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Some new oscillation criteria of fourth-order quasi-linear differential equations with neutral term

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Abstract

In this article, we are interested in studying the asymptotic behavior of fourth-order neutral differential equations. Despite the growing interest in studying the oscillatory behavior of delay differential equations of second-order, fourth-order equations have received less attention. We get more than one criterion to check the oscillation by the generalized Riccati method and the integral average technique. Our results are an extension and complement to some results published in the literature. Examples are given to prove the significance of new theorems.

Keywords: Fourth-order differential equations; Neutral delay; Oscillation; Philos-type oscillation

1 Introduction

In this paper, we investigate the oscillation properties of solutions to the fourth-order neutral differential equations:

$$\left(z(x)\varsigma_{r_1}\left(\delta'''(x)\right)\right)' + \tilde{\omega}(x)\varsigma_{r_2}\left(\beta\left(\theta(x)\right)\right) = 0, \tag{1}$$

where $\varsigma_{r_i}[s] = |s|^{r_i-1}s$, $\delta(x) = \beta(x) + \tilde{y}(x)\beta(\tilde{\theta}(x))$. Throughout this paper, we suppose that:

(S_1) r_1 and r_2 are quotients of odd positive integers,

(S_2) $z, \tilde{y}, \tilde{\omega} \in C[x_0,\infty)$, $z(x) > 0$, $z'(x) \geq 0$, $\tilde{\omega}(x) > 0$, $0 \leq \tilde{y}(x) \leq \tilde{y}_0 < 1$, $\tilde{\theta}$, $\theta \in C[x_0,\infty)$, $\tilde{\theta}(x) \leq x$, $\lim_{x\to\infty} \tilde{\theta}(x) = \lim_{x\to\infty} \theta(x) = \infty$,

and under the assumption

$$\int_{x_0}^{\infty} \frac{1}{z^{1/r_1}(s)}\,\mathrm{d}s = \infty. \tag{2}$$

Definition 1.1 ([1]) Let

$$D = \left\{(x,s) \in \mathbb{R}^2 : x \geq s \geq x_0\right\} \quad \text{and} \quad D_0 = \left\{(x,s) \in \mathbb{R}^2 : x > s \geq x_0\right\}.$$

The function $G_i \in C(D,\mathbb{R})$ fulfills the following conditions:

(i) $G_i(x,s) = 0$ for $x \geq x_0, G_i(x,s) > 0, (x,s) \in D_0$;
(ii) The functions $h, \upsilon \in C^1([x_0,\infty),(0,\infty))$ and $g_i \in C(D_0,\mathbb{R})$ such that

$$\frac{\partial}{\partial s}G_1(x,s) + \frac{\alpha'(s)}{\alpha(s)}G(x,s) = g_1(x,s)G_1^{r_1/(r_1+1)}(x,s) \tag{3}$$

and

$$\frac{\partial}{\partial s}G_2(x,s) + \frac{h'(s)}{h(s)}G_2(x,s) = g_2(x,s)\sqrt{G_2(x,s)}. \tag{4}$$

Theory of oscillation of differential equations is a fertile study area and has attracted the attention of many authors recently. This is due to the existence of many important applications of this theory in neural networks, biology, social sciences, engineering, etc., see [2–10]. Very recently, a great development was found in the study of oscillation of solutions to neutral differential equations, see [11–20]. In particular, quasilinear/Emden–Fowler differential equations have numerous applications in physics and engineering (e.g., quasilinear/Emden–Fowler differential equations arise in the study of p-Laplace equations, porous medium problems, and so on); see, e.g., the papers [5, 21–24] for more details, the papers [5, 6, 25–28] for the oscillation of quasilinear/Emden–Fowler differential equations, and the papers [4, 24, 29–35] for the oscillation and asymptotic behavior of quasilinear/Emden–Fowler differential equations with different neutral coefficients.

