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A solution to nonlinear Fredholm integral equations in the context of w-distances

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Abstract

In this paper we propose a solution to the nonlinear Fredholm integral equations in the context of w-distance. For this purpose, we also provide a fixed point result in the same setting. In addition, we provide best proximity point results. We give examples and present numerical results to approximate fixed points.

MSC: Primary 47H10; secondary 37C25; 54H25; 55M20

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1 Introduction and preliminaries

In nonlinear functional analysis, one of the most interesting topics is the fixed point theory due to its wide application potential. Hundreds and even thousands of fixed point theorems have been proved and published since Banach up to this date. This makes it almost impossible to follow the literature. To change this situation, there is a need to work on more complementary and general results. For this purpose, recently, the concept of a simulation function was defined by Kojasteh et al. [1] in order to combine some existing metric fixed point results. This idea was improved and studied very densely, see e.g. [2–10] and the references therein. This trend has been transformed in best proximity theory by [11–14] and the works referenced therein.

For the sake of self-containment, the definition of simulation function [1] is recalled here:

Definition 1.1 ([1]) A function $\sigma : [0,+\infty) \times [0,+\infty) \to \mathbb{R}$ is called *simulation function* if

1. $\sigma(t,s) < s - t$ for all $t, s > 0$;
2. For the positive sequences (a_n) and (b_n),

$$\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} b_n > 0 \quad \text{implies that} \quad \limsup_{n\to+\infty} \sigma(a_n, b_n) < 0.$$

Note that we removed the superfluous condition $\sigma(0,0) = 0$ from the simulation function definition in [1]. We shall use Σ to indicate the class of all simulation functions. An immediate example of simulation function is $\sigma(t,s) = ks - t$, where $k \in [0,1)$. A function $\psi : [0,+\infty) \to [0,+\infty)$ is called origin-intersect if

$$\psi(s) = 0 \quad \text{if and only if} \quad s = 0.$$

The class of all origin-intersect functions will be denoted by Ψ.

An example of origin-intersect functions is given below.

Example Define $\psi_1, \psi_2 : [0,+\infty) \to [0,+\infty)$ by

$$\psi_1(s) = \frac{s}{4} \quad \text{and} \quad \psi_2(s) = s^2.$$

In what follows, we recall the concept of the quasi-altering distance function.

Definition 1.2 ([15]) We say that $\varphi : [0,+\infty) \to [0,+\infty)$ is *quasi-altering distance function* if

1. φ is continuous
2. φ is origin-intersect.

Remark 1.3 Deduce that ψ_1 and ψ_2 are quasi-altering distance functions.

The following are the examples for simulation functions.

Example Let $\theta_i : [0,+\infty) \to [0,+\infty)$ be quasi-altering distance functions, and we define the mappings $\sigma_i : [0,+\infty) \times [0,+\infty) \to \mathbb{R}$ for $i = 1,2,3$ as follows:

1. Let $\theta_1(\tau) < \tau \leq \theta_2(\tau)$ for all $\tau > 0$. Then

$$\sigma_1(\tau,\nu) = \theta_1(\nu) - \theta_2(\tau) \quad \text{for all } \tau,\nu \in [0,+\infty).$$

2. Suppose that the functions $h, g : [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ are continuous with respect to each component with $h(\tau,\nu) > g(\tau,\nu) > 0$ for all $\tau,\nu > 0$. Then a function

$$\sigma_2(\tau,\nu) = \nu - \frac{h(\tau,\nu)}{g(\tau,\nu)}\tau \quad \text{for all } \tau,\nu \in [0,+\infty).$$

3. $\sigma_3(\tau,\nu) = \nu - \theta_3(\nu) - \tau$ for all $\tau,\nu \in [0,+\infty)$.

Clearly, each σ_i $(i = 1,2,3)$ is a simulation function; also, see e.g. [1–7, 11–13].

Definition 1.4 ([15]) If a quasi-altering distance function $\varphi : [0,+\infty) \to [0,+\infty)$ is non-decreasing, we say that it is an *altering distance function*.

Let $\Phi = \{\varphi : [0,+\infty) \to [0,+\infty)\}$ be the collection of all altering distance functions.

In the following example φ_2 and φ_3 are defined in [16].

Example Let φ_1, φ_2, φ_3 be self-mappings on $[0,+\infty)$ that are defined by

$$\varphi_1(s) = \sqrt{s}, \qquad \varphi_2(s) = se^{3s}, \qquad \varphi_3(s) = \ln(s^2 + 2s + 1).$$

It can be easily verified that φ_1, φ_2, and φ_3 form an altering distance function. Indeed, they are continuous, nondecreasing, and origin-intersect.

The notion of weaker Meir–Keeler function was defined and used effectively by Chen [17]. It was reconsidered by Lakzian and Rhoades [18] as follows.

Definition 1.5 ([18]) A self-mapping ψ, defined on $[0,+\infty)$, is called *weaker Meir–Keeler* if, for each $\varrho > 0$, there exists $\chi > 0$ such that, for $s \in [0,+\infty)$ with $\varrho \leq s < \varrho + \chi$, there exists $n_0 \in \mathbb{N}$ such that $\psi^{n_0}(s) < \varrho$.

Due to [18], we also have:
(i) $\psi(0) = 0$ together with $\psi(s) > 0$ for $s > 0$ and;
(ii) $\{\psi^n(s)\}_{n\in\mathbb{N}}$ is a decreasing sequence for each $s > 0$;
(iii) for a nonnegative sequence (s_n), we have
  (a) $\lim_{n\to+\infty} s_n = 0$ if and only if $\lim_{n\to+\infty} \psi(s_n) = 0$,
  (b) if $\lim_{n\to+\infty} s_n = \ell$, then $\lim_{n\to+\infty} \psi(s_n) < \ell$.

Note that from the above definition we are taking only condition (i) for our ψ i.e. the collection Ψ covers a large number of mappings compared to weaker Meir–Keeler functions. Therefore Ψ is more general than the weaker Meir–Keeler functions.

On the other hand, Kada et al. [19] proposed a new concept that is called w-distance to extend some well-known fixed point results. Indeed, w-distance is kind of a generalization of metric. After that, many authors extended and generalized the fixed point results using w-distance [16, 18, 20–27].

Definition 1.6 ([19]) For a metric space (X,d), a function $q : X \times X \to [0,+\infty)$ is called a *w-distance* on X if
(i) $q(x,y) \leq q(x,z) + q(z,y)$;
(ii) if $x \in X$ and $y_n \to y$ in X, then $q(x,y) \leq \liminf_{n\to+\infty} q(x,y_n)$
(i.e. $q(x,\cdot) : X \to [0,+\infty)$ is lower semi-continuous);
(iii) for each $\varepsilon > 0$, there exists $\delta > 0$ such that $q(x,y) \leq \delta$ and $q(x,z) \leq \delta$ imply $d(y,z) \leq \varepsilon$ for all $x,y,z \in X$.

Note that each metric forms a w-distance. We shall use the triplet (X,d,q) to indicate that $q : X \times X \to [0,+\infty)$ is a w-distance on a metric space (X,d). In addition, if the corresponding metric is complete, we shall use (X^*,d,q). Furthermore, the pair (X^*,d) denotes the complete metric space.

An example of w-distance is below.

Example For a positive real number k and a metric space (X,d), a function $q : X \times X \to [0,+\infty)$, defined by $q(a,b) = k$, forms a w-distance on X. On the other hand, q fails to be a metric since $q(a,a) = k \neq 0$ for any $a,b \in X$.

Example For a normed linear space $(X,\|\cdot\|)$, a function $q : X \times X \to [0,+\infty)$ defined by $q(a,b) = \|b\|$ for any $a,b \in X$ forms a w-distance.

Suppose that $\mathcal{F}_T(X)$ denotes the set of all fixed points of $T : X \to X$. The following theorem is a quote of Theorem 3.2 of Lakzian and Rhodes [18].

Theorem 1.7 ([18]) *For (X^*, d, q), $\varphi \in \Phi$, and a weaker Meir–Keeler function ψ, assume that a self-mapping $T : X \to X$ satisfies*

$$\varphi\bigl(q(Tx, Ty)\bigr) \leq \psi\bigl(\varphi\bigl(M(x,y)\bigr)\bigr) \quad \textit{for all } x, y \in X.$$

If $0 < \psi(t) < t$ for $t > 0$ and one of the following conditions holds:
 (i) *T is continuous*,
 (ii) *for every $w \in X \setminus \mathcal{F}_T(X)$, $\inf\{q(x,w) + q(x,Tx) : x \in X\} > 0$,*
then T possesses a unique fixed point.

In the next section we use Lemma 2 in Lakzian and Rhodes [18].

