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New general integral transform via Atangana–Baleanu derivatives

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Abstract

The current paper is about the investigation of a new integral transform introduced recently by Jafari. Specifically, we explore the applicability of this integral transform on Atangana–Baleanu derivative and the associated fractional integral. It is shown that by applying specific conditions on this integral transform, other integral transforms are deduced. We provide examples to reinforce the applicability of this new integral transform.

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1 Introduction

Fractional calculus is regarded as an extension of integer calculus in the sense that it permits the order of the derivative or integral to be a fraction. The idea of having a fractional order derivative did not make practical sense in the real world, thus the subject of fractional calculus was mainly reserved for mathematicians for a long time since its discovery [1–3].

However, researchers have come to realise that models constructed from fractional calculus can successfully represent real world problems and sometimes yield better results compared to models from the integer calculus. Some useful results from fractional calculus models appear in engineering, physics, biology and economics [1–14].

The definition of the fractional derivative is in itself a developing concept. Numerous definitions have been suggested for the fractional derivative, with almost each definition possessing some form of deficiency. It is generally believed that the choice of the derivative used is dictated by the situation that is being modelled.

Since it was discovered that the fractional derivative can be successfully applied to practical problems, it is the Caputo derivative that has been used the most. The only shortcoming of the Caputo derivative is the singularity issue. Other fractional derivatives that have this singularity problem are Riemann–Liouville, Caputo–Hadamard, and Riesz (see [1–3, 15, 16]). In a bid to address the singularity concern, the Caputo–Fabrizio derivative was proposed [17, 18], this derivative eliminated the singularity problem through the use of an exponential kernel. Atangana and Baleanu replaced the exponential kernel by the Mittag-Leffler function to create another non-singular kernel derivative called the Atangana–Baleanu derivative. More detailed discussions encompassing both theory and applications of the non-singular derivatives are found in [19–24]. In [5, 25–28], the authors discuss the uniqueness and existence of fractional differential equations.

To fully harness the capability of fractional differential equations in modelling problems that arise in the real world, it is imperative that we have methods of solutions that are computationally inexpensive, easily accessible and highly accurate. Integral transforms are some of the techniques that have proven their worth, as they are regarded to be easy to implement and demand reasonable labour in terms of computations.

Integral transforms offer an alternative to integration in the solution of differential equations. The integral transform maps the domain of the original problem into a different domain consisting of an algebraic equation that is normally easy to manipulate. Taking the inverse of the new domain results in the solution of the original problem [29].

There are different types of integral transforms that are used in the solution of differential equations, but it is the Laplace transform that is mostly applied. Most of the integral transforms that have been suggested are extensions of the Laplace transform. Some of the integral transforms that are closely related to the Laplace transform are the Elzaki transform, Sumudu transform, Shehu transform, *etc.* [30–34].

Recently, a more generalized integral transform has been introduced by the second author [35]. Imposing specific conditions on this integral transform yields other integral transforms, for example, the Laplace transform, natural transform, Elzaki transform and Sawi transform [35]. To get a deeper insight into the properties and applications of this new general integral transform, we refer the reader to [35].

Our main intention in this research is to investigate the application of this new generalized integral transform in the solution of differential equations involving the Atangana–Baleanu derivative.

We structure the rest of our work in the following manner. In the next section, we provide some important mathematical concepts that will form the basis for our research. We then follow by presenting the main results of the research in section three. To reinforce the theoretical aspects of our work, we provide applications in the fourth section. A concise summary of our research findings is provided in the last section.

2 Preliminaries

Throughout, set

$$A = \{f(t) : \exists M > 0, k > 0, |f(t)| \leq Me^{kt}, \text{ if } t \geq 0\},$$

and suppose that $f(t)$ is an integrable function defined on the set A.

Definition 1 ([35]) Consider the functions $\varphi(s), \psi(s) : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that $\varphi(s) \neq 0$ for all $s \in \mathbb{R}^+$. The new general integral transform of the function $f(t)$ denoted by $\mathcal{F}_J(s)$ is defined by

$$T\{f(t), s\} = \mathcal{F}_J(s) = \varphi(s) \int_0^\infty f(t) e^{-\psi(s)t}\, dt, \tag{1}$$

with the corresponding inverse,

$$\mathcal{F}_{\mathrm{J}}^{-1}(s) = T^{-1}\left\{\varphi(s)\int_0^\infty f(t)e^{-\psi(s)t}\,dt\right\} = f(t).$$

The integral transform (1) exists for all $\psi(s) > k$. It is simple to check that the new general integral transform is a linear operator and has many properties that are similar to other integral transforms, more detailed discussion on this can be found in [35].