Xing et al. [33] presented criteria for oscillation of the equation

$$\left(z(x)\left(\delta^{(n-1)}(x)\right)^{r_1}\right)' + \tilde{\omega}(x)\beta^{r_1}\left(\theta(x)\right) = 0$$

under the conditions

$$\left(\theta^{-1}(x)\right)' \geq \theta_0 > 0, \qquad \tilde{\theta}'(x) \geq \tilde{\theta}_0 > 0, \qquad \tilde{\theta}^{-1}\left(\theta(x)\right) < x$$

and

$$\liminf_{x\to\infty}\int_{\tilde{\theta}^{-1}(\theta(x))}^{x}\frac{\widehat{\tilde{\omega}}(s)}{z(s)}\left(s^{n-1}\right)^{r_1}\,\mathrm{d}s > \left(\frac{1}{\theta_0} + \frac{\tilde{y}_0^{r_1}}{\theta_0\tilde{\theta}_0}\right) > \frac{((n-1)!)^{r_1}}{\mathrm{e}},$$

where $0 \leq \tilde{y}(x) < \tilde{y}_0 < \infty$ and $\widehat{\tilde{\omega}}(x) := \min\{\tilde{\omega}(\theta^{-1}(x)), \tilde{\omega}(\theta^{-1}(\tilde{\theta}(x)))\}$.

Bazighifan et al. [18], Li and Rogovchenko [25], and Zhang et al. [26, 28] presented oscillation results for fourth-order equation

$$\left(z(x)\left(\delta'''(x)\right)^{r_1}\right)' + \tilde{\omega}(x)\beta^{r_1}\left(\theta(x)\right) = 0$$

under the condition

$$\int_{x_0}^{\infty}\frac{1}{z^{1/r_1}(s)}\,\mathrm{d}s < \infty,$$

and they used the Riccati technique.

Zhang et al. [36] established oscillation criteria for the equation

$$\left(z(x)\left(\delta^{(n-1)}(x)\right)^{r_1}\right)' + \tilde{\omega}(x)f\left(\beta\left(\theta(x)\right)\right) = 0$$

and under the condition

$$\int_{x_0}^{\infty} \left(k\rho(x)E(x) - \frac{1}{4\lambda} \left(\frac{\rho'(x)}{\rho(x)} \right)^2 \eta(x) \right) \mathrm{d}z = \infty.$$

By using the Riccati transformation technique, Chatzarakis et al. [19] established asymptotic behavior for the neutral equation

$$\left(z(x)\left(\delta'''(x)\right)^{r_1}\right)' + \int_a^b \tilde{\omega}(x,s) f\left(\beta\left(\theta(x,s)\right)\right) \mathrm{d}s = 0.$$

In this work, a new oscillation condition is created for fourth-order differential equations with a canonical operator. We use the Riccati technique and the integral averaging technique to prove our results.

Here are the notations used for our study:

$$E_1(x) = \alpha(x)\tilde{\omega}(x)(1-\tilde{y}_0)^{r_2} A_1^{r_2-r_1} \left(\frac{\theta(x)}{x} \right)^{3r_2},$$

$$\Phi(x) = (1-\tilde{y}_0)^{r_2/r_1} h(x) A_2^{r_2/r_1-1}(x) \int_x^{\infty} \left(\frac{1}{z(u)} \int_u^{\infty} \tilde{\omega}(s) \frac{\theta^{r_2}(s)}{s^{r_2}} \,\mathrm{d}s \right)^{1/r_1} \mathrm{d}u$$

and

$$\Theta(x) = r_1 \mu_1 \frac{x^2}{2z^{1/r_1}(x)\alpha^{1/r_1}(x)}.$$

2 Oscillation criteria

We next present the lemmas needed for the proof of the original results.

Lemma 2.1 ([37]) *If $\beta^{(i)}(x) > 0$, $i = 0, 1, \ldots, n$, and $\beta^{(n+1)}(x) < 0$, then*

$$n! \frac{\beta(x)}{x^n} \geq (n-1)! \frac{\beta'(x)}{x^{n-1}}.$$

Lemma 2.2 ([20]) *Let $\beta \in C^n([x_0,\infty),(0,\infty))$. Assume that $\beta^{(n)}(x)$ is of a fixed sign and not identically zero on $[x_0,\infty)$ and that there exists $x_1 \geq x_0$ such that $\beta^{(n-1)}(x)\beta^{(n)}(x) \leq 0$ for all $x \geq x_1$. If $\lim_{x\to\infty} \beta(x) \neq 0$, then for every $\mu \in (0,1)$ there exists $x_\mu \geq x_1$ such that*

$$\beta(x) \geq \frac{\mu}{(n-1)!} x^{n-1} \left| \beta^{(n-1)}(x) \right| \quad \textit{for } x \geq x_\mu.$$