Lemma 1.8 ([18]) *For sequences (x_n) and (y_n) in (X, d, q), we have the following*:
 (i) *If $\lim_{n\to+\infty} q(x_n, x) = \lim_{n\to+\infty} q(x_n, y) = 0$, then $x = y$. In particular, if $q(z,x) = q(z,y) = 0$, then $x = y$, where $x, y, z \in X$.*
 (ii) *For nonnegative sequences (a_n) and (b_n), if $q(x_n, y_n) \leq a_n$ and $q(x_n, y) \leq b_n$ for any $n \in \mathbb{N}$ converging to 0, then (y_n) converges to $y \in X$.*
 (iii) *A sequence (x_n) is Cauchy (or fundamental) if for each $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that $m > n > N_\varepsilon$ implies $q(x_n, x_m) < \varepsilon$, which is equivalent to say $\lim_{n,m\to+\infty} q(x_n, x_m) = 0$.*

Definition 1.9 For (X^*, d, q), we say that q is *ceiling distance* of d if and only if

$$q(x,y) \geq d(x,y)$$

for all $x, y \in X$.

The following examples for ceiling distance are given in [21].

Example ([16]) Each metric forms a ceiling distance of itself.

Example ([21]) Consider $(\mathbb{R}, d(x,y) = |x-y|)$. A mapping $q : X \times X \to [0, +\infty)$, defined by $q(x,y) = \max\{a(y-x), b(x-y)\}$ for all $x, y \in X$, forms a ceiling distance of d, where $a, b \geq 1$.

Example ([21]) Consider again the standard metric $[0, +\infty), d(x,y) = |x-y|$. A mapping $q : X \times X \to [0, +\infty)$, defined by $q(x,y) = \max\{x, y\}$ for all $x, y \in X$, forms a ceiling distance of d.

Define

$$M(x,y) = \max\left\{q(x,y), q(x,Tx), q(y,Ty), \frac{q(x,Ty) + q(Tx,y)}{2}\right\};$$

and

$$m(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty) + d(Tx,y)}{2}\right\}.$$

Now we define a Σ-weighted contraction with respect to σ as follows.

Definition 1.10 For (X^*,d,q), $\varphi\in\Phi$, and $\psi\in\Psi$, $\sigma\in\Sigma$, a mapping $T:X\to X$ is called a Σ-weighted contraction with respect to σ such that

$$\sigma\bigl(\varphi\bigl(q(Tx,Ty)\bigr),\psi\bigl(\varphi\bigl(M(x,y)\bigr)\bigr)\bigr)\geq 0 \quad \text{for all } x,y\in X. \tag{1.1}$$

In this manuscript, we generalize and unify Theorem 1.7 involving simulation functions. Furthermore, motivated by the work in [13], we prove the best proximity theorem for w-distances involving simulation functions. As an application for our fixed point result, we propose a solution for a nonlinear Fredholm integral equation. Also we give examples and numerical approximations to illustrate our main results.

2 Main results

Our first new result in this paper is the following.

Theorem 2.1 *On (X^*,d,q), the function q forms a ceiling distance of d with $q(x,x)=0$ for all $x\in X$. Assume that $T:X\to X$ is a Σ-weighted contraction with respect to σ. If $0<\psi(t)<t$ for $t>0$ and one of the following conditions holds, then T possesses a unique fixed point*:

(i) *T is continuous*;

(ii) *for every $w\in X\setminus\mathcal{F}_T(X)$, $\inf\{q(x,w)+q(x,Tx):x\in X\}>0$.*

Proof Let $x_0\in X$ and define (x_n) by $x_{n+1}=Tx_n$ for all $n\geq 0$. Suppose that $x_{n+1}=x_n$ for some $n\in\mathbb{N}$, then x_n is a fixed point of T. Therefore we assume that $x_{n+1}\neq x_n$ for all $n\in\mathbb{N}\cup\{0\}$. This implies that $d(x_n,x_{n+1})>0$ for all $n\in\mathbb{N}\cup\{0\}$. Using the ceiling distance of d, we get $q(x_n,x_{n+1})>0$ for all $n\in\mathbb{N}\cup\{0\}$. Now, since T is a Σ-weighted contraction, we get

$$\sigma\bigl(\varphi\bigl(q(x_n,x_{n+1})\bigr),\psi\bigl(\varphi\bigl(M(x_{n-1},x_n)\bigr)\bigr)\bigr)\geq 0, \tag{2.1}$$

where

$$\begin{aligned} M(x_{n-1},x_n) &= \max\biggl\{q(x_{n-1},x_n),q(x_{n-1},x_n),q(x_n,x_{n+1}),\frac{q(x_{n-1},x_{n+1})+q(x_n,x_n)}{2}\biggr\} \\ &= \max\biggl\{q(x_{n-1},x_n),q(x_n,x_{n+1}),\frac{q(x_{n-1},x_{n+1})}{2}\biggr\}. \end{aligned}$$

Note that

$$\begin{aligned} \frac{q(x_{n-1},x_{n+1})}{2} &\leq \frac{q(x_{n-1},x_n)+q(x_n,x_{n+1})}{2} \\ &\leq \max\bigl\{q(x_{n-1},x_n),q(x_n,x_{n+1})\bigr\}. \end{aligned}$$

Then we get $M(x_{n-1},x_n)=\max\{q(x_{n-1},x_n),q(x_n,x_{n+1})\}$. Now suppose that $q(x_{n-1},x_n)<q(x_n,x_{n+1})$, equation (2.1) implies that

$$\begin{aligned} 0 &\leq \sigma\bigl(\varphi\bigl(q(x_n,x_{n+1})\bigr),\psi\bigl(\varphi\bigl(q(x_n,x_{n+1})\bigr)\bigr)\bigr) \\ &< \psi\bigl(\varphi\bigl(q(x_n,x_{n+1})\bigr)\bigr)-\varphi\bigl(q(x_n,x_{n+1})\bigr)<0, \end{aligned}$$

which is a contradiction. Therefore, for all $n \in \mathbb{N}$, $q(x_n, x_{n+1}) \leq q(x_{n-1}, x_n)$. Hence the sequence $(q(x_n, x_{n+1}))$ is a decreasing sequence and bounded below by 0, then there exists $r \geq 0$ such that

$$\lim_{n \to +\infty} q(x_n, x_{n+1}) = r. \tag{2.2}$$

Now suppose that $r > 0$; by the Σ-weighted contraction condition, we get

$$\begin{aligned} 0 &\leq \sigma\bigl(\varphi\bigl(q(x_n, x_{n+1})\bigr), \psi\bigl(\varphi\bigl(q(x_{n-1}, x_n)\bigr)\bigr)\bigr) \\ &< \psi\bigl(\varphi\bigl(q(x_{n-1}, x_n)\bigr)\bigr) - \varphi\bigl(q(x_n, x_{n+1})\bigr), \\ \varphi\bigl(q(x_n, x_{n+1})\bigr) &< \psi\bigl(\varphi\bigl(q(x_{n-1}, x_n)\bigr)\bigr) < \varphi\bigl(q(x_{n-1}, x_n)\bigr). \end{aligned} \tag{2.3}$$

Letting $n \to +\infty$, we get $\varphi(q(x_n, x_{n+1})) \to \varphi(r)$, $\psi(\varphi(q(x_{n-1}, x_n))) \to \varphi(r)$, and note that by condition (iii) of simulation function (2.3) becomes

$$0 \leq \limsup_{n \to +\infty} \sigma\bigl(\varphi\bigl(q(x_n, x_{n+1})\bigr), \psi\bigl(\varphi\bigl(q(x_{n-1}, x_n)\bigr)\bigr)\bigr) < 0,$$

which is a contradiction. Therefore $r = 0$. Similarly, we can prove that $q(x_n, x_{n+1}) \to 0$ as $n \to +\infty$. Now we are going to prove that (x_n) is a Cauchy sequence i.e.

$$\lim_{n,m \to +\infty} q(x_n, x_m) = 0. \tag{2.4}$$

Suppose on the contrary that there are $\varepsilon > 0$ and subsequences (x_{m_k}) and (x_{n_k}) of (x_n) with $m_k > n_k \geq k$ such that

$$q(x_{n_k+1}, x_{m_k+1}) \geq \varepsilon \quad \text{for all } k \in \mathbb{N}. \tag{2.5}$$

