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Periodic solutions for second-order difference equations with quadratic–supquadratic condition

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Abstract

In this paper, we consider the existence of multiple periodic solutions for a class of second-order difference equations with quadratic–supquadratic growth condition at infinity. Moreover, we give three examples to illustrate our main result.

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1 Introduction and main result

Discrete equations have been widely employed as mathematical models depicting the nature phenomena in many practical problems including computer sciences, life sciences, mathematical biology, and so on; see [1–5]. Among these discrete equations, discrete nonlinear Schrödinger (DNLS) equations are very important nonlinear lattice models in the nonlinear science, ranging from condensed matter physics to biology [6–11]. Let $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$ stand for the sets of all natural numbers, integers, and real numbers, respectively. For $c,d \in \mathbb{Z}$ with $c \leq d$, we define $\mathbb{Z}[c] = \{c, c+1, \ldots\}$ and $\mathbb{Z}[c,d] = \{c, c+1, \ldots, d\}$. It is well known that by the standing wave assumptions DNLS equations can change into the following nonlinear second-order difference equation [12–18]:

$$\begin{cases} \Delta^2 x_{n-1} + f(n, x_n) = 0, & \forall n \in \mathbb{Z}, \\ f(n, x_n) = \partial_{x_n} F(n, x_n), & \forall n \in \mathbb{Z}, \end{cases} \tag{1.1}$$

where $\Delta x_{n-1} = x_n - x_{n-1}$, $\Delta^2 = \Delta(\Delta)$, $F \in \mathcal{C}^1(\mathbb{R} \times \mathbb{R})$, and $F(n+M, \cdot) = F(n, \cdot)$ for some $M \in \mathbb{N}$.

As is known, the critical-point theory is an important tool when dealing with the existence of solutions of differential equations (see [19]), and for discrete system (1.1), there are some results on the existence of periodic solutions in the last few years: especially, for

$F(n,X)$ with supquadratic growth condition with respect to X at infinity,

$$\lim_{|X|\to+\infty} \frac{F(n,X)}{|X|^2} = +\infty, \tag{1.2}$$

Guo and Yu [18] developed a new approach to obtain the existence and multiplicity of periodic solutions to discrete system (1.1). Later, for $F(n,X)$ with subquadratic growth condition with respect to X at infinity, Guo and Yu [20] proved the existence of nontrivial periodic solutions. For the case of $F(n,X)$ with quadratic–supquadratic growth condition in X at infinity,

$$\liminf_{|X|\to+\infty} \frac{|F(n,X)|}{|X|^2} = \kappa > 0,$$

in 2004, under the assumption that κ depends on M (especially, $\kappa(M) > 2$ for even M), Zhou, Yu, and Guo [21] improved the Guo–Yu method of [18] and obtained the existence of two nontrivial M-periodic solutions for discrete system (1.1); for more details on the existence of multiple nontrivial M-periodic solutions with quadratic–supquadratic condition, we refer to [22, 23]. Moreover, the existence of one nontrivial solution for general nonlinear difference equations, that is, discrete ϕ-Laplacian equations with quadratic–supquadratic condition, is considered in [13, 15, 24]. For other related works, we refer to [14, 16, 17, 25–31].

Note that in [21], for F with quadratic–supquadratic condition with respect to X at infinity, by introducing the smallest and largest eigenvalues of the matrix

$$\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -1 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}_{M\times M}, \tag{1.3}$$

that is,

$$\lambda_{\min} = 2\left(1 - \cos\frac{2\pi}{M}\right), \qquad \lambda_{\max} = \begin{cases} 4 & \text{when } M \text{ is even}, \\ 2(1+\cos\frac{\pi}{M}) & \text{when } M \text{ is odd}, \end{cases} \tag{1.4}$$

Zhou, Yu, and Guo obtained the existence of two nontrivial M-periodic solutions for system (1.1). Four years later, Xue and Tang obtained the following more general result for F with quadratic–supquadratic condition with respect to X at infinity.

Theorem 1.1 ([23, Theorem 2]) *Suppose that $F(n,X)$ satisfies*

(F_1) *There are constants $\delta > 0$ and $k \in [0, [\frac{M}{2}] - 1] \cap \mathbb{Z}$ such that for any $|X| \le \delta$ and $t \in \mathbb{Z}[1,M]$,*

$$\frac{1}{2}\mu_k|X|^2 \le F(n,X) \le \frac{1}{2}\mu_{k+1}|X|^2,$$

where $\mu_k = 2 - 2\cos kw$, $w = \frac{2\pi}{M}$, $M > 2$, and $[\cdot]$ denotes the Gauss function;

