right)^{(\alpha_n-\alpha_{n-1})}\right)^{J_1} \cdots \left(|a_0|\left(\frac{b^\rho - a^\rho}{\rho}\right)^{(\alpha_n-\alpha_0)}\right)^{J_n}}{\Gamma(J_1(\alpha_n-\alpha_{n-1})+\cdots+J_n(\alpha_n-\alpha_0)+\alpha_n+1)} \|g\|_0 \\
&= \kappa E_{(\alpha_n-\alpha_{n-1},\ldots,\alpha_n-\alpha_0,\alpha_n+1)}\left(|a_{n-1}|\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha_n-\alpha_{n-1}}, \right. \\
&\quad \left. \ldots, |a_0|\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha_n-\alpha_0}\right)\|g\|_0,
\end{aligned}$$

where

$$E_{(\alpha_n-\alpha_{n-1},\dots,\alpha_n-\alpha_0,\alpha_n+1)}\left(|a_{n-1}|\left(\frac{b^\rho-a^\rho}{\rho}\right)^{\alpha_n-\alpha_{n-1}},\dots,|a_0|\left(\frac{b^\rho-a^\rho}{\rho}\right)^{\alpha_n-\alpha_0}\right)<\infty$$

is the value at

$$\nu_1=|a_{n-1}|\left(\frac{b^\rho-a^\rho}{\rho}\right)^{\alpha_n-\alpha_{n-1}},\dots,\nu_n=|a_0|\left(\frac{b^\rho-a^\rho}{\rho}\right)^{\alpha_n-\alpha_0}$$

of the multivariate Mittag-Leffler function $E_{(\alpha_n-\alpha_{n-1},\dots,\alpha_n-\alpha_0,\alpha_n+1)}(\nu_1,\dots,\nu_n)$ defined by Eq. (2.1). Thus, the series to the right of Eq. (3.2) is convergent. Clearly, $\varpi(\varkappa)\in AC[a,b]$ since $g\in AC[a,b]$. To confirm that the acquired series is a solution, we substitute it into the left-hand side of Eq. (3.1) as follows:

$$\begin{aligned}
&{}^{C}D_a^{\rho,\alpha_n}\left(\sum_{J=0}^{\infty}(-1)^J\sum_{J_1+\dots+J_n=J}\binom{J}{J_1,J_2,\dots,J_n}\right.\\
&\quad\left.\times a_{n-1}^{J_1}\cdots a_0^{J_n}\,{}^{\rho}I_a^{J_1(\alpha_n-\alpha_{n-1})+\dots+J_n(\alpha_n-\alpha_0)+\alpha_n}g(\varkappa)\right)\\
&\quad+a_{n-1}{}^{C}D_a^{\rho,\alpha_{n-1}}\left(\sum_{J=0}^{\infty}(-1)^J\sum_{J_1+\dots+J_n=J}\binom{J}{J_1,J_2,\dots,J_n}\right.\\
&\quad\left.\times a_{n-1}^{J_1}\cdots a_0^{J_n}\,{}^{\rho}I_a^{J_1(\alpha_n-\alpha_{n-1})+\dots+J_n(\alpha_n-\alpha_0)+\alpha_n}g(\varkappa)\right)\\
&\quad+\dots+a_0{}^{C}D_a^{\rho,\alpha_0}\left(\sum_{J=0}^{\infty}(-1)^J\sum_{J_1+\dots+J_n=J}\binom{J}{J_1,J_2,\dots,J_n}\right.\\
&\quad\left.\times a_{n-1}^{J_1}\cdots a_0^{J_n}\,{}^{\rho}I_a^{J_1(\alpha_n-\alpha_{n-1})+\dots+J_n(\alpha_n-\alpha_0)+\alpha_n}g(\varkappa)\right)\\
&={}^{C}D_a^{\rho,\alpha_n}\left({}^{\rho}I_a^{\alpha_n}g(\varkappa)+\sum_{J=1}^{\infty}(-1)^J\sum_{J_1+\dots+J_n=J}\binom{J}{J_1,J_2,\dots,J_n}\right.\\
&\quad\left.\times a_{n-1}^{J_1}\cdots a_0^{J_n}\,{}^{\rho}I_a^{J_1(\alpha_n-\alpha_{n-1})+\dots+J_n(\alpha_n-\alpha_0)+\alpha_n}g(\varkappa)\right)\\
&\quad+\left(\sum_{J=0}^{\infty}(-1)^J\sum_{J_1+\dots+J_n=J}\binom{J}{J_1,J_2,\dots,J_n}\right.\\
&\quad\left.\times a_{n-1}^{J_1+1}\cdots a_0^{J_n}\,{}^{\rho}I_a^{(J_1+1)(\alpha_n-\alpha_{n-1})+\dots+J_n(\alpha_n-\alpha_0)}g(\varkappa)\right)\\
&\quad+\dots+\left(\sum_{J=0}^{\infty}(-1)^J\sum_{J_1+\dots+J_n=J}\binom{J}{J_1,J_2,\dots,J_n}\right.\\
&\quad\left.\times a_{n-1}^{J_1}\cdots a_0^{J_n+1}\,{}^{\rho}I_a^{J_1(\alpha_n-\alpha_{n-1})+\dots+(J_n+1)(\alpha_n-\alpha_0)}g(\varkappa)\right)
\end{aligned}$$