By choosing m_k to be the smallest integer exceeding n_k for which (2.5) holds, we get

$$q(x_{n_k+1}, x_{m_k}) < \varepsilon. \tag{2.6}$$

Using (2.5) and (2.6), we get

$$\varepsilon \leq q(x_{n_k+1}, x_{m_k+1}) \leq q(x_{n_k+1}, x_{m_k}) + q(x_{m_k}, x_{m_k+1}),$$

and then $q(x_{n_k+1}, x_{m_k+1}) \to \varepsilon$ as $k \to +\infty$. Now,

$$q(x_{n_k+1}, x_{m_k+1}) - q(x_{m_k+1}, x_{m_k}) \leq q(x_{n_k+1}, x_{m_k}) < \varepsilon,$$

and as $k \to +\infty$ we get $q(x_{n_{k+1}}, x_{m_k}) \to \varepsilon$. On the other hand,

$$q(x_{n_k+1}, x_{m_k}) - q(x_{n_k+1}, x_{n_k}) \leq q(x_{n_k}, x_{m_k}) < q(x_{n_k}, x_{n_k+1}) + \varepsilon,$$

and then $q(x_{n_{k+1}}, x_{m_k+1}) \to \varepsilon$. Using the Σ-weighted contraction, we get

$$0 \leq \sigma\bigl(\varphi\bigl(q(x_{n_{k+1}}, x_{m_{k+1}})\bigr), \psi\bigl(\varphi\bigl(M(x_{n_k}, x_{m_k})\bigr)\bigr)\bigr). \tag{2.7}$$

From (2.5) we have $\varepsilon \le M(x_{n_k}, x_{m_k})$ and observe that

$$\begin{aligned}
M(x_{n_k}, x_{m_k}) &= \max\biggl\{q(x_{n_k}, x_{m_k}), q(x_{n_k}, x_{n_{k+1}}), q(x_{m_k}, x_{m_{k+1}}), \\
&\quad \frac{q(x_{n_k}, x_{m_{k+1}}) + q(x_{n_{k+1}}, x_{m_k})}{2}\biggr\} \\
&\le \max\biggl\{q(x_{n_k}, x_{m_k}), q(x_{n_k}, x_{n_{k+1}}), q(x_{m_k}, x_{m_{k+1}}), \\
&\quad \frac{q(x_{n_k}, x_{n_{k+1}}) + 2q(x_{n_{k+1}}, x_{m_{k+1}}) + q(x_{m_{k+1}}, x_{m_k})}{2}\biggr\}.
\end{aligned}$$

Therefore as $k \to +\infty$ we get $M(x_{n_k}, x_{m_k}) \to \varepsilon$, and note that from (2.7) we have

$$\begin{aligned}
&0 < \psi\bigl(\varphi\bigl(M(x_{n_k}, x_{m_k})\bigr)\bigr) - \varphi\bigl(q(x_{n_{k+1}}, x_{m_{k+1}})\bigr), \\
&\varphi\bigl(q(x_{n_{k+1}}, x_{m_{k+1}})\bigr) < \psi\bigl(\varphi\bigl(M(x_{n_k}, x_{m_k})\bigr)\bigr) < \varphi\bigl(M(x_{n_k}, x_{m_k})\bigr),
\end{aligned}$$

as $k \to +\infty$, we get $\psi(\varphi(M(x_{n_k}, x_{m_k}))) \to \varphi(\varepsilon)$. Then, by condition (iii) of the simulation function, we conclude that

$$0 \le \limsup_{n\to+\infty} \sigma\bigl(\varphi\bigl(q(x_{n_{k+1}}, x_{m_{k+1}})\bigr), \psi\bigl(\varphi\bigl(M(x_{n_k}, x_{m_k})\bigr)\bigr)\bigr) < 0,$$

which is a contradiction. Therefore (2.4) holds, and by Lemma 1.8 we conclude that the sequence (x_n) is fundamental (Cauchy). Employing the completeness of X, one can find $s \in X$ such that $x_n \to s$ as $n \to +\infty$.

Now, by (2.4), for every $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that for all $n > N_\varepsilon$ we get $q(x_{N_\varepsilon}, x_n) < \varepsilon$. But $x_n \to s$ and using the lower semi-continuity of T, we get

$$q(x_{N_\varepsilon}, s) \le \liminf_{n\to+\infty} q(x_{N_\varepsilon}, Tx_{n-1}) \le \varepsilon.$$

Therefore $q(x_{N_\varepsilon}, s) \le \varepsilon$. Letting $\varepsilon = \frac{1}{k}$ and $N_\varepsilon = n_k$, we get

$$\lim_{k\to+\infty} q(x_{n_k}, s) = 0. \tag{2.8}$$

Case (i): Assume that $s \neq Ts$. Accordingly, we have

$$0 < \inf\bigl\{q(x, s) + q(x, Tx) : x \in X\bigr\} \le \inf\bigl\{q(x_n, s) + q(x_n, x_{n+1}) : n \in \mathbb{N}\bigr\},$$

which tends to 0 as $n \to +\infty$. We get a contradiction, therefore we conclude that $s = Ts$.

Case (ii): Suppose that $\inf\{q(x, w) + q(x, Tx) : x \in X\} = 0$ for some $w \in X$ such that $w \neq Tw$.

Then there exists a sequence (x_n) such that $q(x_n, w) + q(x_n, Tx_n) \to 0$ as $n \to +\infty$. That is, $q(x_n, w) \to 0$ and $q(x_n, Tx_n) \to 0$ as $n \to +\infty$. By Lemma 1.8, we have $Tx_n \to w$ as $n \to +\infty$. Suppose that $q(Tx_n, T^2x_n) = 0$; then, by the ceiling distance of d, we get $Tx_n = T^2x_n$.

Since $Tx_n \to w$ implies that $T^2x_n \to w$ as $n \to +\infty$, by using the continuity of T, we have

$$Tw = T\Bigl(\lim_{n\to+\infty} Tx_n\Bigr) = \lim_{n\to+\infty} T^2x_n = w, \tag{2.9}$$

which is a contradiction. As a result, $\inf\{q(x,w) + q(x,Tx) : x \in X\} > 0$ if $w \neq Tw$. Then, by case(i), we conclude the desired result.

Now, supposing that $q(Tx_n, T^2x_n) > 0$, then we have $\varphi(q(Tx_n, T^2x_n)) > 0$ and $\psi(\varphi(M(x_n, Tx_n))) > 0$. By the Σ-weighted contraction condition, we get

$$\begin{aligned}
&0 \leq \sigma\bigl(\varphi\bigl(q\bigl(Tx_n, T^2x_n\bigr)\bigr), \psi\bigl(\varphi\bigl(M(x_n, Tx_n)\bigr)\bigr)\bigr) \\
&\quad < \psi\bigl(\varphi\bigl(M(x_n, Tx_n)\bigr)\bigr) - \varphi\bigl(q\bigl(Tx_n, T^2x_n\bigr)\bigr), \\
&\varphi\bigl(q\bigl(Tx_n, T^2x_n\bigr)\bigr) < \psi\bigl(\varphi\bigl(M(x_n, Tx_n)\bigr)\bigr) < \varphi\bigl(M(x_n, Tx_n)\bigr),
\end{aligned}$$

then we get $q(Tx_n, T^2x_n) \to 0$ as $n \to +\infty$. Using the triangular inequality, we have

$$q\bigl(x_n, T^2x_n\bigr) \leq q(x_n, Tx_n) + q\bigl(Tx_n, T^2x_n\bigr).$$

Hence $q(x_n, T^2x_n) \to 0$ as $n \to +\infty$, and Lemma 1.8 implies that $T^2x_n \to w$ as $n \to +\infty$. Taking the continuity of T into account together with (2.9), we get a contradiction. Therefore we conclude that if $w \neq Tw$ then $\inf\{q(x,w) + q(x,Tx) : x \in X\} > 0$, and using case(i) we get the result.

For any fixed point $s \in X$, suppose that $q(s,s) > 0$, and by the Σ-weighted contraction condition, we get

$$\begin{aligned}
&0 \leq \sigma\bigl(\varphi\bigl(q(Ts, Ts)\bigr), \psi\bigl(\varphi\bigl(M(s,s)\bigr)\bigr)\bigr) \\
&\quad = \sigma\bigl(\varphi\bigl(q(Ts, Ts)\bigr), \psi\bigl(\varphi\bigl(q(s,s)\bigr)\bigr)\bigr) \\
&\quad < \psi\bigl(\varphi\bigl(q(s,s)\bigr)\bigr) - \varphi\bigl(q(Ts, Ts)\bigr), \\
&\varphi\bigl(q(Ts, Ts)\bigr) < \psi\bigl(\varphi\bigl(q(s,s)\bigr)\bigr) < \varphi\bigl(q(s,s)\bigr),
\end{aligned}$$

a contradiction. Therefore $q(s,s) = 0$.

To prove the uniqueness of the fixed point, suppose that t is another fixed point of T and $q(s,t) > 0$.