(F_2) *For* $t \in \mathbb{Z}[1, M]$, *there exists a constant* $\beta \in (\frac{\lambda_{\max}}{2}, +\infty)$ *such that*

$$\lim_{|X|\to+\infty} \inf \frac{F(n,X)}{|X|^2} \geq \beta.$$

Then system (1.1) *has at least two nontrivial M-periodic solutions.*

In condition (F_2), $\lambda_{\max}$ is dependent on M; when M is even, $\beta > 2$, and when $M \geq 3$ is odd, $\beta \geq 1 + \cos\pi/M \geq 3/2$. Therefore the constant β is at least greater than $3/2$. Moreover, in condition (F_1), if $k = 0$, then when $M = 4$, $0 \leq F(n,X) \leq 1 - \cos(2\pi/4) = 1$, and when $M > 4$, $0 \leq F(n,X) \leq 1 - \cos(2\pi/M) < 1$. Clearly, the range of parameter values in (F_1) and (F_2) will play a critical role in discrete model (1.1) when proving the existence of periodic solutions. However, all parameters in Theorem 1.1 are limited. In this paper, we want to establish an existence result for periodic solutions without this limitation. The main result of this paper is the following:

Theorem 1.2 *Let* $M \geq 4$, *and let* $F(n,X)$ *satisfy the following conditions*:

(F_1') *There exist constants* $\delta > 0$ *and* $0 < \alpha < 1$ *such that*

$$0 \leq F(n,X) \leq \alpha|X|^2 \quad \text{for } n \in \mathbb{N}, X \in \mathbb{R} \text{ and } |X| \leq \delta.$$

(F_2') *For* $t \in \mathbb{Z}[1, M]$, *there exists a constant* $\beta \in (1, +\infty)$ *such that*

$$\lim_{|X|\to+\infty} \inf \frac{F(n,X)}{|X|^2} \geq \beta.$$

Then system (1.1) *has at least two nontrivial M-periodic solutions.*

Remark 1.1 (i) Since $M \geq 4$, the parameter β in Theorem 1.1 can only take values in $(1 + \cos(\pi/5), +\infty)$, but it can take any value in $(1, +\infty)$ in Theorem 1.2. Moreover, if $M > 4$ and $k = 0$, then the parameter α in Theorem 1.1 can only take values in $(0, 1 - \cos(2\pi/M))$, but in the present paper, we prove that this parameter can take any value in $(0, 1)$. In this sense, we extend the ranges of parameters.

(ii) In Theorem 1.1, if $k \neq 0$, then $F(n,X) = O(|X|^2)$ as $|X| \to 0$, but in Theorem 1.2, as $|X| \to 0$, both $F(n,X) = O(|X|^2)$ and $F(n,X) = o(|X|^2)$ are admissible.

Remark 1.2 (i) In 2020, under the assumptions that $M \geq 5$ and $F(n,X) \to -\infty$ as $|X| \to +\infty$, by using an extended mountain pass theorem, we obtained the existence of two nontrivial M-periodic solutions for quadratic–supquadratic vector field $F(n,X)$ (see [22, Theorem 1.1]) in X at infinity. Different from the method in [22, Theorem 1.1], now by constructing a new functional $J_1(x)$, two new orthogonal direct sum decompositions, and Linking theorem [19, Theorem 5.3], under the assumptions that $M \geq 4$ and $F(n,X) \to +\infty$ as $|X| \to +\infty$, we also obtain the existence of two nontrivial M-periodic solutions for quadratic–supquadratic condition in X at infinity.

(ii) The method improved here may be applied to the general difference equations [13, 15, 21, 24, 28, 29], and under general quadratic–supquadratic growth conditions at infinity, we may also obtain the existence of multiple periodic solutions.

Now we give three examples to explain Theorem 1.2. First, we give an example for completely quadratic condition with respect to X at infinity.

Example 1 Let F be given by

$$F(n,X) = a_1|X|^2\bigl(\phi(n) + D\bigr),$$

where a_1 is an arbitrary constant that belongs to $(0,1)$, a constant $D > 0$, and $\phi(n)$ is a continuous M-periodic function satisfying $|\phi(n)| < D$ for every n. Now $F(n,X)$ satisfies all assumptions in Theorem 1.2. Thus (1.1) has at least two nontrivial M-periodic solutions.

Second, we give an example for quadratic–supquadratic condition with respect to X at infinity.

Example 2 Let F be given by

$$F(n,X) = \bigl(a_4|X|^2 - a_5|X|^4 + a_6|X|^6\bigr)\bigl(\phi(n) + D\bigr),$$

where a_4, a_5, a_6 are arbitrary constants that belong to $(0,1)$, a constant $D > 0$, and $\phi(n)$ is as in Example 1. Then $F(n,X)$ satisfies all assumptions in Theorem 1.2. Thus (1.1) has at least two nontrivial M-periodic solutions.