$$
\begin{aligned}
&= g(\varkappa) + \sum_{J=1}^{\infty}(-1)^J \sum_{J_1+\cdots+J_n=J} \binom{J}{J_1, J_2, \ldots, J_n} \\
&\quad \times a_{n-1}^{J_1} \cdots a_0^{J_n}\, {}^{\rho}I_a^{J_1(\alpha_n-\alpha_{n-1})+\cdots+J_n(\alpha_n-\alpha_0)} g(\varkappa) \\
&\quad + \sum_{J=0}^{\infty}(-1)^J \sum_{J_1+\cdots+J_n=J} \binom{J}{J_1, J_2, \ldots, J_n} \\
&\quad \times a_{n-1}^{J_1+1} \cdots a_0^{J_n}\, {}^{\rho}I_a^{(J_1+1)(\alpha_n-\alpha_{n-1})+\cdots+J_n(\alpha_n-\alpha_0)} g(\varkappa) \\
&\quad + \cdots + \sum_{J=0}^{\infty}(-1)^J \sum_{J_1+\cdots+J_n=J} \binom{J}{J_1, J_2, \ldots, J_n} \\
&\quad \times a_{n-1}^{J_1} \cdots a_0^{J_n+1}\, {}^{\rho}I_a^{J_1(\alpha_n-\alpha_{n-1})+\cdots+(J_n+1)(\alpha_n-\alpha_0)} g(\varkappa) \\
&= g(\varkappa)
\end{aligned}
$$

by the deletion. Observe that all series are absolutely convergent and the term rearrangements are possible for the deletion. In fact,

$$
\begin{aligned}
&- \sum_{J_1+\cdots+J_n=1} \binom{J}{J_1, J_2, \ldots, J_n} a_{n-1}^{J_1} \cdots a_0^{J_n}\, {}^{\rho}I_a^{J_1(\alpha_n-\alpha_{n-1})+\cdots+J_n(\alpha_n-\alpha_0)} g(\varkappa) \\
&\quad + \sum_{J_1+\cdots+J_n=0} \binom{J}{J_1, J_2, \ldots, J_n} a_{n-1}^{J_1+1} \cdots a_0^{J_n}\, {}^{\rho}I_a^{(J_1+1)(\alpha_n-\alpha_{n-1})+\cdots+J_n(\alpha_n-\alpha_0)} g(\varkappa) + \cdots \\
&\quad + \sum_{J_1+\cdots+J_n=0} \binom{J}{J_1, J_2, \ldots, J_n} a_{n-1}^{J_1} \cdots a_0^{J_n+1}\, {}^{\rho}I_a^{J_1(\alpha_n-\alpha_{n-1})+\cdots+(J_n+1)(\alpha_n-\alpha_0)} g(\varkappa) \\
&= 0.
\end{aligned}
$$

The remnant terms cancel each other comparatively. Obviously, the uniqueness follows directly from the fact that

$$
{}^{C}D_a^{\rho,\alpha_n} \varpi(\varkappa) + a_{n-1}\, {}^{C}D_a^{\rho,\alpha_{n-1}} \varpi(\varkappa) + \cdots + a_0\, {}^{C}D_a^{\rho,\alpha_0} \varpi(\varkappa) = 0
$$

only has solution zero by Babenko's approach. This finalizes the proof. □

*Remark* 3.2 A solution of Eq. (3.1) in $AC_0[a,b]$ is said to be stable if $\forall \epsilon > 0\ \exists \delta > 0$ such that $\|\varpi\|_0 < \epsilon$ if $\|g\|_0 < \delta$. Applying the inequality

$$
\begin{aligned}
\|\varpi\|_0 \le \kappa E_{(\alpha_n-\alpha_{n-1},\ldots,\alpha_n-\alpha_0,\alpha_n+1)} \Bigg( &|a_{n-1}| \left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha_n-\alpha_{n-1}}, \\
&\ldots, |a_0| \left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha_n-\alpha_0} \Bigg) \|g\|_0,
\end{aligned}
\tag{3.4}
$$

we obtain that $\varpi$ is stable.