Theorem 1 *The integral transform $\mathcal{F}_{\mathrm{J}}(s)$ of the derivative of $f(t)$ is given as* [35],

$$T\bigl\{f^{(n)}(t),s\bigr\} = \psi^n(s)\mathcal{F}_{\mathrm{J}}(s) - \varphi(s)\sum_{k=0}^{n-1}\psi^{n-1-k}(s)f^{(k)}(0), \quad \varphi(s),\psi(s) > 0\ \forall n \in \mathbb{N}. \tag{2}$$

Theorem 2 *If $\mathcal{F}_{\mathrm{J}}(s)$ and $\mathcal{H}_{\mathrm{J}}(s)$ are general integral transforms of $f(t)$ and $h(t)$, respectively, then*

$$T\{f*h\} = \frac{1}{\varphi(s)}\mathcal{F}_{\mathrm{J}}(s)\cdot\mathcal{H}_{\mathrm{J}}(s).$$

Moreover,

$$T^{-1}\{f\cdot h\} = \varphi(s)T^{-1}\{f\} * T^{-1}\{h\}.$$

Proof We have

$$f*h = \int_0^\infty f(\tau)h(t-\tau)\,d\tau.$$

Using the new general transform and the Leibniz theorem, we obtain

$$\begin{aligned} T\{f*h\} &= T\left\{\int_0^\infty f(\tau)h(t-\tau)\,d\tau\right\} = \varphi(s)\int_0^\infty\left[\int_0^\infty f(\tau)h(t-\tau)\,d\tau\right]e^{-\psi(s)t}\,dt \\ &= \varphi(s)\int_0^\infty f(\tau)\left[\int_0^\infty h(t-\tau)e^{-\psi(s)t}\,dt\right]d\tau, \end{aligned}$$

by setting $u = t-\tau$, we get

$$\begin{aligned} T\{f*h\} &= \varphi(s)\int_0^\infty f(\tau)e^{-\psi(s)\tau}\left[\int_0^\infty h(u)e^{-\psi(s)u}\,dt\right]d\tau \\ &= \varphi(s)\int_0^\infty f(\tau)e^{-\psi(s)\tau}\,d\tau \times \frac{1}{\varphi(s)}\mathcal{H}_{\mathrm{J}}(s) \\ &= \frac{1}{\varphi(s)}\mathcal{F}_{\mathrm{J}}(s)\cdot\mathcal{H}_{\mathrm{J}}(s). \end{aligned}$$

Furthermore, the convolution of the inverse transform is

$$T\bigl\{T^{-1}\{f\} * T^{-1}\{h\}\bigr\} = \frac{1}{\varphi(s)}f\cdot h.$$

Table 1 Integral transform of some basic functions

$f(t)$	$T\{f(t),s\} = \mathcal{F}_J(s)$
c	$\frac{\varphi(s)}{\psi(s)}c, c \in \mathbb{R}$
t	$\frac{\varphi(s)}{\psi(s)^2}$
t^α	$\frac{\Gamma(\alpha+1)\varphi(s)}{\psi(s)^{\alpha+1}}, \alpha > 0$
$\sin t$	$\frac{\varphi(s)}{\psi(s)^2+1}$
$\cos t$	$\frac{\varphi(s)\psi(s)}{\psi(s)^2+1}$

Hence,

$$T^{-1}\{f \cdot h\} = \varphi(s)T^{-1}\{f\} * T^{-1}\{h\}. \qquad \square$$

Table 1 shows the general integral transforms of some basic functions.

The two parameter Mittag-Leffler function is stated as [3]

$$E_{\eta,\sigma}(t) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\eta k + \sigma)}, \quad z \in \mathbb{C}, \mathrm{Re}(\eta) > 0 \text{ and } \mathrm{Re}(\sigma) > 0.$$

A variant of the Mittag-Leffler function is given by Prabhakar as [36]

$$E^{\gamma}_{\eta,\sigma}(z) = \sum_{k=0}^{\infty} \frac{\gamma_k}{\Gamma(\eta k + \sigma)} \frac{z^k}{k!}, \quad z \in \mathbb{C}, \mathrm{Re}(\eta) > 0, \mathrm{Re}(\sigma) > 0 \text{ and } \mathrm{Re}(\gamma) > 0,$$

γ_k is the Pochhammer symbol.

Definition 2 ([19, 22]) The Atangana–Baleanu fractional derivative is defined by

$${}^{ABC}_{a}D^{\eta}_{t}\big(f(t)\big) = \frac{\mathcal{K}(\eta)}{1-\eta} \int_a^t f'(x) E_\eta\left(\frac{\eta}{\eta-1}(t-x)^\eta\right) dx, \tag{3}$$

another version of the Atangana–Baleanu derivative is stated as

$${}^{ABR}_{a}D^{\eta}_{t}\big(f(t)\big) = \frac{\mathcal{K}(\eta)}{1-\eta} \frac{d}{dt} \int_a^t f(x) E_\eta\left(\frac{\eta}{\eta-1}(t-x)^\eta\right) dx, \tag{4}$$

where $\eta \in (0,1)$ and $\mathcal{K}(\eta)$ represents the normalization function with the property $\mathcal{K}(0) = \mathcal{K}(1) = 1$.

Definition 3 ([19, 22]) The fractional integral associate to the fractional derivative of Atangana–Baleanu is defined by

$${}^{AB}_{a}I^{\eta}_{t}\big(f(t)\big) = \frac{1-\eta}{\mathcal{K}(\eta)} f(t) + \frac{\eta}{\mathcal{K}(\eta)\Gamma(\eta)} \int_a^t f(x)(t-x)^{\eta-1}\, dx. \tag{5}$$

When $\eta = 0$ we recover the initial function, and if $\eta = 1$, we obtain the ordinary integral.