Remark 1.3 Since the constants a_1 and a_4 in Examples 1 and 2 are arbitrary, these examples cannot be solved by Theorem 1.1.

Finally, we give an example for completely supquadratic condition with respect to X at infinity.

Example 3 Let F be given by

$$F(n,X) = \bigl(a_2|X|^4 + a_3|X|^6\bigr)\bigl(\phi(n) + D\bigr),$$

where a_2 and a_3 are arbitrary constants that belong to $(0,1)$, a constant $D > 0$, and $\phi(n)$ is as in Example 1. Then $F(n,X)$ satsifies all assumptions in Theorem 1.2. Thus (1.1) has at least two nontrivial M-periodic solutions.

2 Some useful lemmas

To use the critical point theory to study the existence of periodic solutions to (1.1), we introduce some notions and notations.

- Let S be the set of sequences, that is, $S = \{s = \{s_j\} = (\dots, s_{-j}, \dots, s_0, \dots, s_j, \dots), s_j \in \mathbb{R}, j \in \mathbb{Z}\}$. When $x, y \in S$ and $a, b \in \mathbb{R}$, $ax + by$ is given by $ax + by = \{ax_j + by_j\}_{j=-\infty}^{+\infty}$. For any given positive integer M, E_M is a subspace of S defined as

$$E_M = \bigl\{x = \{x_j\} \in S \mid x_{j+M} = x_j, j \in \mathbb{Z}\bigr\}.$$

Then, with the common Euclid inner product ($\|x\| = (\sum_{j=1}^{M}(x_j)^2)^{\frac{1}{2}}$), E_M is an M-dimensional Hilbert space. Let

$$\|x\|_\alpha = \left(\sum_{k=1}^{M}|x_k|^\alpha\right)^{\frac{1}{\alpha}}, \quad \alpha \in (1, \infty).$$

Then

$$\frac{1}{M}\|x\|_4 \le \|x\| \le M\|x\|_4, \qquad \frac{1}{M}\|x\|_{\frac{3}{2}} \le \|x\| \le M\|x\|_{\frac{3}{2}}, \quad \forall x \in E_M.$$

- Let H be a real Hilbert space. $J \in \mathcal{C}^1(H)$ is said to satisfy the PS condition if any sequence $\{x^{(j)}\} \subset H$ for which $\{J(x^{(j)})\}$ is bounded and $J'(x^{(j)}) \to 0$ as $j \to \infty$ possesses a convergent subsequence in H.
- Different from the known literature used to study the existence of periodic solutions for discrete system (1.1), the result of this paper is not related to the complicated smallest and largest eigenvalues of matrix (1.3), that is, $\lambda_{\min}$ and $\lambda_{\max}$ in (1.4), and now for any $x \in E_M$ and $\Delta x = (\Delta x_1, \Delta x_2, \dots, \Delta x_M)^\top \in \mathbb{R}^M$, we let

$$\sum_{j=1}^{M}(\Delta x_j)^2 + 2\Delta x_{n-1}\Delta x_n = (\Delta x_1, \Delta x_2, \dots, \Delta x_M)^\top P(\Delta x_1, \Delta x_2, \dots, \Delta x_M),$$

where

$$P = \begin{pmatrix}
1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
\vdots & \cdots & \ddots & \ddots & 0 & 0 & \cdots & \cdots & \cdots & \vdots \\
\vdots & \cdots & \cdots & 0 & 1 & 1 & 0 & \cdots & \cdots & \vdots \\
\vdots & \cdots & \cdots & 0 & 1 & 1 & 0 & \cdots & \cdots & \vdots \\
\vdots & \cdots & \cdots & \vdots & 0 & 0 & \ddots & 0 & \cdots & \vdots \\
\vdots & \cdots & \cdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1
\end{pmatrix}_{M\times M}.$$

Remark 2.1 Obviously, the matrix P is different from (1.3), and it is easy to get that the eigenvalues of P are $\underbrace{1, \dots, 1}_{M-2}$, 0, 2. Besides, we compute that the matrix P has M linearly independent eigenvectors.

- To obtain the general existence of periodic solutions for second-order difference equation (1.1) with new quadratic–supquadratic condition, we also let $H_1 = \{(x_1, x_2, \dots, x_M)^\top \in E_M \mid \Delta x_1 = \cdots = \Delta x_{n-2} = \Delta x_{n+1} = \cdots = \Delta x_M = 0, \Delta x_{n-1} = -\Delta x_n = -w \in \mathbb{R}, M \ge 4\}$, and $H_2 = H_1^\perp$. Then we construct the first new orthogonal direct sum decomposition, $E_M = H_1 \oplus H_2$.