Lemma 2.3 ([27]) *Let $a \geq 0$. Then*

$$X\beta - Y\beta^{(a+1)/a} \leq a^a (a+1)^{-(a+1)} Y^{-a} X^{a+1},$$

where $Y > 0$ and X are constants.

Lemma 2.4 ([38])

$$\textit{Assume that } \beta \textit{is an eventually positive solution of } (1). \tag{5}$$

Then

$$\begin{aligned} &\textit{Case } (\mathbf{N}_1): \quad \delta^{(j)}(x) > 0 \quad \textit{for } j = 0,1,2,3, \\ &\textit{Case } (\mathbf{N}_2): \quad \delta^{(j)}(x) > 0 \quad \textit{for } j = 0,1,3 \textit{ and } \delta''(x) < 0, \end{aligned}$$

for $x \geq x_1$, *where* $x_1 \geq x_0$ *is sufficiently large.*

Lemma 2.5 *Let* (5) *hold. Then*

$$\bigl(z(x)\bigl(\delta'''(x)\bigr)^{r_1}\bigr)' \leq -G(x)\bigl(\delta'''\bigl(\theta(x)\bigr)\bigr)^{r_2}, \tag{6}$$

where

$$G(x) = \tilde{\omega}(x)(1-\tilde{y}_0)^{r_2}\left(\frac{\mu}{6}\theta^3(x)\right)^{r_2}.$$

Proof Let (5) hold. From the definition of δ, we get

$$\begin{aligned} \beta(x) &\geq \delta(x) - \tilde{y}_0\beta\bigl(\tilde{\theta}(x)\bigr) \\ &\geq \delta(x) - \tilde{y}_0\delta\bigl(\tilde{\theta}(x)\bigr) \\ &\geq (1-\tilde{y}_0)\delta(x), \end{aligned}$$

which with (1) gives

$$\bigl(z(x)\bigl(\delta'''(x)\bigr)^{r_1}\bigr)' + \tilde{\omega}(x)(1-\tilde{y}_0)^{r_2}\delta^{r_2}\bigl(\theta(x)\bigr) \leq 0. \tag{7}$$

Using Lemma 2.2, we see that

$$\delta(x) \geq \frac{\mu}{6}x^3\delta'''(x). \tag{8}$$

Combining (7) and (8), we find

$$\bigl(z(x)\bigl(\delta'''(x)\bigr)^{r_1}\bigr)' + \tilde{\omega}(x)(1-\tilde{y}_0)^{r_2}\left(\frac{\mu}{6}\theta^3(x)\right)^{r_2}\bigl(\delta'''\bigl(\theta(x)\bigr)\bigr)^{r_2} \leq 0.$$

Thus, (6) holds. This completes the proof. □

Lemma 2.6 *Let* (5) *hold. If* δ *satisfies* $(\mathbf{N}_1)$, *then*

$$B'(x) \leq \frac{\alpha'(x)}{\alpha(x)}B(x) - E_1(x) - r_1\mu_1\frac{x^2}{2z^{1/r_1}(x)\alpha^{1/r_1}(x)}B^{\frac{r_1+1}{r_1}}(x), \tag{9}$$

if δ *satisfies* $(\mathbf{N}_2)$, *then*

$$A'(x) \leq -\Phi(x) + \frac{h'(x)}{h(x)}A(x) - \frac{1}{h(x)}A^2(x), \tag{10}$$

where

$$B(x) := \alpha(x)\frac{z(x)(\delta'''(x))^{r_1}}{\delta^{r_1}(x)} > 0 \tag{11}$$

and

$$A(x) := h(x)\frac{\delta'(x)}{\delta(x)}, \quad x \geq x_1. \tag{12}$$

Proof Let (5) and ($\mathbf{N}_1$) hold. From (11) and (7), we find

$$B'(x) \leq \frac{\alpha'(x)}{\alpha(x)}B(x) - \alpha(x)\tilde{\omega}(x)(1-\tilde{y}_0)^{r_2}\frac{\delta^{r_2}(\theta(x))}{\delta^{r_1}(x)} - r_1\alpha(x)\frac{z(x)(\delta'''(x))^{r_1}}{\delta^{r_1+1}(x)}\delta'(x). \tag{13}$$