3 New general transform for Atangana–Baleanu fractional derivatives

We present the main results of our research in this section.

Lemma 1 *Let* $0 < \eta < 1$ *and* $\lambda \in \mathbb{R}$ *such that* $\psi(s) < |\lambda|^{\frac{1}{\eta}}$, *then*

$$T\{t^{\sigma-1}E^{\gamma}_{\eta,\sigma}(\lambda t^{\eta}),s\} = \frac{\varphi(s)}{\psi(s)^{\sigma}}\frac{1}{(1-\frac{\lambda}{\psi(s)^{\eta}})^{\gamma}}, \quad \psi(s) > 0. \tag{6}$$

Proof The new general integral transform of the function $t^{\sigma-1}E^{\gamma}_{\eta,\sigma}(\lambda t^{\eta})$ yields

$$\begin{aligned} T\{t^{\sigma-1}E^{\gamma}_{\eta,\sigma}(\lambda t^{\eta}),s\} &= \varphi(s)\int_0^{\infty} t^{\sigma-1}E^{\gamma}_{\eta,\sigma}(\lambda t^{\eta})e^{-\psi(s)t}\,dt \\ &= \varphi(s)\int_0^{\infty} t^{\sigma-1}\sum_{k=0}^{\infty}\frac{\gamma_k}{\Gamma(\eta k+\sigma)}\frac{(\lambda t^{\eta})^k}{k!}e^{-\psi(s)t}\,dt \\ &= \sum_{k=0}^{\infty}\frac{\gamma_k}{\Gamma(\eta k+\sigma)}\frac{\lambda^k}{k!}\varphi(s)\int_0^{\infty} t^{\eta k+\sigma-1}e^{-\psi(s)t}\,dt \\ &= \sum_{k=0}^{\infty}\frac{\gamma_k}{\Gamma(\eta k+\sigma)}\frac{\lambda^k}{k!}T\{t^{\eta k+\sigma-1}\}, \end{aligned}$$

thus,

$$\begin{aligned} T\{t^{\sigma-1}E^{\gamma}_{\eta,\sigma}(\lambda t^{\eta}),s\} &= \sum_{k=0}^{\infty}\frac{\gamma_k}{\Gamma(\eta k+\sigma)}\frac{\lambda^k}{k!}T\{t^{\eta k+\sigma-1}\} \\ &= \sum_{k=0}^{\infty}\frac{\gamma_k}{\Gamma(\eta k+\sigma)}\frac{\lambda^k}{k!}\frac{\Gamma(\eta k+\sigma)\varphi(s)}{\psi(s)^{\eta k+\sigma}} \\ &= \frac{\varphi(s)}{\psi(s)^{\sigma}}\sum_{k=0}^{\infty}\frac{\gamma_k}{k!}\left(\frac{\lambda}{\psi(s)^{\eta}}\right)^k, \end{aligned}$$

since $q(s) < |\lambda|^{\frac{1}{\eta}}$, it follows that

$$T\{t^{\sigma-1}E^{\gamma}_{\eta,\sigma}(\lambda t^{\eta}),s\} = \frac{\varphi(s)}{\psi(s)^{\sigma}}\frac{1}{(1-\frac{\lambda}{\psi(s)^{\eta}})^{\gamma}}.$$

□

Corollary 1 *Under the same conditions of Lemma* (1), *we have the new general transform of the function* $E_{\eta}(\lambda t^{\eta})$ *as*

$$T\{E_{\eta}(\lambda t^{\eta})\} = \frac{\varphi(s)}{\psi(s)}\frac{\psi(s)^{\eta}}{\psi(s)^{\eta}-\lambda},$$

and the new general transform of the function $t^{\sigma-1}E_{\eta}(\lambda t^{\eta})$ *as*

$$T\{t^{\sigma-1}E_{\eta}(\lambda t^{\eta})\} = \frac{\varphi(s)\psi(s)^{\eta-\sigma}}{\psi(s)^{\eta}-\lambda}.$$

Proof We have

$$T\{E_\eta(\lambda t^\eta)\} = T\{E^1_{\eta,1}(\lambda t^\eta), s\} = \frac{\varphi(s)}{\psi(s)} \frac{1}{1 - \frac{\lambda}{\psi(s)^\eta}}$$

and

$$T\{t^{\sigma-1}E_\eta(\lambda t^\eta)\} = T\{t^{\sigma-1}E^1_{\eta,\sigma}(\lambda t^\eta), s\} = \frac{\varphi(s)\psi(s)^{\eta-\sigma}}{\psi(s)^\eta - \lambda}. \qquad \square$$