Now, using the Σ-weighted contraction condition, we have

$$\begin{aligned}
&0 \leq \sigma\bigl(\varphi\bigl(q(Ts, Tt)\bigr), \psi\bigl(\varphi\bigl(q(s,t)\bigr)\bigr)\bigr) \\
&\quad = \sigma\bigl(\varphi\bigl(q(s,t)\bigr), \psi\bigl(\varphi\bigl(q(s,t)\bigr)\bigr)\bigr) \\
&\quad < \psi\bigl(\varphi\bigl(q(s,t)\bigr)\bigr) - \varphi\bigl(q(s,t)\bigr), \\
&\varphi\bigl(q(s,t)\bigr) < \psi\bigl(\varphi\bigl(q(s,t)\bigr)\bigr) < \varphi\bigl(q(s,t)\bigr),
\end{aligned}$$

which implies a contradiction. Therefore we get $q(s,t) = 0$, hence by Lemma 1.8 we get $s = t$. □

Example Consider the standard metric $(\mathbb{R}, d(x,y) = |x - y|)$. Let $T : X \to X$ be a mapping defined by

$$Tx = \begin{cases} \frac{x^2}{4} & \text{if } x \in [0,1], \\ 0 & \text{if } x > 1. \end{cases}$$

Note that T is not continuous and hence the Banach contraction principle is not applicable. On the other hand, T satisfies condition (i) of Theorem 2.1. Consider $\varphi, \psi : [0,+\infty) \to [0,+\infty)$ by

$$\varphi(t) = \frac{t^2}{4}$$

and

$$\psi(t) = \frac{t}{2}.$$

It is easy to verify that $\varphi \in \Phi$, $\psi \in \Psi$, and $\psi(t) < t$ for all $t > 0$.

Now we define a w-distance $q : X \times X \to [0,+\infty)$ by

$$q(x,y) = \begin{cases} \max\{x,y\} & \text{if } x \neq y, \\ 0 & \text{if } x = y \end{cases}$$

for all $x, y \in X$, and also we define the simulation function $\sigma : [0,+\infty) \times [0,+\infty) \to \mathbb{R}$ such that

$$\sigma(t,s) = \frac{s}{2} - t$$

for all $s, t > 0$. We can easily verify that q is a ceiling distance of d, and by example (1) of simulation function, we conclude that $\sigma \in \Sigma$. Now we have to show that T satisfies (1.1). Note that

$$M(x,y) = \begin{cases} \max\{x,y\} = q(x,y) & \text{if } x \neq y, \\ x & \text{if } x = y. \end{cases}$$

Case (i): If $x, y \in [0,1]$ with $x \neq y$, then we have

$$\begin{aligned} \varphi\bigl(q(x,y)\bigr) &= \varphi\bigl(\max\{x,y\}\bigr) = \max\left\{\frac{x^2}{4}, \frac{y^2}{4}\right\}, \\ \psi\bigl(\varphi\bigl(q(x,y)\bigr)\bigr) &= \max\left\{\frac{x^2}{8}, \frac{y^2}{8}\right\}, && (2.10) \\ \varphi\bigl(q(Tx,Ty)\bigr) &= \varphi\left(\max\left\{\frac{x^2}{4}, \frac{y^2}{4}\right\}\right) = \max\left\{\frac{x^4}{64}, \frac{y^4}{64}\right\}. && (2.11) \end{aligned}$$

Then, by (2.10) and (2.11), equation (1.1) becomes

$$\begin{aligned} \sigma\bigl(\varphi\bigl(q(Tx,Ty)\bigr), \psi\bigl(\varphi\bigl(M(x,y)\bigr)\bigr)\bigr) &= \frac{1}{2}\bigl(\psi\bigl(\varphi\bigl(M(x,y)\bigr)\bigr) - \varphi\bigl(q(Tx,Ty)\bigr)\bigr) \\ &= \max\left\{\frac{x^2}{16}, \frac{y^2}{16}\right\} - \max\left\{\frac{x^4}{64}, \frac{y^4}{64}\right\} \geq 0. \end{aligned}$$

Case (ii): If $x, y > 1$ with $x \neq y$, then we have

$$\varphi\bigl(q(x,y)\bigr) = \varphi\bigl(\max\{x,y\}\bigr) = \max\left\{\frac{x^2}{4}, \frac{y^2}{4}\right\},$$

$$\psi\left(\varphi\left(q(x,y)\right)\right) = \max\left\{\frac{x^2}{8}, \frac{y^2}{8}\right\}, \tag{2.12}$$

$$\varphi\left(q(Tx,Ty)\right) = 0. \tag{2.13}$$

Then, by (2.12) and (2.13), equation (1.1) becomes

$$\begin{aligned}\sigma\left(\varphi\left(q(Tx,Ty)\right), \psi\left(\varphi\left(M(x,y)\right)\right)\right) &= \frac{1}{2}\left(\psi\left(\varphi\left(M(x,y)\right)\right)\right) - \varphi\left(q(Tx,Ty)\right) \\ &= \max\left\{\frac{x^2}{16}, \frac{y^2}{16}\right\} \geq 0.\end{aligned}$$

Case (iii): Suppose, without loss of generality, that $x \in [0,1]$, $y > 1$, then we have

$$\varphi\left(q(x,y)\right) = \varphi\left(\max\{x,y\}\right) = \max\left\{\frac{x^2}{4}, \frac{y^2}{4}\right\},$$

$$\psi\left(\varphi\left(q(x,y)\right)\right) = \max\left\{\frac{x^2}{8}, \frac{y^2}{8}\right\}, \tag{2.14}$$

$$\varphi\left(q(Tx,Ty)\right) = \varphi\left(\max\left\{\frac{x^2}{4}, 0\right\}\right) = \frac{x^4}{64}. \tag{2.15}$$

Then, by (2.12) and (2.13), equation (1.1) becomes

$$\begin{aligned}\sigma\left(\varphi\left(q(Tx,Ty)\right), \psi\left(\varphi\left(M(x,y)\right)\right)\right) &= \frac{1}{2}\left(\psi\left(\varphi\left(M(x,y)\right)\right)\right) - \varphi\left(q(Tx,Ty)\right) \\ &= \max\left\{\frac{x^2}{16}, \frac{y^2}{16}\right\} \geq 0.\end{aligned}$$

Case (iv): If $x, y \in X$ with $x = y$ implies that

$$\varphi\left(M(x,x)\right) = \frac{x^2}{4},$$

$$\psi\left(\varphi\left(M(x,x)\right)\right) = \frac{x^2}{8},$$

$$\varphi\left(q(Tx,Tx)\right) = \varphi(0) = 0. \tag{2.16}$$

Then equation (1.1) becomes

$$\begin{aligned}\sigma\left(\varphi\left(q(Tx,Tx)\right), \psi\left(\varphi\left(M(x,x)\right)\right)\right) &= \frac{1}{2}\left(\psi\left(\varphi\left(M(x,x)\right)\right)\right) - \varphi\left(q(Tx,Tx)\right) \\ &= \frac{x^2}{16} - 0 \geq 0.\end{aligned}$$

Therefore, the hypotheses of Theorem 2.1 are satisfied, and hence T has a unique fixed point. Here $\sigma(0,0) = 0$, then we get that $x = 0$ is the unique fixed point of T.

We have given the numerical results for the above example in Table 1, and also the convergence behavior of the above iterations is shown in Fig. 1.

Taking $q = d$ in Theorem 2.1, we obtain the following result.

Table 1 Picard iterations

x_n	$x_0 = 0.25$	$x_0 = 0.5$	$x_0 = 0.75$	$x_0 = 1$
x_1	1.5625e–02	6.2500e–02	1.4062e–01	2.5000e–01
x_2	6.1035e–05	9.7656e–04	4.9438e–03	1.5625e–02
x_3	9.3132e–10	2.3842e–07	6.1104e–06	6.1035e–05
x_4	2.1684e–19	1.4211e–14	9.3343e–12	9.3132e–10
x_5	1.1755e–38	5.0487e–29	2.1782e–23	2.1684e–19
x_6	3.4545e–77	6.3724e–58	1.1862e–46	1.1755e–38
x_7	2.9833e–154	1.0152e–115	3.5174e–93	3.4545e–77
x_8	2.2251e–308	2.5765e–231	3.0930e–186	2.9833e–154
x_9	0	0	0	2.2251e–308
x_{10}	0	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

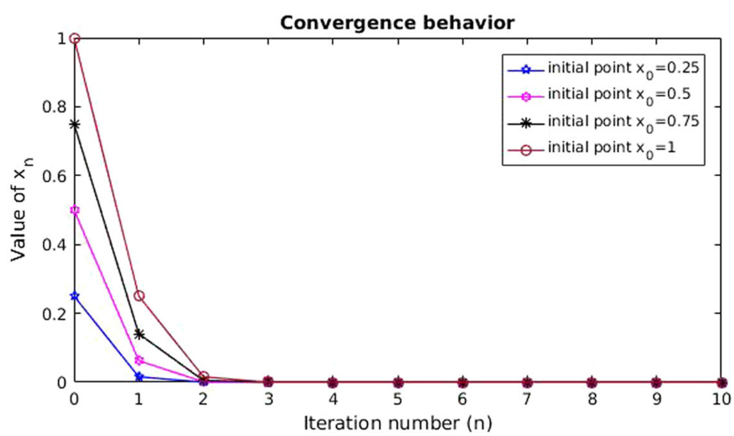

Figure 1 Convergence behavior

Corollary 2.2 *Let T be a self-mapping on (X^*, d). If T satisfies*

$$\sigma\big(\varphi\big(d(Tx,Ty)\big),\psi\big(\varphi\big(m(x,y)\big)\big)\big) \geq 0 \quad \textit{for all } x,y \in X,$$

where $\varphi \in \Phi$ and $\psi \in \Psi$. If $0 < \psi(t) < t$ for $t > 0$ and either assumption (i) *or* (ii) *defined in Theorem 2.1 holds, then T possesses a unique fixed point.*

In Theorem 2.1 and Corollary 2.2, taking $\sigma(t,s) = \lambda s - t$, where $\lambda \in [0,1)$, then we obtain the following results.