Using Lemma 2.1, we find

$$\delta(x) \geq \frac{x}{3}\delta'(x),$$

and hence

$$\frac{\delta(\theta(x))}{\delta(x)} \geq \frac{\theta^3(x)}{x^3}. \tag{14}$$

It follows from Lemma 2.2 that

$$\delta'(x) \geq \frac{\mu_1}{2}x^2\delta'''(x) \tag{15}$$

for all $\mu_1 \in (0,1)$ and every sufficiently large x. Thus, by (13), (14), and (15), we get

$$\begin{aligned} B'(x) \leq{}& \frac{\alpha'(x)}{\alpha(x)}B(x) - \alpha(x)\tilde{\omega}(x)(1-\tilde{y}_0)^{r_2}\delta^{r_2-r_1}(x)\left(\frac{\theta(x)}{x}\right)^{3r_2} \\ & - r_1\mu_1\frac{x^2}{2z^{1/r_1}(x)\alpha^{1/r_1}(x)}B^{\frac{r_1+1}{r_1}}(x). \end{aligned}$$

Since $\delta'(x) > 0$, there exist $x_2 \geq x_1$ and $A_1 > 0$ such that

$$\delta(x) > A_1. \tag{16}$$

Thus, we obtain

$$\begin{aligned} B'(x) \leq{}& \frac{\alpha'(x)}{\alpha(x)}B(x) - \alpha(x)\tilde{\omega}(x)(1-\tilde{y}_0)^{r_2}A^{r_2-r_1}\left(\frac{\theta(x)}{x}\right)^{3r_2} \\ & - r_1\mu_1\frac{x^2}{2z^{1/r_1}(x)\alpha^{1/r_1}(x)}B^{\frac{r_1+1}{r_1}}(x), \end{aligned}$$

which yields

$$B'(x) \leq \frac{\alpha'(x)}{\alpha(x)}B(x) - E_1(x) - r_1\mu_1\frac{x^2}{2z^{1/r_1}(x)\alpha^{1/r_1}(x)}B^{\frac{r_1+1}{r_1}}(x).$$

Thus, (9) holds.

Let ($\mathbf{N}_2$) hold. Integrating (7) from x to u, we find

$$z(u)\bigl(\delta'''(u)\bigr)^{r_1} - z(x)\bigl(\delta'''(x)\bigr)^{r_1} \le -\int_x^u \tilde{\omega}(s)(1-\tilde{y}_0)^{r_2}\delta^{r_2}\bigl(\theta(s)\bigr)\,\mathrm{d}s. \tag{17}$$

From Lemma 2.1, we obtain

$$\delta(x) \ge x\delta'(x),$$

and hence

$$\delta\bigl(\theta(x)\bigr) \ge \frac{\theta(x)}{x}\delta(x). \tag{18}$$

For (17), letting $u \to \infty$ and using (18), we get

$$z(x)\bigl(\delta'''(x)\bigr)^{r_1} \ge (1-\tilde{y}_0)^{r_2}\delta^{r_2}(x)\int_x^\infty \tilde{\omega}(s)\frac{\theta^{r_2}(s)}{s^{r_2}}\,\mathrm{d}s. \tag{19}$$

Integrating (19) from x to ∞, we find

$$\delta''(x) \le -(1-\tilde{y}_0)^{r_2/r_1}\delta^{r_2/r_1}(x)\int_x^\infty \left(\frac{1}{z(u)}\int_u^\infty \tilde{\omega}(s)\frac{\theta^{r_2}(s)}{s^{r_2}}\,\mathrm{d}s\right)^{1/r_1}\mathrm{d}u. \tag{20}$$