Remark 1 Lemma (1) generalizes the results obtained in [37] for the Sumudu transform and in [38] for the Shehu transform of the function $t^{\sigma-1}E^\gamma_{\eta,\sigma}(\lambda t^\eta)$. Indeed, when $\varphi(s) = \frac{1}{s}$ and $\psi(s) = \frac{1}{s}$ the Sumudu transform is

$$S\{t^{\sigma-1}E^\gamma_{\eta,\sigma}(\lambda t^\eta)\} = \frac{\frac{1}{s}}{(\frac{1}{s})^\sigma} \frac{1}{(1 - \frac{\lambda}{(\frac{1}{s})^\eta})^\gamma} = s^{\sigma-1}(1 - \lambda s^\eta)^{-\gamma},$$

for $\varphi(s) = 1$ and $\psi(s) = \frac{s}{u}$, we obtain the Shehu transform,

$$SH\{t^{\sigma-1}E^\gamma_{\eta,\sigma}(\lambda t^\eta)\} = \frac{1}{(\frac{s}{u})^\sigma} \frac{1}{(1 - \frac{\lambda}{(\frac{s}{u})^\eta})^\gamma} = \left(\frac{u}{s}\right)^\sigma \left(1 - \lambda\left(\frac{u}{s}\right)^\eta\right)^{-\gamma}.$$

Obviously, when $\gamma = 1$, we get the Sumudu and Shehu transform of the function $t^{\sigma-1}E_{\eta,\sigma}(\lambda t^\eta)$.

Henceforth, we suppose that the function $f(t) \in A \cap H^1(a,b)$ such that $H^1(a,b)$ is a Sobolev space of order one defined by

$$H^1(a,b) = \{f \in L^2(a,b) : f' \in L^2(a,b)\}.$$

Theorem 3 *The new general integral transform of the Atangana–Baleanu derivative stated in* (3) *is given as*

$$T\{{}^{ABC}_0D^\eta_t(f(t))\} = \frac{\mathcal{K}(\eta)}{1-\eta} \frac{\psi(s)^\eta}{\psi(s)^\eta + \frac{\eta}{1-\eta}} \left[\mathcal{F}_J(s) - \frac{\varphi(s)}{\psi(s)} f(0)\right].$$

Proof Let $\mathcal{F}_J(s)$ be the new general transform of the function $f(t)$. Let us observe that in definition (3) we have a convolution integral,

$$\int_0^t f'(x)E_\eta\left(\frac{\eta}{\eta-1}(t-x)^\eta\right)dx = f'(t) * E_\eta\left(\frac{\eta}{\eta-1}t^\eta\right)$$

then one has

$$\begin{aligned} T\{{}^{ABC}_0D^\eta_t(f(t))\} &= T\left\{\frac{\mathcal{K}(\eta)}{1-\eta}\int_0^t f'(x)E_\eta\left(\frac{\eta}{\eta-1}(t-x)^\eta\right)dx\right\} \\ &= \frac{\mathcal{K}(\eta)}{1-\eta} T\left\{f'(t) * E_\eta\left(\frac{\eta}{\eta-1}t^\eta\right)\right\} \end{aligned}$$

$$= \frac{\mathcal{K}(\eta)}{1-\eta}\frac{1}{\varphi(s)}T\{f'(t)\}\cdot T\left\{E_\eta\left(\frac{\eta}{\eta-1}t^\eta\right)\right\}.$$

Using Theorem 1 and applying the result obtained in Corollary 1, then

$$\begin{aligned} T\{{}_0^{ABC}D_t^\eta(f(t))\} &= \frac{\mathcal{K}(\eta)}{1-\eta}\frac{1}{\varphi(s)}\left[\psi(s)\mathcal{F}_J(s)-\varphi(s)f(0)\right]\frac{\varphi(s)}{\psi(s)}\frac{\psi(s)^\eta}{\psi(s)^\eta+\frac{\eta}{1-\eta}} \\ &= \frac{\mathcal{K}(\eta)}{1-\eta}\frac{\psi(s)^\eta}{\psi(s)^\eta+\frac{\eta}{1-\eta}}\left[\mathcal{F}_J(s)-\frac{\varphi(s)}{\psi(s)}f(0)\right]. \end{aligned}$$

□

Corollary 2

- *If $\varphi(s)=1$ and $\psi(s)=s$, then this new transform gives the Laplace transform, see* [19, 20],

$$\begin{aligned} L\{{}_0^{ABC}D_t^\eta(f(t))\} &= \frac{\mathcal{K}(\eta)}{1-\eta}\times\frac{s^\eta}{s^\eta+\frac{\eta}{1-\eta}}\left[\mathcal{F}_J(s)-\frac{1}{s}f(0)\right] \\ &= \frac{\mathcal{K}(\eta)}{1-\eta}\frac{s^\eta\mathcal{F}_J(s)-s^{\eta-1}f(0)}{s^\eta+\frac{\eta}{1-\eta}}. \end{aligned}$$

- *If $\varphi(s)=s$ and $\psi(s)=\frac{1}{s}$, we have the Elzaki transform* [39],

$$\begin{aligned} E\{{}_0^{ABR}D_t^\eta(u(t))\} &= \frac{\mathcal{K}(\eta)}{1-\eta}\frac{(\frac{1}{s})^\eta}{(\frac{1}{s})^\eta+\frac{\eta}{1-\eta}}\left[\mathcal{F}_J(s)-\frac{s}{\frac{1}{s}}f(0)\right] \\ &= \frac{\mathcal{K}(\eta)}{1-\eta}\frac{1}{1+\frac{\eta}{1-\eta}s^\eta}\left[\mathcal{F}_J(s)-s^2f(0)\right]. \end{aligned}$$