*Example* 3.3 The Katugampola-type FIDE

$$
{}^{C}D_a^{\rho,0.9} \varpi(\varkappa) + 2\, {}^{C}D_a^{\rho,0.7} \varpi(\varkappa) - {}^{C}D_a^{\rho,0.4} \varpi(\varkappa) = \left(\frac{\varkappa^\rho - a^\rho}{\rho}\right)^{\beta}
$$

has the solution

$$\varpi(\varkappa) = \sum_{J=0}^{\infty}(-1)^{J} \sum_{J_1+J_2=J} \binom{J}{J_1, J_2} (2)^{J_1}(-1)^{J_2} \times \frac{\Gamma(\beta+1)}{\Gamma(\beta+0.2J_1+0.5J_2+1.9)} \times \left(\frac{\varkappa^{\rho}-a^{\rho}}{\rho}\right)^{\beta+0.2J_1+0.5J_2+0.9}$$

in $AC_0[a,b]$. Indeed, in view of Theorem 3.1, we have

$$\varpi(\varkappa) = \sum_{J=0}^{\infty}(-1)^{J} \sum_{J_1+J_2=J} \binom{J}{J_1, J_2} (2)^{J_1}(-1)^{J_2} \times {}^{\rho}I_a^{0.2J_1+0.5J_2+0.9}\left(\frac{\varkappa^{\rho}-a^{\rho}}{\rho}\right)^{\beta}.$$

Using Property 2.7, we obtain

$$\varpi(\varkappa) = \sum_{J=0}^{\infty}(-1)^{J} \sum_{J_1+J_2=J} \binom{J}{J_1, J_2} (2)^{J_1}(-1)^{J_2} \times \frac{\Gamma(\beta+1)}{\Gamma(\beta+0.2J_1+0.5J_2+1.9)} \times \left(\frac{\varkappa^{\rho}-a^{\rho}}{\rho}\right)^{\beta+0.2J_1+0.5J_2+0.9}.$$

*Example* 3.4 The Katugampola-type FIDE

$${}^{C}D_a^{\rho,0.8}\varpi(\varkappa) + {}^{C}D_a^{\rho,0.7}\varpi(\varkappa) - 3\,{}^{C}D_a^{\rho,0.2}\varpi(\varkappa) = e^{\varkappa^{\rho}}$$

has the solution

$$\varpi(\varkappa) = e^{a^{\rho}} \sum_{J=0}^{\infty}(-1)^{J} \sum_{J_1+J_2=J} \binom{J}{J_1, J_2} (-3)^{J_2} \times \left(\frac{\varkappa^{\rho}-a^{\rho}}{\rho}\right)^{0.1J_1+0.6J_2+0.8} \times \sum_{i=0}^{\infty}\frac{(\varkappa^{\rho}-a^{\rho})^{i}}{\Gamma(0.1J_1+0.6J_2+0.8+i+1)}$$

in $AC_0[a,b]$. Indeed, according to Theorem 3.1, we obtain

$$\varpi(\varkappa) = \sum_{J=0}^{\infty}(-1)^{J} \sum_{J_1+J_2=J} \binom{J}{J_1, J_2} (1)^{J_1}(-3)^{J_2} \times {}^{\rho}I_a^{0.1J_1+0.6J_2+0.8}e^{\varkappa^{\rho}}.$$

Using Lemma 2.11, we get

$$\begin{aligned}\varpi(\varkappa) &= \sum_{J=0}^{\infty}(-1)^{J}\sum_{J_1+J_2=J}\binom{J}{J_1, J_2}(-3)^{J_2}\\ &\quad\times e^{a^{\rho}}\left(\frac{\varkappa^{\rho}-a^{\rho}}{\rho}\right)^{0.1J_1+0.6J_2+0.8}\\ &\quad\times \sum_{i=0}^{\infty}\frac{(\varkappa^{\rho}-a^{\rho})^{i}}{\Gamma(0.1J_1+0.6J_2+0.8+i+1)}.\end{aligned}$$

The next theorem proves the uniqueness result of Eq. (1.1).