From the definition of $A(x)$, we see that $A(x) > 0$ for $x \ge x_1$, and using (16) and (20), we find

$$\begin{aligned}
A'(x) &= \frac{h'(x)}{h(x)}A(x) + h(x)\frac{\delta''(x)}{\delta(x)} - h(x)\left(\frac{\delta'(x)}{\delta(x)}\right)^2 \\
&\le \frac{h'(x)}{h(x)}A(x) - \frac{1}{h(x)}A^2(x) \\
&\quad - (1-\tilde{y}_0)^{r_2/r_1}h(x)\delta^{r_2/r_1-1}(x)\int_x^\infty \left(\frac{1}{z(u)}\int_u^\infty \tilde{\omega}(s)\frac{\theta^{r_2}(s)}{s^{r_2}}\,\mathrm{d}s\right)^{1/r_1}\mathrm{d}u.
\end{aligned}$$

Since $\delta'(x) > 0$, there exist $x_2 \ge x_1$ and $A_2 > 0$ such that

$$\delta(x) > A_2.$$

Thus, we obtain

$$A'(x) \le -\Phi(x) + \frac{h'(x)}{h(x)}A(x) - \frac{1}{h(x)}A^2(x).$$

Thus, (10) holds. The proof of the theorem is completed. □

Now, we present some Philos-type oscillation criteria for (1).

Theorem 2.7 *Let* (25) *hold. If* $\alpha, h \in C^1([x_0,\infty),\mathbb{R})$ *such that*

$$\limsup_{x\to\infty}\frac{1}{G(x,x_1)}\int_{x_1}^x G(x,s)E_1(s) - \frac{g_1^{r_1+1}(x,s)G_1^{r_1}(x,s)}{(r_1+1)^{r_1+1}}\frac{2^{r_1}z(s)\alpha(s)}{(\mu_1 s^2)^{r_1}}\,\mathrm{d}s = \infty \tag{21}$$

for all $\mu_2 \in (0,1)$, *and*

$$\limsup_{x\to\infty} \frac{1}{G_2(x,x_1)} \int_{x_1}^{x} \left(G_2(x,s)\Phi(s) - \frac{h(s)g_2^2(x,s)}{4} \right) \mathrm{d}s = \infty, \tag{22}$$

then (1) *is oscillatory.*

Proof Let β be a nonoscillatory solution of (1), we see that $\beta > 0$. Assume that $(\mathbf{N}_1)$ holds. Multiplying (9) by $G(x,s)$ and integrating the resulting inequality from x_1 to x; we obtain

$$\begin{aligned} \int_{x_1}^{x} G(x,s)E_1(s)\,\mathrm{d}s \le{}& B(x_1)G(x,x_1) + \int_{x_1}^{x} \left(\frac{\partial}{\partial s} G(x,s) + \frac{\alpha'(s)}{\alpha(s)} G(x,s) \right) B(s)\,\mathrm{d}s \\ & - \int_{x_1}^{x} \Theta(s)G(x,s)B^{\frac{r_1+1}{r_1}}(s)\,\mathrm{d}s. \end{aligned}$$

From (3), we get

$$\begin{aligned} \int_{x_1}^{x} G(x,s)E_1(s)\,\mathrm{d}s \le{}& B(x_1)G(x,x_1) + \int_{x_1}^{x} g_1(x,s)G_1^{r_1/(r_1+1)}(x,s)B(s)\,\mathrm{d}s \\ & - \int_{x_1}^{x} \Theta(s)G(x,s)B^{\frac{r_1+1}{r_1}}(s)\,\mathrm{d}s. \end{aligned} \tag{23}$$

Using Lemma 2.3 with $V = \Theta(s)G(x,s)$, $U = g_1(x,s)G_1^{r_1/(r_1+1)}(x,s)$, and $\beta = B(s)$, we get

$$\begin{aligned} & g_1(x,s)G_1^{r_1/(r_1+1)}(x,s)B(s) - \Theta(s)G(x,s)B^{\frac{r_1+1}{r_1}}(s) \\ & \quad \le \frac{g_1^{r_1+1}(x,s)G_1^{r_1}(x,s)}{(r_1+1)^{r_1+1}} \frac{2^{r_1}z(x)\alpha(x)}{(\mu_1 x^2)^{r_1}}, \end{aligned}$$

which with (23) gives

$$\frac{1}{G(x,x_1)} \int_{x_1}^{x} \left(G(x,s)E_1(s) - \frac{g_1^{r_1+1}(x,s)G_1^{r_1}(x,s)}{(r_1+1)^{r_1+1}} \frac{2^{r_1}z(s)\alpha(s)}{(\mu_1 s^2)^{r_1}} \right) \mathrm{d}s \le B(x_1),$$

which contradicts (21).