- *If $\varphi(s)=\psi(s)=\frac{1}{s}$, the new transform coincides with the Sumudu transform* [40],

$$\begin{aligned} S\{{}_0^{ABC}D_t^\eta(u(t))\} &= \frac{\mathcal{K}(\eta)}{1-\eta}\frac{\frac{1}{s^\eta}}{\frac{1}{s^\eta}+\frac{\eta}{1-\eta}}\left[\mathcal{F}_J(s)-f(0)\right] \\ &= \frac{\mathcal{K}(\eta)}{1-\eta}\frac{1}{1+\frac{\eta}{1-\eta}s^\eta}\left[\mathcal{F}_J(s)-f(0)\right]. \end{aligned}$$

- *If $\varphi(s)=1$ and $\psi(s)=\frac{s}{u}$, then the Shehu transform of the Atangana–Baleanu fractional derivative in Caputo sense* [40] *is obtained,*

$$\begin{aligned} SH\{{}_0^{ABC}D_t^\eta(u(t))\} &= \frac{\mathcal{K}(\eta)}{1-\eta}\frac{(\frac{s}{u})^\eta}{(\frac{s}{u})^\eta+\frac{\eta}{1-\eta}}\left[\mathcal{F}_J(s)-\frac{1}{\frac{s}{u}}f(0)\right] \\ &= \frac{\mathcal{K}(\eta)}{1-\eta}\frac{(\frac{s}{u})^\eta\mathcal{F}_J(s)-(\frac{s}{u})^{\eta-1}f(0)}{(\frac{s}{u})^\eta+\frac{\eta}{1-\eta}}. \end{aligned}$$

Theorem 4 *The new general integral transform of the Atangana–Baleanu derivative stated in* (4) *is given as*

$$T\{{}_0^{ABR}D_t^\eta(u(t))\} = \frac{\mathcal{K}(\eta)}{1-\eta}\frac{\psi(s)^\eta}{\psi(s)^\eta+\frac{\eta}{1-\eta}}\mathcal{F}_J(s).$$

Proof Let $\mathcal{F}_{\mathrm{J}}(s)$ be the generalized transform of the function $f(t)$. We have

$$\begin{aligned} T\left\{{}_0^{ABR}D_t^{\eta}\bigl(f(t)\bigr)\right\} &= T\left\{\frac{\mathcal{K}(\eta)}{1-\eta}\frac{d}{dt}\int_a^t f(x)E_{\eta}\left(\frac{\eta}{\eta-1}(t-x)^{\eta}\right)dx\right\} \\ &= \frac{\mathcal{K}(\eta)}{1-\eta}T\left\{\frac{d}{dt}\left[f(t)*E_{\eta}\left(\frac{\eta}{\eta-1}t^{\eta}\right)\right]\right\} \\ &= \frac{\mathcal{K}(\eta)}{1-\eta}\left[\psi(s)T\left\{f(t)*E_{\eta}\left(\frac{\eta}{\eta-1}t^{\eta}\right)\right\}-\varphi(s)T\bigl\{f(0)*E_{\eta}(0)\bigr\}\right], \end{aligned}$$

hence,

$$T\left\{{}_0^{ABR}D_t^{\eta}\bigl(f(t)\bigr)\right\} = \frac{\mathcal{K}(\eta)}{1-\eta}\frac{\psi(s)^{\eta}}{\psi(s)^{\eta}+\frac{\eta}{1-\eta}}\mathcal{F}_{\mathrm{J}}(s).$$

□

Corollary 3

- *If* $\varphi(s)=1$ *and* $\psi(s)=s$, *then the Laplace transform is given by* [19, 20]

$$L\left\{{}_0^{ABR}D_t^{\eta}\bigl(f(t)\bigr)\right\} = \frac{\mathcal{K}(\eta)}{1-\eta}\frac{s^{\eta}}{s^{\eta}+\frac{\eta}{1-\eta}}\mathcal{F}_{\mathrm{J}}(s).$$

- *If* $\varphi(s)=s$ *and* $\psi(s)=\frac{1}{s}$, *we have the Elzaki transform* [39],

$$E\left\{{}_0^{ABC}D_t^{\eta}\bigl(f(t)\bigr)\right\} = \frac{\mathcal{K}(\eta)}{1-\eta}\frac{1}{1+\frac{\eta}{1-\eta}s^{\eta}}\mathcal{F}_{\mathrm{J}}(s).$$

- *If* $\varphi(s)=\psi(s)=\frac{1}{s}$, *we get the Sumudu transform as follows* [40]:

$$S\left\{{}_0^{ABC}D_t^{\eta}\bigl(f(t)\bigr)\right\} = \frac{\mathcal{K}(\eta)}{1-\eta}\frac{(\frac{1}{s})^{\eta}}{(\frac{1}{s})^{\eta}+\frac{\eta}{1-\eta}}\mathcal{F}_{\mathrm{J}}(s).$$