**Theorem 3.5** *Suppose that $\mathbb{F}:[a,b]\times\mathbb{R}\to\mathbb{R}$ is a continuous function, and there exists a constant $C$ such that*

$$\left|\mathbb{F}(\varkappa,\varpi_1)-\mathbb{F}(\varkappa,\varpi_2)\right|\le C|\varpi_1-\varpi_2|,\quad \varkappa\in[a,b],\varpi_1,\varpi_2\in\mathbb{R}.$$

*In addition, if*

$$\begin{aligned}&C\kappa E_{(\alpha_n-\alpha_{n-1},\dots,\alpha_n-\alpha_0,\alpha_n+1)}\left(|a_{n-1}|\left(\frac{b^{\rho}-a^{\rho}}{\rho}\right)^{(\alpha_n-\alpha_{n-1})},\dots,|a_0|\left(\frac{b^{\rho}-a^{\rho}}{\rho}\right)^{(\alpha_n-\alpha_0)}\right)\\ &\quad<1,\end{aligned}\tag{3.5}$$

*then problem FIDE (1.1) has a unique solution on $AC_0[a,b]$.*

*Proof* Define the operator $\mathfrak{L}$ on $AC_0[a,b]$ by

$$\begin{aligned}\mathfrak{L}(\varpi) &= \sum_{J=0}^{\infty}(-1)^{J}\sum_{J_1+\dots+J_n=J}\binom{J}{J_1,J_2,\dots,J_n}a_{n-1}^{J_1}\cdots a_0^{J_n}\\ &\quad\times {}^{\rho}I_a^{J_1(\alpha_n-\alpha_{n-1})+\dots+J_n(\alpha_n-\alpha_0)+\alpha_n}\int_a^{\varkappa}\mathbb{F}\big(\tau,\varpi'(\tau)\big)\,d\tau.\end{aligned}$$

Let $\varpi\in AC_0[a,b]$. Then

$$\int_a^{\varkappa}\mathbb{F}\big(\tau,\varpi'(\tau)\big)\,d\tau\in AC_0[a,b],$$

as $\varpi'(\tau)\in L(a,b)$ and $\mathbb{F}(\tau,\varpi'(\tau))\in L(a,b)$. Obviously,

$$\begin{aligned}\left\|\int_a^{\varkappa}\mathbb{F}\big(\tau,\varpi'(\tau)\big)\,d\tau\right\|_0 &= \int_a^b\left|\mathbb{F}\big(\varkappa,\varpi'(\varkappa)\big)\right|d\varkappa\\ &= \int_a^b\left|\mathbb{F}\big(\varkappa,\varpi'(\varkappa)\big)-\mathbb{F}(\varkappa,0)+\mathbb{F}(\varkappa,0)\right|d\varkappa\\ &\le \int_a^b\left|\mathbb{F}\big(\varkappa,\varpi'(\varkappa)\big)-\mathbb{F}(\varkappa,0)\right|d\varkappa+\int_a^b\left|\mathbb{F}(\varkappa,0)\right|d\varkappa\\ &\le C\int_a^b\left|\varpi'(\varkappa)\right|d\varkappa+\int_a^b\left|\mathbb{F}(\varkappa,0)\right|d\varkappa<\infty.\end{aligned}$$

Inequality (3.4) shows that

$$\|\mathfrak{L}(\varpi)\|_0 < \infty \quad \text{and} \quad \mathfrak{L}(\varpi)(a) = 0.$$

Moreover, $\mathfrak{L}(\varpi)$ is absolutely continuous on $[a,b]$ by Theorem 3.1. Consequently, $\mathfrak{L} : AC_0[a,b] \to AC_0[a,b]$. It remains to show that $\mathfrak{L}$ is a contraction. To this end, let $\varpi, \varpi^* \in AC_0[a,b]$. Then

$$\begin{aligned}\|\mathfrak{L}(\varpi) - \mathfrak{L}(\varpi^*)\|_0 \le \kappa E_{(\alpha_n-\alpha_{n-1},\dots,\alpha_n-\alpha_0,\alpha_n+1)}\biggl(|a_{n-1}|\biggl(\frac{b^\rho - a^\rho}{\rho}\biggr)^{\alpha_n-\alpha_{n-1}},\\ \dots, |a_0|\biggl(\frac{b^\rho - a^\rho}{\rho}\biggr)^{\alpha_n-\alpha_0}\biggr)\\ \times \biggl\|\int_a^\varkappa \mathbb{F}(\tau, \varpi'(\tau))\,d\tau - \int_a^\varkappa \mathbb{F}(\tau, \varpi^{*\prime}(\tau))\,d\tau\biggr\|_0.\end{aligned}$$