Corollary 2.3 *On (X^*, d, q), the function q forms a ceiling distance of d with $q(x,x) = 0$ for all $x \in X$. Assume that $T : X \to X$ satisfies*

$$\varphi\big(q(Tx,Ty)\big) \leq \lambda\psi\big(\varphi\big(M(x,y)\big)\big) \quad \textit{for all } x,y \in X,$$

where $\lambda \in [0,1)$, $\varphi \in \Phi$, and $\psi \in \Psi$. If $0 < \psi(t) < t$ for $t > 0$ and either assumption (i) *or* (ii) *defined in Theorem 2.1 holds, then T possesses a unique fixed point.*

Corollary 2.4 *Let T be a self-mapping on (X^*, d). If T satisfies*

$$\varphi\big(d(Tx,Ty)\big) \leq \lambda\psi\big(\varphi\big(m(x,y)\big)\big) \quad \textit{for all } x,y \in X,$$

where $\lambda \in [0,1)$, $\varphi \in \Phi$, *and* $\psi \in \Psi$. *If* $0 < \psi(t) < t$ *for* $t > 0$ *and either assumption* (i) *or* (ii) *defined in Theorem* 2.1 *holds, then* T *possesses a unique fixed point.*

Remark 2.5 Suppose that $\lambda \in (0,1)$ in Corollary 2.3 and Corollary 2.4, and define $\psi^*(t) := \lambda\psi(t)$. Note that $\psi^* \in \Psi$, therefore from Corollary 2.3 and Corollary 2.4 we get Theorem 3.2 and Corollary 4 of [18] respectively.

This implies that our results generalize and unify the results of Lakzian and Rhoades [18].

In Corollary 2.3 and Corollary 2.4, let $\psi(t) = st$ where $s \in [0,1)$, and taking $k = \lambda s$, we derived the following results and noted that $k \in [0,1)$.

Corollary 2.6 *On* (X^*, d, q), *the function* q *forms a ceiling distance of* d *with* $q(x,x) = 0$ *for all* $x \in X$. *Assume that* $T : X \to X$ *satisfies*

$$\varphi\bigl(q(Tx,Ty)\bigr) \le k\varphi\bigl(M(x,y)\bigr) \quad \textit{for all } x,y \in X,$$

where $k \in [0,1)$, $\varphi \in \Phi$, *and* $\psi \in \Psi$. *If* $0 < \psi(t) < t$ *for* $t > 0$ *and either assumption* (i) *or* (ii) *defined in Theorem* 2.1 *holds, then* T *possesses a unique fixed point.*

Corollary 2.7 *Let* T *be a self-mapping on* (X^*, d) *satisfying*

$$\varphi\bigl(d(Tx,Ty)\bigr) \le k\varphi\bigl(m(x,y)\bigr) \quad \textit{for all } x,y \in X,$$

where $k \in [0,1)$, $\varphi \in \Phi$, *and* $\psi \in \Psi$. *If* $0 < \psi(t) < t$ *for* $t > 0$ *and either assumption* (i) *or* (ii) *defined in Theorem* 2.1 *holds, then* T *possesses a unique fixed point.*

Taking $\varphi(t) = t$ in Corollary 2.6 and Corollary 2.7, we obtain the following results.

Corollary 2.8 *On* (X^*, d, q), *the function* q *forms a ceiling distance of* d *with* $q(x,x) = 0$ *for all* $x \in X$. *Assume that* $T : X \to X$ *satisfies*

$$q(Tx,Ty) \le kM(x,y) \quad \textit{for all } x,y \in X,$$

where $k \in [0,1)$. *If* $0 < \psi(t) < t$ *for* $t > 0$ *and either assumption* (i) *or* (ii) *defined in Theorem* 2.1 *holds, then* T *possesses a unique fixed point.*

Corollary 2.9 *Let* T *be a self-mapping on* (X^*, d) *satisfying*

$$d(Tx,Ty) \le km(x,y) \quad \textit{for all } x,y \in X,$$

where $k \in [0,1)$. *If* $0 < \psi(t) < t$ *for* $t > 0$ *and either assumption* (i) *or* (ii) *defined in Theorem* 2.1 *holds, then* T *possesses a unique fixed point.*

3 Best proximity point theorems

Let A, B be a nonempty subset of a metric space (X, d). Suppose that $T : A \to B$ is a non-self mapping. A point $z \in A$ is called *a best proximity point* if

$$d(z, Tz) = d(A, B) := \inf\{d(x, y) : x \in A, y \in B\}.$$

It is evident that the case $d(A, B) = 0$ turns the best proximity point problem into the fixed point problem.

In the field of optimization theory the best proximity point is an interesting topic for researchers [28–35]. Now let $A_0 = \{x \in A : d(x, y) = d(A, B)$ for some $y \in B\}$ and $B_0 = \{y \in B : d(x, y) = d(A, B)$ for some $x \in A\}$. The necessary and sufficient condition for A_0 and B_0 to be nonempty is given in [34].

Definition 3.1 On (X^*, d, q), a mapping $T : A \to B$ is called a Σ-*weighted proximal contraction* with respect to σ if there is $\sigma \in \Sigma$ such that

$$\left.\begin{aligned} d(v, Ty) = d(A, B) \\ d(u, Tx) = d(A, B) \end{aligned}\right\} \quad \text{implies} \quad \sigma\big(\varphi\big(q(u, v)\big), \psi\big(\varphi\big(q(x, y)\big)\big)\big) \geq 0 \tag{3.1}$$

for all $x, y, u, v \in A$, where $\varphi \in \Phi$ and $\psi \in \Psi$.

Our second main result in this paper is the following.

Theorem 3.2 *On (X^*, d, q), assume that A, B are two subsets of X such that A_0 is nonempty and closed. Assume that $T : A \to B$ T is a Σ-weighted proximal contraction with respect to σ and satisfies*:

(i) $T(A_0) \subseteq B_0$;

(ii) *q is a ceiling distance of d such that $q(x, x) = 0$ for each $x \in X$.*

Suppose also that one of the following cases holds:

1. *T is continuous*;
2. $\inf\{q(x, w) + q(x, y) : x \in X\} > 0$ *for every $w, y \in X$ with $d(w, Tw) \neq d(A, B)$ and* $d(y, Tx) = d(A, B)$.

If $0 < \psi(t) < t$ for $t > 0$, then there exists a unique $u \in A_0$ such that $d(u, Tu) = d(A, B)$. Moreover, for each $x_0 \in A_0$, there exists a sequence $(x_n) \subseteq A_0$ converging to x such that $d(x_{n+1}, Tx_n) = d(A, B)$ for all $n \in \mathbb{N} \cup \{0\}$.

Proof Let $x_0 \in A_0$, since $T(A_0) \subseteq B_0$, there exists $x_1 \in A_0$ such that $d(x_1, Tx_0) = d(A, B)$. Similarly, for $x_1 \in A_0$ there exists $x_2 \in A_0$ such that $d(x_2, Tx_1) = d(A, B)$.

Suppose that there exists $n_0 \in \mathbb{N}$ such that $q(x_{n_0-1}, x_{n_0}) = 0$. By using the ceiling distance of d, we get $x_{n_0} = x_{n_0-1}$, then we get $d(x_{n_0-1}, Tx_{n_0-1}) = d(A, B)$ i.e. x_{n_0-1} is a best proximity point of T.