- *If* $\varphi(s)=1$ *and* $\psi(s)=\frac{s}{u}$, *then the Shehu transform of the Atangana–Baleanu fractional derivative in Riemann–Liouville sense* [40] *is given as*

$$SH\left\{{}_0^{ABC}D_t^{\eta}\bigl(f(t)\bigr)\right\} = \frac{\mathcal{K}(\eta)}{1-\eta}\frac{(\frac{s}{u})^{\eta}}{(\frac{s}{u})^{\eta}+\frac{\eta}{1-\eta}}\mathcal{F}_{\mathrm{J}}(s).$$

Lemma 2 *The new general integral transform of the function* $t^{\eta-1}$ *is given as*

$$T\left\{t^{\eta-1}\right\} = \frac{\Gamma(\eta)\varphi(s)}{\psi(s)^{\eta}}, \quad \eta>0.$$

Proof Applying the new general integral transform of the function $t^{\eta-1}$, we get

$$T\left\{t^{\eta-1}\right\} = \varphi(s)\int_0^{\infty} t^{\eta-1}e^{-\psi(s)t}\,dt.$$

Let $\Gamma(\eta)$ be the Gamma function defined by

$$\Gamma(\eta) = \int_0^{\infty} u^{\eta-1}e^{-u}\,du.$$

Setting $u = \psi(s)t$ implies $du = \psi(s)\,dt$, then

$$\Gamma(\eta) = \int_0^\infty \big(\psi(s)t\big)^{\eta-1} e^{-\psi(s)t}\psi(s)\,dt = \psi(s)^\eta \int_0^\infty t^{\eta-1}e^{-\psi(s)t}\,dt,$$

thus,

$$\int_0^\infty t^{\eta-1}e^{-\psi(s)t}\,dt = \frac{\Gamma(\eta)}{\psi(s)^\eta}.$$

Hence,

$$T\{t^{\eta-1}\} = \frac{\Gamma(\eta)\varphi(s)}{\psi(s)^\eta}.$$ □

Remark 2 Through the result in Lemma (2), we can deduce the Sumudu transform of the function $t^{\eta-1}$ (see [41]) and the Shehu transform (see [40]).

Theorem 5 *The general transform of the Atangana–Baleanu fractional integral of the function $f(t)$ is given as*

$$T\big\{{}_0^{AB}I_t^\eta\big(f(t)\big)\big\} = \frac{1}{\mathcal{K}(\eta)}\frac{(1-\eta)\psi(s)^\eta+\eta}{\psi(s)^\eta}\mathcal{F}_J(s).$$

Proof Let $\mathcal{F}_J(s)$ be the new general transform of the function $f(t)$, we have

$$\begin{aligned} T\big\{{}_0^{AB}I_t^\eta\big(f(t)\big)\big\} &= \frac{1-\eta}{\mathcal{K}(\eta)}T\big\{f(t)\big\} + \frac{\eta}{\mathcal{K}(\eta)\Gamma(\eta)}T\left\{\int_0^t f(x)(t-x)^{\eta-1}\,dx\right\} \\ &= \frac{1-\eta}{\mathcal{K}(\eta)}T\big\{f(t)\big\} + \frac{\eta}{\mathcal{K}(\eta)\Gamma(\eta)}T\big\{f(t)*t^{\eta-1}\big\}. \end{aligned}$$

According to the convolution theorem (2) and Lemma (2), we obtain,

$$\begin{aligned} T\big\{{}_0^{AB}I_t^\eta\big(f(t)\big)\big\} &= \frac{1-\eta}{\mathcal{K}(\eta)}\mathcal{F}_J(s) + \frac{\eta}{\mathcal{K}(\eta)\Gamma(\eta)}\frac{1}{\varphi(s)}T\big\{f(t)\big\}T\big\{t^{\eta-1}\big\} \\ &= \frac{1-\eta}{\mathcal{K}(\eta)}\mathcal{F}_J(s) + \frac{\eta}{\mathcal{K}(\eta)\Gamma(\eta)}\frac{1}{\varphi(s)}\mathcal{F}_J(s)\frac{\Gamma(\eta)\varphi(s)}{\psi(s)^\eta}, \end{aligned}$$

then

$$T\big\{{}_0^{AB}I_t^\eta\big(f(t)\big)\big\} = \frac{1}{\mathcal{K}(\eta)}\frac{(1-\eta)\psi(s)^\eta+\eta}{\psi(s)^\eta}\mathcal{F}_J(s).$$ □

4 Applications

We now substantiate the theoretical aspects that we developed in the previous section by providing practical examples.