Since

$$\begin{aligned}\biggl\|\int_a^\varkappa \mathbb{F}(\tau, \varpi'(\tau))\,d\tau - \int_a^\varkappa \mathbb{F}(\tau, \varpi^{*\prime}(\tau))\,d\tau\biggr\|_0 &= \int_a^b \bigl|\mathbb{F}(\varkappa, \varpi'(\varkappa)) - \mathbb{F}(\varkappa, \varpi^{*\prime}(\varkappa))\bigr|\,d\varkappa\\ &\le C\int_a^b |\varpi' - \varpi^{*\prime}|\,d\varkappa\\ &= C\|\varpi - \varpi^*\|_0,\end{aligned}$$

we get

$$\begin{aligned}\|\mathfrak{L}(\varpi) - \mathfrak{L}(\varpi^*)\|_0 \le C\kappa E_{(\alpha_n-\alpha_{n-1},\dots,\alpha_n-\alpha_0,\alpha_n+1)}\biggl(|a_{n-1}|\biggl(\frac{b^\rho - a^\rho}{\rho}\biggr)^{\alpha_n-\alpha_{n-1}},\\ \dots, |a_0|\biggl(\frac{b^\rho - a^\rho}{\rho}\biggr)^{\alpha_n-\alpha_0}\biggr)\|\varpi - \varpi^*\|_0.\end{aligned}$$

Inequality (3.5) shows that $\mathfrak{L}$ is contractive. The proof is done. □

## 4 Ulam–Hyers stability (UHS)

Here, we develop and give some recent results on the UHS and generalized UHS of system (1.1). For $\epsilon > 0$ and $\varpi_1 \in AC_0[a,b]$, we consider the following inequality:

$$\begin{aligned}&\biggl|{}^C D_a^{\rho,\alpha_n}\varpi_1(\varkappa) + a_{n-1}{}^C D_a^{\rho,\alpha_{n-1}}\varpi_1(\varkappa) + \cdots + a_0{}^C D_a^{\rho,\alpha_0}\varpi_1(\varkappa)\\ &\quad - \int_a^\varkappa \mathbb{F}(\tau, \varpi'(\tau))\,d\tau\biggr| \le \epsilon, \quad \varkappa \in [a,b].\end{aligned} \tag{4.1}$$

*Remark* 4.1 Let $\epsilon > 0$. The function $\varpi_1 \in AC_0[a,b]$ satisfies (4.1) if and only if there exists a small perturbation $\zeta(\varkappa) \in AC_0[a,b]$ with $\zeta(0) = 0$ such that

(i) $\|\zeta\|_0 = \int_a^\varkappa |\zeta'(\tau)|\,d\tau \le \epsilon$, for $\varkappa \in [a,b]$,

(ii) For $\varkappa \in [a,b]$,

$$\begin{cases} {}^{C}D_a^{\rho,\alpha_n}\varpi_1(\varkappa) + a_{n-1}{}^{C}D_a^{\rho,\alpha_{n-1}}\varpi_1(\varkappa) + \cdots + a_0{}^{C}D_a^{\rho,\alpha_0}\varpi_1(\varkappa) \\ \quad = \int_a^{\varkappa} \mathbb{F}(\tau,\varpi_1'(\tau))\,d\tau + \int_a^{\varkappa} |\zeta'(\tau)|\,d\tau. \end{cases} \tag{4.2}$$

**Lemma 4.2** *The solution of perturbed problem* (4.2) *with the condition* $\varpi_1(a) = 0$ *satisfies the following inequality*:

$$\|\varpi_1 - Z_{\mathbb{F}}\|_0 \le \kappa E_{(\alpha_n-\alpha_{n-1},\dots,\alpha_n-\alpha_0,\alpha_n+1)}\left(|a_{n-1}|\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha_n-\alpha_{n-1}}, \dots, |a_0|\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha_n-\alpha_0}\right)\epsilon,$$

*where*

$$Z_{\mathbb{F}}(\varkappa) := \sum_{J=0}^{\infty}(-1)^J \sum_{J_1+\cdots+J_n=J} \binom{J}{J_1, J_2, \dots, J_n} \times a_{n-1}^{J_1}\cdots a_0^{J_n}\, {}^{\rho}I_a^{J_1(\alpha_n-\alpha_{n-1})+\cdots+J_n(\alpha_n-\alpha_0)+\alpha_n} \int_a^{\varkappa} \mathbb{F}\bigl(\tau,\varpi_1'(\tau)\bigr)\,d\tau$$

*and* $\kappa$ *is defined by* (3.3).