Now suppose that $q(x_{n-1}, x_n) > 0$ for all $n \in \mathbb{N}$, and using the Σ-weighted proximal contraction condition, we get

$$\begin{aligned} 0 &\leq \sigma\big(\varphi\big(q(x_n, x_{n+1})\big), \psi\big(\varphi\big(q(x_{n-1}, x_n)\big)\big)\big) \\ &< \psi\big(\varphi\big(q(x_{n-1}, x_n)\big)\big) - \varphi\big(q(x_n, x_{n+1})\big), \end{aligned} \tag{3.2}$$

$$\varphi\big(q(x_n,x_{n+1})\big) < \psi\big(\varphi\big(q(x_{n-1},x_n)\big)\big) < \varphi\big(q(x_{n-1},x_n)\big). \tag{3.3}$$

Therefore, since φ is increasing, we get $q(x_n,x_{n+1}) < q(x_{n-1},x_n)$ for all $n \in \mathbb{N}$. Thus the sequence $(q(x_n,x_{n+1}))$ is decreasing and bounded below by 0. Hence there exists $r \geq 0$ such that

$$\lim_{n\to+\infty} q(x_n,x_{n+1}) = r. \tag{3.4}$$

Now suppose that $r > 0$, by (3.2) and (3.3) we have $\varphi(r) < \varphi(r)$ as $n \to +\infty$, which is a contradiction. Then we conclude that $r = 0$. Now we claim that (x_n) is a Cauchy sequence,

$$\lim_{n,m\to+\infty} q(x_n,x_m) = 0. \tag{3.5}$$

By using the proof of Theorem 2.1, we get

$$\lim_{k\to+\infty} q(x_{n_k},x_{m_k}) = \varepsilon \tag{3.6}$$

and

$$\lim_{k\to+\infty} q(x_{n_{k+1}},x_{m_{k+1}}) = \varepsilon. \tag{3.7}$$

Using the Σ-weighted proximal contraction condition and by (3.6) and (3.7) we have

$$\begin{aligned}
&0 \leq \sigma\big(\varphi\big(q(x_{n_{k+1}},x_{m_{k+1}})\big), \psi\big(\varphi\big(q(x_{n_k},x_{m_k})\big)\big)\big),\\
&0 < \psi\big(\varphi\big(q(x_{n_k},x_{m_k})\big)\big) - \varphi\big(q(x_{n_{k+1}},x_{m_{k+1}})\big),\\
&\varphi\big(q(x_{n_{k+1}},x_{m_{k+1}})\big) < \psi\big(\varphi\big(q(x_{n_k},x_{m_k})\big)\big) < \varphi\big(q(x_{n_k},x_{m_k})\big),
\end{aligned}$$

and as $k \to +\infty$ we get $\psi(\varphi(q(x_{n_k},x_{m_k}))) \to \varphi(\varepsilon)$. Then, by condition (iii) of the simulation function, we conclude that

$$0 \leq \limsup_{n\to+\infty} \sigma\big(\varphi\big(q(x_{n_{k+1}},x_{m_{k+1}})\big), \psi\big(\varphi\big(q(x_{n_k},x_{m_k})\big)\big)\big) < 0,$$

which is a contradiction. As a result, (3.5) holds. Moreover, by Lemma 1.8 the sequence (x_n) is Cauchy in A_0. Regarding the completeness of A_0, there exists $u \in A_0$ such that $x_n \to u$ as $n \to +\infty$.

Using the proof of Theorem 2.1, for every $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that for all $n > N_\varepsilon$ we get $q(x_{N_\varepsilon},x_n) < \varepsilon$. Also, by letting $\varepsilon = \frac{1}{k}$ and $N_\varepsilon = n_k$, we get

$$\lim_{k\to+\infty} q(x_{n_k},u) = 0.$$

Case (i): Suppose that $d(u,Tu) \neq d(A,B)$. Then we have

$$0 < \inf\big\{q(x,u) + q(x,y) : x \in X\big\} \leq \inf\big\{q(x_n,u) + q(x_n,x_{n+1})\big\},$$

which tends to 0 as $n \to +\infty$, which is a contradiction. Accordingly, we find $d(u,Tu) = d(A,B)$.

Case (ii): We presume that there exists $w \in X$ with $d(w, Tw) \neq d(A,B)$ and $\inf\{q(x,w) + q(x,y) : x \in X\} = 0$, where $d(y, Tx) = d(A,B)$.

Then there exist two sequences (x_n) and (y_n) such that $q(x_n, w) + q(x_n, y_n) \to 0$ as $n \to +\infty$, where $d(y_n, Tx_n) = d(A,B)$ for all $n \in \mathbb{N}$.

That is $q(x_n, w) \to 0$ and $q(x_n, y_n) \to 0$ as $n \to +\infty$. Then, by Lemma 1.8, we have $y_n \to w$ as $n \to +\infty$.

From the hypothesis note that $q(x_n, x_n) = 0$ for all $n \in \mathbb{N}$. Thus $q(x_n, x_n) \to 0$ as $n \to +\infty$. Consequently, by Lemma 1.8, we get $x_n \to w$ as $n \to +\infty$. From the continuity of T, we have $Tx_n \to Tw$ as $n \to +\infty$.

From $d(y_n, Tx_n) = d(A,B)$ we get $d(w, Tw) = d(A,B)$, which is a contradiction. Therefore if $d(w, Tw) \neq d(A,B)$ and $d(y, Tx) = d(A,B)$, then $\inf\{q(x,w) + q(x,y) : x \in X\} > 0$. Thus by case(i) we get the desired result.

To prove the uniqueness of the best proximity point, suppose that there exists $v \in A_0$ such that $d(v, Tv) = d(A,B)$. From the Σ-weighted proximal contraction condition we get

$$\begin{aligned} &0 \leq \sigma\bigl(\varphi\bigl(q(u,v)\bigr), \psi\bigl(\varphi\bigl(q(u,v)\bigr)\bigr)\bigr) \\ &\quad < \psi\bigl(\varphi\bigl(q(u,v)\bigr)\bigr) - \varphi\bigl(q(u,v)\bigr), \\ &\varphi\bigl(q(u,v)\bigr) < \psi\bigl(\varphi\bigl(q(u,v)\bigr)\bigr) < \varphi\bigl(q(u,v)\bigr), \end{aligned}$$

a contradiction. Therefore $u \in A_0$ is the unique best proximity point of T.

Also note that $q(u,u) = 0$. Supposing the contrary, we have $q(u,u) > 0$, and using the Σ-weighted proximal contraction condition, we get

$$\begin{aligned} &0 \leq \sigma\bigl(\varphi\bigl(q(u,u)\bigr), \psi\bigl(\varphi\bigl(q(u,u)\bigr)\bigr)\bigr) \\ &\quad < \psi\bigl(\varphi\bigl(q(u,u)\bigr)\bigr) - \varphi\bigl(q(u,u)\bigr), \\ &\varphi\bigl(q(u,u)\bigr) < \psi\bigl(\varphi\bigl(q(u,u)\bigr)\bigr) < \varphi\bigl(q(u,u)\bigr), \end{aligned}$$

which implies a contradiction, and hence $q(u,u) = 0$. □

Example Let $X = \mathbb{R}^2$, with the metric $d : X \times X \to \mathbb{R}$ defined by

$$d(x,y) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

for all $x = (x_1, y_1), y = (x_2, y_2) \in X$ and the w-distance q defined by

$$q(x,y) = \begin{cases} \|y\| & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

Let $A = \{(0,x) : 0 \leq x \leq 1\}$ and $B = \{\frac{1}{8}(1,y) : 0 \leq y \leq 1\}$. Define a mapping $T : A \to B$ by

$$T(0,x) = \frac{1}{8}(1,x).$$

Note that $A_0 = \{\frac{1}{8}(0,x) : 0 \leq x \leq 1\}$ and $B_0 = B$, and also we have $d(A,B) = \frac{1}{8}$.

Let us define two functions $\psi,\varphi:[0,+\infty)\to[0,+\infty)$ by

$$\varphi(t)=\frac{t^2}{4}$$

and

$$\psi(t)=\frac{t}{2}.$$

Now define the simulation function $\sigma:[0,+\infty)\times[0,+\infty)\to\mathbb{R}$ such that

$$\sigma(t,s)=\frac{s}{2}-t.$$

Suppose that $x\neq y$ where $x=(0,y_1)$ and $y=(0,y_2)$, then we have $Tx=\frac{1}{8}(1,y_1)$ and $Ty=\frac{1}{8}(1,y_2)$, also there exist $u,v\in A_0$ such that $u=\frac{1}{8}(0,y_1)$ and $v=\frac{1}{8}(0,y_2)$. Thus, $d(u,Tx)=d(v,Ty)=d(A,B)$.

$$\begin{aligned}
&q(x,y)=\|y\|=y_2,\\
&\varphi\bigl(q(x,y)\bigr)=\frac{y_2^2}{4},\\
&\psi\bigl(\varphi\bigl(q(x,y)\bigr)\bigr)=\frac{y_2^2}{8}, && (3.8)\\
&q(u,v)=\|v\|=\frac{y_2}{8},\\
&\varphi\bigl(q(u,v)\bigr)=\frac{y_2^2}{256}. && (3.9)
\end{aligned}$$

Then, by (3.8) and (3.9), equation (3.1) becomes

$$\begin{aligned}
\sigma\bigl(\varphi\bigl(q(u,v)\bigr),\psi\bigl(\varphi\bigl(q(x,y)\bigr)\bigr)\bigr)&=\frac{1}{2}\psi\bigl(\varphi\bigl(q(x,y)\bigr)\bigr)-\varphi\bigl(q(u,v)\bigr)\\
&=\frac{y_2^2}{16}-\frac{y_2^2}{256}\geq 0.
\end{aligned}$$

Also note that if $x=y$ then we get $u=v$, which implies that $\sigma(0,0)=0$. Then the hypotheses of Theorem 3.2 are satisfied, and hence T has a unique best proximity point. So, $x=(0,0)$ is a unique best proximity point of T i.e. $d((0,0),\frac{1}{8}(1,0))=d(A,B)=\frac{1}{8}$.