Example 1 Consider the following fractional initial value problem:

$${}_0^{ABC}D_t^\eta\big(u(t)\big) = f(t), \quad u(0) = c, c \in \mathbb{R}. \tag{7}$$

Applying the integral transform on both sides of Eq. (7),

$$T\left\{{}_0^{ABC}D_t^{\eta}\big(u(t)\big)\right\} = T\big\{f(t)\big\},$$

gives

$$\frac{\mathcal{K}(\eta)}{1-\eta}\frac{\psi(s)^{\eta}}{\psi(s)^{\eta}+\frac{\eta}{1-\eta}}\left[\mathcal{U}_{\mathrm{J}}(s)-\frac{\varphi(s)}{\psi(s)}u(0)\right] = \mathcal{F}_{\mathrm{J}}(s).$$

Therefore,

$$\mathcal{U}_{\mathrm{J}}(s) = \frac{1-\eta}{\mathcal{K}(\eta)}\frac{\psi(s)^{\eta}+\frac{\eta}{1-\eta}}{\psi(s)^{\eta}}\mathcal{F}_{\mathrm{J}}(s)+\frac{\varphi(s)}{\psi(s)}u(0),$$

taking the inverse general transform, we obtain

$$u(t) = T^{-1}\left\{\frac{1-\eta}{\mathcal{K}(\eta)}\frac{\psi(s)^{\eta}+\frac{\eta}{1-\eta}}{\psi(s)^{\eta}}\mathcal{F}_{\mathrm{J}}(s)+\frac{\varphi(s)}{\psi(s)}u(0)\right\}.$$

If $f(t) = t$, the equivalence of (7) is

$${}_0^{ABC}D_t^{\eta}\big(u(t)\big) = t, \quad u(0) = c, c \in \mathbb{R}, \tag{8}$$

whose solution is

$$\begin{aligned} u(t) &= T^{-1}\left\{\frac{1-\eta}{\mathcal{K}(\eta)}\left(1+\frac{\eta}{1-\eta}\frac{1}{\psi(s)^{\eta}}\right)\frac{\varphi(s)}{\psi(s)}+\frac{\varphi(s)}{\psi(s)}u(0)\right\} \\ &= \frac{1-\eta}{\mathcal{K}(\eta)}T^{-1}\left\{\frac{\varphi(s)}{\psi(s)}+\frac{\eta}{1-\eta}\frac{\varphi(s)}{\psi(s)^{\eta+1}}+\frac{\varphi(s)}{\psi(s)}u(0)\right\}. \end{aligned}$$

The above equation reduces to

$$u(t) = \frac{1-\eta}{\mathcal{K}(\eta)}\left(1+c+\frac{\eta}{1-\eta}\frac{1}{\Gamma(\eta+1)}t^{\eta}\right).$$

Example 2 Consider the following nonlinear fractional differential equation:

$${}_0^{ABC}D_t^{\eta}\big(u(t)\big)+u(t) = f(t). \tag{9}$$

By employing the new general transform in Eq. (9), we get

$$T\left\{{}_0^{ABC}D_t^{\eta}\big(u(t)\big)\right\}+T\big\{u(t)\big\} = T\big\{f(t)\big\},$$

thus,

$$\frac{\mathcal{K}(\eta)}{1-\eta}\frac{\psi(s)^{\eta}}{\psi(s)^{\eta}+\frac{\eta}{1-\eta}}\left[\mathcal{U}_{\mathrm{J}}(s)-\frac{\varphi(s)}{\psi(s)}u(0)\right]+\mathcal{U}_{\mathrm{J}}(s) = \mathcal{F}_{\mathrm{J}}(s).$$

Therefore,

$$\left(\frac{\mathcal{K}(\eta)}{1-\eta}\frac{\psi(s)^{\eta}}{\psi(s)^{\eta}+\frac{\eta}{1-\eta}}+1\right)\mathcal{U}_{\mathrm{J}}(s)=\frac{\mathcal{K}(\eta)}{1-\eta}\frac{\psi(s)^{\eta}}{\psi(s)^{\eta}+\frac{\eta}{1-\eta}}\frac{\varphi(s)}{\psi(s)}u(0)+\mathcal{F}_{\mathrm{J}}(s),$$

it follows that

$$\begin{aligned}\mathcal{U}_{\mathrm{J}}(s)&=\frac{\mathcal{K}(\eta)}{\mathcal{K}(\eta)+1-\eta}\frac{\psi(s)^{\eta}}{\psi(s)^{\eta}+\frac{\eta}{\mathcal{K}(\eta)+1-\eta}}\frac{\varphi(s)}{\psi(s)}u(0)+\frac{1-\eta}{\mathcal{K}(\eta)+1-\eta}\frac{\psi(s)^{\eta}+\eta}{\psi(s)^{\eta}+\frac{\eta}{\mathcal{K}(\eta)+1-\eta}}\mathcal{F}_{\mathrm{J}}(s),\\ \mathcal{U}_{\mathrm{J}}(s)&=\frac{\mathcal{K}(\eta)}{\mathcal{K}(\eta)+1-\eta}\frac{\psi(s)^{\eta}}{\psi(s)^{\eta}+\frac{\eta}{\mathcal{K}(\eta)+1-\eta}}\frac{\varphi(s)}{\psi(s)}u(0)+\frac{1-\eta}{\mathcal{K}(\eta)+1-\eta}\mathcal{F}_{\mathrm{J}}(s)\\ &\quad+\frac{\eta}{(\mathcal{K}(\eta)+1-\eta)^2}\frac{1}{\psi(s)^{\eta}+\frac{\eta}{\mathcal{K}(\eta)+1-\eta}}\mathcal{F}_{\mathrm{J}}(s).\end{aligned}$$