*Proof* By Theorem 3.1, the solution of perturbed problem (4.2) is given by

$$\varpi_1(\varkappa) = \sum_{J=0}^{\infty}(-1)^J \sum_{J_1+\cdots+J_n=J} \binom{J}{J_1, J_2, \dots, J_n} a_{n-1}^{J_1}\cdots a_0^{J_n} \times {}^{\rho}I_a^{J_1(\alpha_n-\alpha_{n-1})+\cdots+J_n(\alpha_n-\alpha_0)+\alpha_n}\left[\int_a^{\varkappa} \mathbb{F}\bigl(\tau,\varpi_1'(\tau)\bigr)\,d\tau + \int_a^{\varkappa}\bigl|\zeta'(\tau)\bigr|\,d\tau\right]. \tag{4.3}$$

From Eq. (4.3), Remark 4.1, and Eq. (3.4), we get

$$\begin{aligned} \|\varpi_1 - Z_{\mathbb{F}}\|_0 &\le \kappa E_{(\alpha_n-\alpha_{n-1},\dots,\alpha_n-\alpha_0,\alpha_n+1)}\left(|a_{n-1}|\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha_n-\alpha_{n-1}}, \dots, |a_0|\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha_n-\alpha_0}\right) \\ &\quad \times \left\|\int_a^{\varkappa} \mathbb{F}\bigl(\tau,\varpi_1'(\tau)\bigr)\,d\tau + \int_a^{\varkappa}\bigl|\zeta'(\tau)\bigr|\,d\tau - \int_a^{\varkappa} \mathbb{F}\bigl(\tau,\varpi_1'(\tau)\bigr)\,d\tau\right\|_0 \\ &\le \kappa E_{(\alpha_n-\alpha_{n-1},\dots,\alpha_n-\alpha_0,\alpha_n+1)}\left(|a_{n-1}|\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha_n-\alpha_{n-1}}, \dots, |a_0|\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha_n-\alpha_0}\right)\|\zeta\|_0 \\ &\le \kappa E_{(\alpha_n-\alpha_{n-1},\dots,\alpha_n-\alpha_0,\alpha_n+1)}\left(|a_{n-1}|\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha_n-\alpha_{n-1}}, \dots, |a_0|\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha_n-\alpha_0}\right)\epsilon. \end{aligned}$$

□

**Theorem 4.3** (UHS) *Assume that the assumptions of Theorem 3.5 and (4.1) hold. Then problem (1.1) is UH stable.*

*Proof* Let $\epsilon > 0$ and $\varpi_1 \in AC_0[a,b]$ satisfy (4.1), and let $\varpi \in AC_0[a,b]$ be a unique solution of

$$\begin{cases} {}^{C}D_a^{\rho,\alpha_n}\varpi(\varkappa) + a_{n-1}{}^{C}D_a^{\rho,\alpha_{n-1}}\varpi(\varkappa) + \cdots + a_0{}^{C}D_a^{\rho,\alpha_0}\varpi(\varkappa) = \int_a^{\varkappa}\mathbb{F}(\tau,\varpi'(\tau))\,d\tau, \\ \varpi(a) = \varpi_1(a) = 0, \end{cases}$$

that is,

$$\begin{aligned} \varpi(\varkappa) &= \varpi(a) + \sum_{J=0}^{\infty}(-1)^J \sum_{J_1+\cdots+J_n=J} \binom{J}{J_1, J_2, \ldots, J_n} \\ &\quad \times a_{n-1}^{J_1}\cdots a_0^{J_n}\,{}^{\rho}I_a^{J_1(\alpha_n-\alpha_{n-1})+\cdots+J_n(\alpha_n-\alpha_0)+\alpha_n}\left[\int_a^{\varkappa}\mathbb{F}\big(\tau,\varpi'(\tau)\big)\,d\tau\right]. \end{aligned}$$

Since $\varpi(a) = \varpi_1(a) = 0$, we get

$$\begin{aligned} \varpi(\varkappa) &= \sum_{J=0}^{\infty}(-1)^J \sum_{J_1+\cdots+J_n=J} \binom{J}{J_1, J_2, \ldots, J_n} \\ &\quad \times a_{n-1}^{J_1}\cdots a_0^{J_n}\,{}^{\rho}I_a^{J_1(\alpha_n-\alpha_{n-1})+\cdots+J_n(\alpha_n-\alpha_0)+\alpha_n}\left[\int_a^{\varkappa}\mathbb{F}\big(\tau,\varpi'(\tau)\big)\,d\tau\right]. \end{aligned}$$