Taking $q=d$ in Theorem 3.2, we get the following result.

Corollary 3.3 *Let (X^*,d) and A,B be two nonempty subsets of X such that A_0 is nonempty and closed. Suppose that the mapping $T:A\to B$ satisfies:*

(i) *there exists $\sigma\in\Sigma$ such that*

$$\left.\begin{aligned}d(u,Tx)=d(A,B)\\ d(v,Ty)=d(A,B)\end{aligned}\right\}\quad\Rightarrow\quad\sigma\bigl(\varphi\bigl(d(u,v)\bigr),\psi\bigl(\varphi\bigl(d(x,y)\bigr)\bigr)\bigr)\geq 0 \tag{3.10}$$

for all $x,y,u,v\in A$, where $\varphi\in\Phi$ and $\psi\in\Psi$.

(ii) $T(A_0) \subseteq B_0$;

(iii) *Suppose that either*

(a) $\inf\{d(x,w)+d(x,y): x \in X\} > 0$ *for every* $w, y \in X$ *with* $d(w,Tw) \neq d(A,B)$ *and* $d(y,Tx) = d(A,B)$

(*or*)

(b) T *is continuous.*

If $0 < \psi(t) < t$ *for* $t > 0$, *then there exists unique* $u \in A_0$ *such that* $d(u,Tu) = d(A,B)$. *Moreover, for each* $x_0 \in A_0$, *there exists a sequence* $(x_n) \subseteq A_0$ *converging to* x *such that* $d(x_{n+1}, Tx_n) = d(A,B)$ *for all* $n \in \mathbb{N} \cup \{0\}$.

4 Application to Fredholm integral equations

The solution of a nonlinear Fredholm (Volterra) integral equation has been one of the hot topics in the last decades [36–40]. In this section, we shall provide a solution for the nonlinear Fredholm integral equation via our observed fixed point result. We shall consider $C[a,b]$, the metric space of continuous real-valued functions defined on $[a,b]$, and examine

$$x(t) = \varphi(t) + \int_a^b K\bigl(t, x(s)\bigr)\,ds, \tag{4.1}$$

where $x \in C[a,b]$ with $a, b \in \mathbb{R}$ such that $a < b$ and $\varphi : [a,b] \to \mathbb{R}$, $K : [a,b] \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

Theorem 4.1 *Let* $\psi, \varphi : [0,+\infty) \to [0,+\infty)$ *with* $\psi \in \Psi$ *and* $\varphi \in \Phi$ *such that* $\psi(t) > \varphi(t)$ *for all* $t > 0$ *and* $\psi(t) < t$ *for all* $t \geq 0$. *If*

$$\bigl|K(t_1, w_1)\bigr| + \bigl|K(t_2, w_2)\bigr| \leq \frac{\varphi(|w_1| + |w_2|)}{2(b-a)} - \frac{(|\varphi(t_1)| + |\varphi(t_2)|)}{(b-a)}$$

for all $t_1, t_2 \in [a,b]$ *and for all* $w_1, w_2 \in \mathbb{R}$, *then equation* (4.1) *possesses a unique solution.*

Proof Set $T : C[a,b] \to C[a,b]$ as

$$(Tx)(t) = \varphi(t) + \int_a^b K\bigl(t, x(s)\bigr)\,ds$$

for all $x \in X := C[a,b]$ with the metric

$$d(x,y) = \sup_{t \in [a,b]} \bigl|x(t) - y(t)\bigr|$$

for all $x, y \in X$. Note that the metric space $(C[a,b], d)$ is complete. Now, we define the function $q : X \times X \to [0,+\infty)$ by

$$q(x,y) = \begin{cases} \sup_{t \in [a,b]} |x(t)| + \sup_{t \in [a,b]} |y(t)| & \text{if } x \neq y, \\ 0 & \text{if } x = y \end{cases}$$

for all $x, y \in X$. Clearly, q is a w-distance on X and a ceiling distance of d. Now we have to indicate that T satisfies the Σ-weighted contraction condition (1.1). Let $\sigma(t,s) = \frac{s}{2} - t$ in

(1.1) and assume that $x, y \in X$ and $t_1, t_2 \in [a, b]$. Thus we get

$$\begin{aligned}
&\left|(Tx)(t_1)\right| + \left|(Ty)(t_2)\right| \\
&= \left|\varphi(t_1) + \int_a^b K\big(t_1, x(s)\big)\,ds\right| + \left|\varphi(t_2) + \int_a^b K\big(t_2, y(s)\big)\,ds\right| \\
&\le \left|\varphi(t_1)\right| + \left|\int_a^b K\big(t_1, x(s)\big)\,ds\right| + \left|\varphi(t_2)\right| + \left|\int_a^b K\big(t_2, y(s)\big)\,ds\right| \\
&\le \left|\varphi(t_1)\right| + \left|\varphi(t_2)\right| + \int_a^b \left|K\big(t_1, x(s)\big)\right|ds + \int_a^b \left|K\big(t_2, y(s)\big)\right|ds \\
&= \left|\varphi(t_1)\right| + \left|\varphi(t_2)\right| + \int_a^b \left(\left|K\big(t_1, x(s)\big)\right| + \left|K\big(t_2, y(s)\big)\right|\right)ds \\
&\le \left|\varphi(t_1)\right| + \left|\varphi(t_2)\right| + \int_a^b \left(\frac{\varphi(|x(s)| + |y(s)|)}{2(b-a)} - \frac{(|\varphi(t_1)| + |\varphi(t_2)|)}{(b-a)}\right)ds \\
&\le \left|\varphi(t_1)\right| + \left|\varphi(t_2)\right| + \frac{1}{b-a}\left[\int_a^b \frac{\varphi(q(x,y))}{2}\,ds - \int_a^b \left(\left|\varphi(t_1)\right| + \left|\varphi(t_2)\right|\right)ds\right] \\
&= \frac{\varphi(q(x,y))}{2} \le \frac{\varphi(M(x,y))}{2}.
\end{aligned}$$

From this we have

$$\sup_{t\in[a,b]} \left|(Tx)(t)\right| + \sup_{t\in[a,b]} \left|(Ty)(t)\right| \le \frac{\varphi(M(x,y))}{2},$$

which implies that

$$q(Tx, Ty) \le \frac{\varphi(M(x,y))}{2}$$

for all $x, y \in X$ with $x \neq y$. Therefore, we get

$$\begin{aligned}
&\varphi\big(q(Tx, Ty)\big) \le \frac{\varphi(\varphi(M(x,y)))}{2} \le \frac{\psi(\varphi(M(x,y)))}{2}, \\
&\frac{\psi(\varphi(M(x,y)))}{2} - \varphi\big(q(Tx, Ty)\big) \ge 0, \\
&\sigma\left(\varphi\big(q(Tx, Ty)\big), \psi\left(\varphi\big(M(x,y)\big)\right)\right) \ge 0
\end{aligned}$$

for all $x, y \in X$ with $x \neq y$. For $x = y$, it is easy to verify that T satisfies the Σ-weighted contraction condition (1.1). Hence T satisfies the Σ-weighted contraction condition (1.1). Consequently, T fulfills all hypotheses of Theorem 2.1, then T possesses a unique fixed point. From this we conclude that there is a unique solution for the nonlinear Fredholm integral equation (4.1). Hence the proof. □

Example Consider the Fredholm integral equation such that

$$x(t) = \frac{t}{4} + \frac{3}{2}\int_0^1 \big(s^2 t + st^2\big)x(s)\,ds, \tag{4.2}$$

where $x \in C[0,1]$ and satisfies the following condition:

$$\left|s^2t_1 + st_1^2\right|\left|x(s)\right| + \left|s^2t_2 + st_2^2\right|\left|y(s)\right| \leq \frac{(|x(s)| + |y(s)|)}{12} - \frac{(|t_1| + |t_2|)}{6}$$

for all $t_1, t_2 \in [0,1]$ and for all $w_1, w_2 \in \mathbb{R}$.