Assume that $(\mathbf{N}_2)$ holds. Multiplying (10) by $G_2(x,s)$ and integrating the resulting inequality from x_1 to x, we find

$$\begin{aligned} \int_{x_1}^{x} G_2(x,s)\Phi(s)\,\mathrm{d}s \le{}& A(x_1)G_2(x,x_1) \\ & + \int_{x_1}^{x} \left(\frac{\partial}{\partial s} G_2(x,s) + \frac{h'(s)}{h(s)} G_2(x,s) \right) A(s)\,\mathrm{d}s \\ & - \int_{x_1}^{x} \frac{1}{h(s)} G_2(x,s)A^2(s)\,\mathrm{d}s. \end{aligned}$$

Thus,

$$\int_{x_1}^{x} G_2(x,s)\Phi(s)\,\mathrm{d}s \le A(x_1)G_2(x,x_1) + \int_{x_1}^{x} g_2(x,s)\sqrt{G_2(x,s)}A(s)\,\mathrm{d}s$$

$$- \int_{x_1}^{x} \frac{1}{h(s)} G_2(x,s) A^2(s)\,\mathrm{d}s$$
$$\le A(x_1) G_2(x,x_1) + \int_{x_1}^{x} \frac{h(s) g_2^2(x,s)}{4}\,\mathrm{d}s,$$

and so

$$\frac{1}{G_2(x,x_1)} \int_{x_1}^{x} \left(G_2(x,s)\Phi(s) - \frac{h(s) g_2^2(x,s)}{4} \right) \mathrm{d}s \le A(x_1),$$

which contradicts (22). The proof of the theorem completed. □

Corollary 2.8 *Let* (25) *hold. If* $\alpha, h \in C^1([x_0,\infty),\mathbb{R})$ *such that*

$$\int_{x_0}^{\infty} \left(E_1(s) - \frac{2^{r_1}}{(r_1+1)^{r_1+1}} \frac{z(s)(\alpha'(s))^{r_1+1}}{\mu_1^{r_1} s^{2r_1} \alpha^{r_1}(s)} \right) \mathrm{d}s = \infty \tag{24}$$

and

$$\int_{x_0}^{\infty} \left(\Phi(s) - \frac{(h'(s))^2}{4h(s)} \right) \mathrm{d}s = \infty \tag{25}$$

for some $\mu_1 \in (0,1)$ *and every* $A_1, A_2 > 0$, *then* (1) *is oscillatory.*

Example 2.9 Consider the equation

$$\left(\beta + \frac{1}{2} \beta\left(\frac{1}{3}x\right) \right)^{(4)} + \frac{\tilde{\omega}_0}{x^4} \beta\left(\frac{1}{2}x\right) = 0, \quad x \ge 1, \tilde{\omega}_0 > 0. \tag{26}$$

Let $r_1 = r_2 = 1$, $z(x) = 1$, $\tilde{y}(x) = 1/2$, $\tilde{\theta}(x) = x/3$, $\theta(x) = x/2$, and $\tilde{\omega}(x) = \tilde{\omega}_0/x^4$. Hence, it is easy to see that

$$\int_{x_0}^{\infty} \frac{1}{z^{1/r_1}(s)}\,\mathrm{d}s = \infty, \qquad E_1(x) = \frac{\tilde{\omega}_0}{16s}$$