By virtue of Lemma 4.2 and Eq. (3.4), we have

$$\begin{aligned} \|\varpi_1 - \varpi\|_0 &\le \|\varpi_1 - Z_{\mathbb{F}}\|_0 + \|Z_{\mathbb{F}} - \varpi\|_0 \\ &\le \kappa E_{(\alpha_n-\alpha_{n-1},\ldots,\alpha_n-\alpha_0,\alpha_n+1)}\left(|a_{n-1}|\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha_n-\alpha_{n-1}},\right. \\ &\quad \left.\ldots, |a_0|\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha_n-\alpha_0}\right)\epsilon \\ &\quad + \kappa E_{(\alpha_n-\alpha_{n-1},\ldots,\alpha_n-\alpha_0,\alpha_n+1)}\left(|a_{n-1}|\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha_n-\alpha_{n-1}},\right. \\ &\quad \left.\ldots, |a_0|\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha_n-\alpha_0}\right) \\ &\quad \times \left\|\int_a^{\varkappa}\mathbb{F}\big(\tau,\varpi_1'(\tau)\big)\,d\tau - \int_a^{\varkappa}\mathbb{F}\big(\tau,\varpi'(\tau)\big)\,d\tau\right\|_0. \end{aligned}$$

Using the assumptions of Theorem 3.5, we obtain

$$\left\|\int_a^{\varkappa}\mathbb{F}\big(\tau,\varpi_1'(\tau)\big)\,d\tau - \int_a^{\varkappa}\mathbb{F}\big(\tau,\varpi'(\tau)\big)\,d\tau\right\|_0 \le C\|\varpi_1 - \varpi\|_0.$$

Consequently,

$$\|\varpi_1 - \varpi\|_0 \le \kappa E_{(\alpha_n-\alpha_{n-1},\ldots,\alpha_n-\alpha_0,\alpha_n+1)}\left(|a_{n-1}|\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha_n-\alpha_{n-1}},\right.$$

$$
\begin{aligned}
&\dots, |a_0|\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha_n-\alpha_0}\right)\epsilon \\
&+ \kappa E_{(\alpha_n-\alpha_{n-1},\dots,\alpha_n-\alpha_0,\alpha_n+1)}\left(|a_{n-1}|\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha_n-\alpha_{n-1}},\right. \\
&\left.\dots, |a_0|\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha_n-\alpha_0}\right)C\|\varpi_1 - \varpi\|_0.
\end{aligned}
$$

By dint of inequality (3.5), we conclude that

$$
\|\varpi_1 - \varpi\|_0 \le C_{\mathbb{F}}\epsilon,
$$

where $C_{\mathbb{F}} := \frac{\mathfrak{R}}{1-\mathfrak{R}C}$ and

$$
\mathfrak{R} := \kappa E_{(\alpha_n-\alpha_{n-1},\dots,\alpha_n-\alpha_0,\alpha_n+1)}\left(|a_{n-1}|\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha_n-\alpha_{n-1}}, \dots, |a_0|\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha_n-\alpha_0}\right). \qquad \square
$$

**Conclusion 4.4** Under assumptions of Theorem 4.3, if we set $\Phi(\epsilon) = C_{\mathbb{F}}\epsilon$ such that $\Phi(0) = 0$, then problem (1.1) is generalized Ulam–Hyers stable.

*Example* 4.5 Let $a = 1$ and $b = \sqrt[\rho]{1+\rho}$. Then there is a unique solution for the following Katugampola-type FIDE:

$$
\begin{aligned}
&{}^C D_a^{\rho,0.9}\varpi(\varkappa) - {}^C D_a^{\rho,0.4}\varpi(\varkappa) + {}^C D_a^{\rho,0.3}\varpi(\varkappa) - {}^C D_a^{\rho,0.1}\varpi(\varkappa) \\
&= \int_a^{\varkappa}\left(\frac{e^{\tau^2}}{C(3+e^{\tau^2})}\sin\varpi'(\tau) + e^{\cos\tau} + \ln(1+\sqrt{\tau})\right)d\tau, \qquad (4.4)
\end{aligned}
$$

where the constant $C$ is to be determined.

Clearly, the function

$$
\mathbb{F}(\varkappa, z) = \frac{e^{\varkappa^2}}{C(3+e^{\varkappa^2})}\sin z + e^{\cos\varkappa} + \ln(1+\sqrt{\varkappa})
$$

is a continuous function from $[1, \sqrt[\rho]{1+\rho}] \times \mathbb{R}$ to $\mathbb{R}$ and satisfies

$$
\begin{aligned}
\left|\mathbb{F}(\varkappa, z_1) - \mathbb{F}(\varkappa, z_2)\right| &= \left|\frac{e^{\varkappa^2}}{C(3+e^{\varkappa^2})}\sin z_1 - \frac{e^{\varkappa^2}}{C(3+e^{\varkappa^2})}\sin z_1\right| \\
&\le \frac{e^{\varkappa^2}}{C(3+e^{\varkappa^2})}|\sin z_1 - \sin z_1| \\
&\le \frac{e^{\varkappa^2}}{C(3+e^{\varkappa^2})}|z_1 - z_1| \le \frac{1}{C}|z_1 - z_1|.
\end{aligned}
$$