Using the inverse transform, the convolution theorem (2) and Corollary 1, we get

$$T^{-1}\left\{\frac{\psi(s)^{\eta}}{\psi(s)^{\eta}+\frac{\eta}{\mathcal{K}(\eta)+1-\eta}}\frac{\varphi(s)}{\psi(s)}\right\}=E_{\eta}\left(-\frac{\eta}{\mathcal{K}(\eta)+1-\eta}t^{\eta}\right),$$

also,

$$\begin{aligned}T^{-1}\left\{\frac{1}{\psi(s)^{\eta}+\frac{\eta}{\mathcal{K}(\eta)+1-\eta}}\mathcal{F}_{\mathrm{J}}(s)\right\}&=T^{-1}\left\{\frac{\varphi(s)}{\psi(s)^{\eta}+\frac{\eta}{\mathcal{K}(\eta)+1-\eta}}\right\}*f(t)\\ &=t^{\eta-1}E_{\eta}\left(-\frac{\eta}{\mathcal{K}(\eta)+1-\eta}t^{\eta}\right)*f(t).\end{aligned}$$

Then

$$\begin{aligned}u(t)&=T^{-1}\{\mathcal{U}_{\mathrm{J}}(s)\}\\ &=\frac{\mathcal{K}(\eta)}{\mathcal{K}(\eta)+1-\eta}E_{\eta}\left(-\frac{\eta}{\mathcal{K}(\eta)+1-\eta}t^{\eta}\right)u(0)+\frac{1-\eta}{\mathcal{K}(\eta)+1-\eta}f(t)\\ &\quad+\frac{\eta}{(\mathcal{K}(\eta)+1-\eta)^2}t^{\eta-1}E_{\eta}\left(-\frac{\eta}{\mathcal{K}(\eta)+1-\eta}t^{\eta}\right)*f(t).\end{aligned}$$

Note that, when $\eta\rightarrow 1$ and $f(t)=e^{-2t}$, then the exact solution is

$$u(t)=\bigl(1+u(0)\bigr)e^{-t}-e^{-2t}.$$

Example 3 We now consider Eq. (9) with the Riemann–Liouville derivative,

$${}_{0}^{\mathrm{ABR}}D_t^{\eta}\bigl(u(t)\bigr)+u(t)=f(t). \tag{10}$$

Applying the new general transform in Eq. (10) yields

$$\left(\frac{\mathcal{K}(\eta)}{1-\eta}\frac{\psi(s)^{\eta}}{\psi(s)^{\eta}+\frac{\eta}{1-\eta}}+1\right)\mathcal{U}_{\mathrm{J}}(s)=T\bigl\{f(t)\bigr\}.$$

Therefore,

$$\mathcal{U}_J(s) = \frac{(1-\eta)\psi(s)^\eta + \eta}{(\mathcal{K}(\eta) + 1 - \eta)\psi(s)^\eta + \eta}\mathcal{F}_J(s),$$

$$u(t) = T^{-1}\left\{\frac{(1-\eta)\psi(s)^\eta + \eta}{(\mathcal{K}(\eta) + 1 - \eta)\psi(s)^\eta + \eta}\mathcal{F}_J(s)\right\}.$$

If $f(t) = \sin(t)$, (10) becomes

$${}_0^{ABR}D_t^\eta\big(u(t)\big) + u(t) = \sin(t), \quad 0 < \eta \leq 1,$$

whose exact solution is

$$u(t) = T^{-1}\left\{\frac{(1-\eta)\psi(s)^\eta + \eta}{(\mathcal{K}(\eta) + 1 - \eta)\psi(s)^\eta + \eta}\frac{\varphi(s)}{\psi(s)^2 + 1}\right\}.$$

In particular, when $\eta \to 1$, we obtain

$$\begin{aligned} u(t) &= T^{-1}\left\{\frac{1}{\psi(s) + 1}\frac{\varphi(s)}{\psi(s)^2 + 1}\right\} \\ &= T^{-1}\left\{0.5\frac{\varphi(s)}{\psi(s)+1} - 0.5\frac{\psi(s)\varphi(s)}{\psi(s)^2+1} + 0.5\frac{\varphi(s)}{\psi(s)^2+1}\right\}, \end{aligned}$$

with the exact solution

$$u(t) = 0.5\big(e^{-t} - \cos t + \sin t\big).$$

5 Conclusion

We explored the feasibility of applying the generalized integral transform (Jafari transform) in fractional calculus, the Atangana–Baleanu derivative with its corresponding integral is used as a case in point. It is proved that imposing certain conditions on the Jafari transform leads to other integral transforms. To prove the applicability of this generalised integral transform, practical examples are given. This generalised integral transform resembles other integral transforms in that it is easy to implement and offers the convenience of using a table in the solution procedure of differential equations.

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