and

$$\Phi(x) := \frac{\tilde{\omega}_0}{24}.$$

If we put $\alpha(s) := x^3$ and $h(x) := x^2$, then we find

$$\int_{x_0}^{\infty} \left(E_1(s) - \frac{2^{r_1}}{(r_1+1)^{r_1+1}} \frac{z(s)(\alpha'(s))^{r_1+1}}{\mu_1^{r_1} s^{2r_1} \alpha^{r_1}(s)} \right) \mathrm{d}s$$
$$= \int_{x_0}^{\infty} \left(\frac{\tilde{\omega}_0}{16s} - \frac{9}{2\mu_1 s} \right) \mathrm{d}s$$

and

$$\int_{x_0}^{\infty} \left(\Phi(s) - \frac{(h'(s))^2}{4h(s)} \right) \mathrm{d}s$$

$$= \int_{x_0}^{\infty} \left(\frac{\tilde{\omega}_0}{24} - 1 \right) \mathrm{d}s.$$

Thus,

$$\tilde{\omega}_0 > 72 \tag{27}$$

and

$$\tilde{\omega}_0 > 24. \tag{28}$$

From Corollary 2.8, equation (26) is oscillatory if $\tilde{\omega}_0 > 72$.

Example 2.10 Consider the equation

$$\left(x\left(\beta + \tilde{y}_0 \beta(\gamma x) \right)''' \right)' + \frac{\tilde{\omega}_0}{x^3} \beta(\eta x) = 0, \quad x \geq 1, \tag{29}$$

where $\tilde{y}_0 \in [0,1), \gamma, \eta \in (0,1)$, and $\tilde{\omega}_0 > 0$. Let $r_1 = r_2 = 1$, $z(x) = x$, $\tilde{y}(x) = \tilde{y}_0$, $\tilde{\theta}(x) = \gamma x$, $\theta(x) = \eta x$, and $\tilde{\omega}(x) = \tilde{\omega}_0 / x^3$. Hence, if we set $\alpha(s) := x^2$ and $h(x) := x$, then we get

$$E_1(x) = \frac{\tilde{\omega}_0 (1 - \tilde{y}_0) \eta^3}{x}, \qquad \Phi(x) = \frac{\tilde{\omega}_0 (1 - \tilde{y}_0) \eta}{4x}.$$

Thus, (24) and (25) become

$$\int_{x_0}^{\infty} \left(E_1(s) - \frac{2^{r_1}}{(r_1+1)^{r_1+1}} \frac{z(s)(\alpha'(s))^{r_1+1}}{\mu_1^{r_1} s^{2r_1} \alpha^{r_1}(s)} \right) \mathrm{d}s$$
$$= \int_{x_0}^{\infty} \left(\frac{\tilde{\omega}_0 (1 - \tilde{y}_0) \eta^3}{s} - \frac{2}{\mu_1 s} \right) \mathrm{d}s$$

and

$$\int_{x_0}^{\infty} \left(\Phi(s) - \frac{(h'(s))^2}{4h(s)} \right) \mathrm{d}s$$
$$= \int_{x_0}^{\infty} \left(\frac{\tilde{\omega}_0 (1 - \tilde{y}_0) \eta}{4s} - \frac{1}{4s} \right) \mathrm{d}s.$$

So,

$$\tilde{\omega}_0 > \frac{2}{(1 - \tilde{y}_0) \eta^3} \tag{30}$$

and

$$\tilde{\omega}_0 > \frac{1}{(1 - \tilde{y}_0) \eta}.$$

From Corollary 2.8, equation (26) is oscillatory if (30) holds.

3 Conclusion

In this work, we proved some new oscillation theorems for (1). New oscillation results are established that complement related contributions to the subject. We used the Riccati technique and the integral averages technique to get some new results to oscillation of equation (1) under the condition $\int_{x_0}^{\infty} \frac{1}{z^{1/r_1}(s)}\,\mathrm{d}s = \infty$. We may say that, in future work, we will study this type of equation under the condition

$$\int_{x_0}^{\infty} \frac{1}{z^{1/r_1}(s)}\,\mathrm{d}s < \infty.$$

Also we will try to introduce some important oscillation criteria of differential equations of fourth-order and under

$$\delta(x) = \beta(x) + \tilde{y}(x)\sum_{i=1}^{j} \beta_i\bigl(\tilde{\theta}(x)\bigr).$$

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