Now let $X = C[0,1]$ with the metric $d : X \times X \to [0,+\infty)$ given by

$$d(x,y) = \sup_{t\in[0,1]} \left|x(t) - y(t)\right| \tag{4.3}$$

for all $x, y \in X$. It is clear that (X,d) is a complete metric space. Now, we define the function $q : X \times X \to [0,+\infty)$ by

$$q(x,y) = \begin{cases} \sup_{t\in[0,1]}|x(t)| + \sup_{t\in[0,1]}|y(t)| & \text{if } x \neq y, \\ 0 & \text{if } x = y \end{cases} \tag{4.4}$$

for all $x, y \in X$. Clearly, q is a w-distance on X and a ceiling distance of d. Now, define a mapping $T : X \to X$ by

$$(Tx)(t) = \frac{t}{4} + \frac{3}{2}\int_0^1 \left(s^2t + st^2\right)x(s)\,ds$$

for all $x \in X$. Next we define two functions $\psi, \varphi : [0,+\infty) \to [0,+\infty)$ by

$$\psi(t) = \frac{t}{2}$$

and

$$\varphi(t) = \frac{t}{4}.$$

It is easy to verify that $\varphi \in \Phi$ and $\psi \in \Psi$. Note that $\psi(t) < t$ for all $t > 0$ and $\psi(t) > \varphi(t)$ for all $t > 0$.

$$\begin{aligned}
&\left|(Tx)(t_1)\right| + \left|(Ty)(t_2)\right| \\
&\quad = \left|\frac{t_1}{4} + \frac{3}{2}\int_0^1 \left(s^2t_1 + st_1^2\right)x(s)\,ds\right| + \left|\frac{t_2}{4} + \frac{3}{2}\int_0^1 \left(s^2t_2 + st_2^2\right)y(s)\,ds\right| \\
&\quad \leq \left|\frac{t_1}{4}\right| + \frac{3}{2}\int_0^1 \left|s^2t_1 + st_1^2\right|\left|x(s)\right|ds + \left|\frac{t_2}{4}\right| + \frac{3}{2}\int_0^1 \left|s^2t_2 + st_2^2\right|\left|y(s)\right|ds \\
&\quad \leq \left|\frac{t_1}{4}\right| + \left|\frac{t_2}{4}\right| + \frac{3}{2}\int_0^1 \left(\left|s^2t_1 + st_1^2\right|\left|x(s)\right| + \left|s^2t_2 + st_2^2\right|\left|y(s)\right|\right)ds \\
&\quad \leq \left|\frac{t_1}{4}\right| + \left|\frac{t_2}{4}\right| + \frac{3}{2}\int_0^1 \left(\frac{(|x(s)| + |y(s)|)}{12} - \frac{(|t_1| + |t_2|)}{6}\right)ds \\
&\quad \leq \left|\frac{t_1}{4}\right| + \left|\frac{t_2}{4}\right| + \int_0^1 \frac{\varphi(q(x,y))}{2}\,ds - \int_0^1 \frac{(|t_1| + |t_2|)}{4}\,ds \\
&\quad = \frac{\varphi(q(x,y))}{2}.
\end{aligned}$$

From this we get

$$\sup_{t\in[a,b]}\left|(Tx)(t)\right| + \sup_{t\in[a,b]}\left|(Ty)(t)\right| \le \frac{\varphi(q(x,y))}{2}.$$

Then we have

$$q(Tx,Ty) \le \frac{\varphi(q(x,y))}{2}$$

for all $x, y \in X$ with $x \neq y$. Therefore, we get

$$\begin{aligned}
&\varphi\big(q(Tx,Ty)\big) \le \frac{\varphi(\varphi(q(x,y)))}{2} \le \frac{\psi(\varphi(q(x,y)))}{2},\\
&\frac{\psi(\varphi(q(x,y)))}{2} - \varphi\big(q(Tx,Ty)\big) \ge 0,\\
&\sigma\Big(\varphi\big(q(Tx,Ty)\big), \psi\big(\varphi\big(q(x,y)\big)\big)\Big) \ge 0
\end{aligned}$$

for all $x, y \in X$ with $x \neq y$. For $x = y$, it is easy to verify that T satisfies the Σ-weighted contraction condition (1.1). Hence T satisfies the Σ-weighted contraction condition (1.1), therefore by Theorem 2.1 T has a unique fixed point. Thus, we conclude that equation (4.2) possesses a unique solution.

5 Application to the problem of buckling of a rod

In this section, we study the solution for the buckling of a rod problem as an application of our derived fixed point result. For this consider a homogeneous thin rod whose ends are pinned, the left end is fixed, and the right end is free to move along the x axis. The rod coincides with the portion of the x axis between 0 and l when it is unloaded. Under a compressive load P a possible state for the rod is that of pure compression, but for sufficiently large P transverse deflections occur. Assume that the buckling occurs in the $x - y$ plane.

Buckling of a rod leads to the following boundary value problem:

$$u'' + \alpha u = 0, \quad 0 < S < l; \qquad u(0) = u(l) = 0. \tag{5.1}$$

Here $\alpha = \frac{P}{EI}$, where the constants E and I are respectively Young's modulus and the moment of inertia. The problem can be translated into the Fredholm integral equation

$$u(x) = \alpha \int_0^l G(x,\xi)u(\xi)\,d\xi, \tag{5.2}$$

where $G(x,\xi)$ is the Green's function associated with (5.1) given by

$$G(x,\xi) = \begin{cases} \frac{(l-\xi)x}{l}, & 0 \le x < \xi, \\ \frac{(l-x)\xi}{l}, & \xi < x \le l. \end{cases} \tag{5.3}$$

Let $X = C[0,l]$ be a set of all real-valued continuous functions on $[0,l]$ with the metric

$$d(x,y) = \sup_{t\in[a,b]}\left|x(t) - y(t)\right|$$

for all $x, y \in X$. Note that the metric space $(C[a,b], d)$ is complete and the w-distance $q : X \times X \to [0, +\infty)$ is defined as follows:

$$q(x,y) = \begin{cases} \sup_{t\in[a,b]}|x(t)| + \sup_{t\in[a,b]}|y(t)| & \text{if } x \neq y, \\ 0 & \text{if } x = y \end{cases}$$

for all $x, y \in X$.

Theorem 5.1 *The problem of buckling of a rod leads to the second order boundary value problem* (5.1). *Consider the mapping* $T : X \to X$ *defined by*

$$Tu(x) = \alpha \int_0^l G(x,\xi)u(\xi)\,d\xi, \tag{5.4}$$

where $G(x,\xi)$ *is a Green's function* (5.3) *related to* (5.1). *For* $\alpha l^2 < 2$, *there exists a unique fixed point for the Fredholm integral equation* (5.4) *which provides a solution for* (5.1).

Proof It is clear that a map $T : X \to X$ is well defined. Define two functions $\psi : [0, +\infty) \to [0, +\infty)$ and $\varphi : [0, +\infty) \to [0, +\infty)$ by $\psi(t) = \frac{t}{2}$ and $\varphi(t) = t$. One can easily verify that $\varphi \in \Phi$ and $\psi \in \Psi$. Also $\psi(t) < t$ for all $t > 0$.

Let $\sigma : [0, +\infty) \times [0, +\infty) \to \mathbb{R}$ defined by $\sigma(t,s) = \frac{s}{2} - t$. If $u, v \in X$, then we have

$$\begin{aligned} \left|Tu(x)\right| + \left|Tv(x)\right| &\leq \alpha \int_0^l G(x,\xi)\left|u(\xi)\right| + \left|v(\xi)\right| d\xi \\ &\leq \alpha q(u,v) \int_0^l G(x,\xi)\,d\xi, \\ q(Tu,Tv) &\leq \alpha q(u,v)\frac{l^2}{8} < \frac{1}{4}q(u,v), \\ \varphi\bigl(q(Tu,Tv)\bigr) &\leq \frac{1}{4}\varphi\bigl(q(u,v)\bigr), \\ \varphi\bigl(q(Tu,Tv)\bigr) &\leq \frac{1}{2}\psi\bigl(\varphi\bigl(q(u,v)\bigr)\bigr), \\ \frac{1}{2}\psi\bigl(\varphi\bigl(q(u,v)\bigr)\bigr) &- \varphi\bigl(q(Tu,Tv)\bigr) \geq 0, \\ \sigma\bigl(\varphi\bigl(q(Tu,Tv)\bigr), \psi\bigl(\varphi\bigl(q(u,v)\bigr)\bigr)\bigr) &\geq 0, \end{aligned}$$

where $\varphi(t) = t$, $\psi(t) = \frac{t}{2}$, and $\sigma(t,s) = \frac{s}{2}$.

T satisfies the hypothesis of Theorem 2.1. Therefore, for $\alpha l^2 < 2$, equation (5.1) or (5.4) has one and only one solution. □

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