Obviously $\frac{b^\rho - a^\rho}{\rho} = 1$ and

$$
\begin{aligned}
&\sum_{J=0}^{\infty}\sum_{J_1+J_2+J_3=J}\binom{J}{J_1, J_2, J_3} \\
&\quad\times\left(|-1|\left(\frac{b^\rho - a^\rho}{\rho}\right)^{0.5}\right)^{J_1}\left(|1|\left(\frac{b^\rho - a^\rho}{\rho}\right)^{0.6}\right)^{J_2}\left(|-1|\left(\frac{b^\rho - a^\rho}{\rho}\right)^{0.8}\right)^{J_3}
\end{aligned}
$$

$$\begin{aligned}
&\times \frac{1}{\Gamma(0.5\jmath_1 + 0.6\jmath_2 + 0.8\jmath_3 + 1.9)} \\
&= \sum_{\jmath=0}^{\infty} \sum_{\jmath_1+\jmath_2+\jmath_3=\jmath} \binom{\jmath}{\jmath_1, \jmath_2, \jmath_3} \frac{1}{\Gamma(0.5\jmath_1 + 0.6\jmath_2 + 0.8\jmath_3 + 1.9)} \\
&= E_{(0.5,0.6,0.8,1.9)}(1,1,1).
\end{aligned}$$

Then we choose a positive $C$ such that

$$C\kappa E_{(0.5,0.6,0.8,1.9)}(1,1,1) < 1.$$

According to Theorem 3.5, Eq. (4.4) has a unique solution.

Furthermore, by Theorem 4.3, for any solution $\varpi_1(\varkappa) \in AC_0[a,b]$ of the inequality

$$\begin{aligned}
&\bigg| {}^{C}D_a^{\rho,0.9}\varpi_1(\varkappa) - {}^{C}D_a^{\rho,0.4}\varpi_1(\varkappa) + {}^{C}D_a^{\rho,0.3}\varpi_1(\varkappa) - {}^{C}D_a^{\rho,0.1}\varpi_1(\varkappa) \\
&\quad - \int_a^{\varkappa} \left( \frac{e^{\tau^2}}{C(3+e^{\tau^2})} \sin \varpi_1'(\tau) + e^{\cos\tau} + \ln(1+\sqrt{\tau}) \right) d\tau \bigg| \le \epsilon, \quad \varkappa \in [a,b],
\end{aligned} \tag{4.5}$$

there exists a unique solution $\varpi(\varkappa) \in AC_0[a,b]$ of Eq. (4.4) such that

$$\|\varpi_1 - \varpi\|_0 \le C_{\mathbb{F}}\epsilon,$$

where $C_{\mathbb{F}} := \frac{\mathfrak{R}}{1-\mathfrak{R}C} > 0$, $\mathfrak{R} = \kappa E_{(0.5,0.6,0.8,1.9)}(1,1,1)$, and $\kappa = \frac{1}{\Gamma(0.5\jmath_1+0.6\jmath_2+0.8\jmath_3+1.9)}$. Hence Eq. (4.4) is UH stable.

*Remark* 4.6 The results obtained in this work will remain valid if we use generalized Riemann–Liouville-type instead of generalized Caputo-type in the proposed problem (1.1). Specifically, in problem (1.1), if we replace Caputo derivatives with Riemann–Liouville derivatives, then Lemma 2.9 is valid with respect to ${}^{RL}D_a^{\rho,\alpha}(\cdot)$, due to $\vartheta(a) = 0$ whenever $\vartheta \in AC_0[a,b]$.

## 5 Conclusions

Using the Banach space $AC_0[a,b]$, Banach's fixed point technique, and Babenko's method, we have obtained the uniqueness of solutions for nonlinear FIDE (1.1) with generalized Caputo fractional derivatives. Moreover, we have proven various types of stability analysis of the suggested problem. Also, some pertinent examples are given to substantiate the main results. The reported results in this study extend and develop the presented study by Li [38].

As future work, we are thinking of extending the current results to include more generalized operators such as $\psi$-Caputo [52].

**Acknowledgements**
The authors T. Abdeljawad and W. Shatanawi would like to thank Prince Sultan University for the moral support to this work through the research group Nonlinear Analysis Methods in Applied Mathematics (NAMAM) group number RG-DES-2017-01-17. The authors also thank the reviewers for their valuable remarks on our work.

**Funding**
Funding source is not available.

**Availability of data and materials**
This is not applicable in our paper.

**Competing interests**
The authors declare that they have no competing interests.

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All authors contributed this manuscript and approved the final version.

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## Publisher's Note



Received: 8 March 2021 Accepted: 28 July 2021 Published online: 10 August 2021

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