Remark 2.2 Most literatures used the method in [18, 21] and H_1 is trivial, that is, $H_1 = \{\underbrace{(v,v,\dots,v)}_{M}^{\top} \mid v \in \mathbb{R}\}$, but in this paper, when $M \geq 4$, H_1 is nontrivial, and $\{\underbrace{(v,v,\dots,v)}_{M}^{\top} \mid v \in \mathbb{R}\} \subseteq H_1$, but $\{\underbrace{(v,v,\dots,v)}_{M}^{\top} \mid v \in \mathbb{R}\} \neq H_1$.

Let $M \geq 4$ be a integer, and let $n \in \mathbb{Z}[1,M]$. To prove Theorem 1.2, we define the new functional $J_1(x)$, related to the matrix P, on E_M as follows:

$$\begin{aligned} J_1(x) &= \frac{1}{2}\|\Delta x\|^2 - \sum_{j=1}^{M} F(j,x_j) - M^4\Biggl[\sum_{j\neq n}|x_j|^4 + \frac{1}{16}|x_{n+1}+x_{n-1}|^4\Biggr] \\ &\quad + M^{-\frac{3}{2}}\Biggl[\sum_{j\neq n}|x_j|^{\frac{3}{2}} + \frac{1}{2\sqrt{2}}|x_{n+1}+x_{n-1}|^{\frac{3}{2}}\Biggr] - \frac{1}{2}(\Delta x)^{\top}P(\Delta x). \end{aligned} \tag{2.1}$$

Remark 2.3 We have the following identity:

$$\begin{aligned} \frac{\partial[(\Delta x)^{\top}P(\Delta x)]}{\partial x_n} &= \frac{\partial[\sum_{j\neq n}|x_j|^4 + \frac{1}{16}|x_{n+1}+x_{n-1}|^4]}{\partial x_n} \\ &= \frac{\partial[\sum_{j\neq n}|x_j|^{\frac{3}{2}} + \frac{1}{2\sqrt{2}}|x_{n+1}+x_{n-1}|^{\frac{3}{2}}]}{\partial x_n} = 0. \end{aligned}$$

Clearly, $J_1 \in \mathcal{C}^1(E_M)$. For any $x = \{x_j\}_{j\in\mathbb{Z}} \in E_M$, according to Remark 2.3, we compute that

$$\frac{\partial J_1}{\partial x_n} = -\bigl[\Delta^2 x_{n-1} + f(n,x_n)\bigr], \quad \forall n \in \mathbb{Z}[1,M].$$

So, the existence of critical points of J_1 on E_M may imply the existence of periodic solutions of system (1.1).

We now give some useful lemmas, which will serve us later.

Lemma 2.1 *Let* (F_2') *be valid. Then there exists a constant* $\gamma' \in \mathbb{R}$ *such that* $F(n,X) \geq \beta|X|^2 - \gamma'$ *for* $n \in \mathbb{Z}$, $X \in \mathbb{R}$.

Proof By (F_2'), for $\beta > 1$, there exist constants $\rho > 0$ and $\gamma > 0$ such that

$$F(n,X) \geq \beta|X|^2 - \gamma \quad \text{for } n \in \mathbb{Z}, |X| \geq \rho.$$

Then letting $\gamma_1 = \max\{|F(n,X) - \beta|X|^2 + \gamma| : n \in \mathbb{Z}, |X| \leq \rho\}$ and $\gamma' = \max\{\gamma, \gamma_1 + \gamma\}$, we obtain

$$F(n,X) \geq \beta|X|^2 - \gamma' \quad \text{for } n \in \mathbb{Z}, X \in \mathbb{R}.$$

The proof is complete. □

Lemma 2.2 *Let* (F_2') *be in force. Then* $J_1(x)$ *is bounded from above on* E_M.

Proof Let J_1 be given by (2.1). By Lemma 2.1, for all $x \in E_M$, we have

$$\begin{aligned} J_1(x) &\le \frac{1}{2}\|\Delta x\|^2 - \sum_{j=1}^{M}\bigl(\beta(x_j)^2 - \gamma'\bigr) - M^4\left[\sum_{j\neq n}|x_j|^4 + \frac{1}{16}|x_{n+1}+x_{n-1}|^4\right] \\ &\quad + M^{-\frac{3}{2}}\left[\sum_{j\neq n}|x_j|^{\frac{3}{2}} + \frac{1}{2\sqrt{2}}|x_{n+1}+x_{n-1}|^{\frac{3}{2}}\right] - \frac{1}{2}(\Delta x)^\top P(\Delta x) \\ &= \frac{1}{2}\|\Delta x\|^2 - \beta\|x\|^2 - M^4\left[\sum_{j\neq n}|x_j|^4 + \frac{1}{16}|x_{n+1}+x_{n-1}|^4\right] \\ &\quad + M^{-\frac{3}{2}}\left[\sum_{j\neq n}|x_j|^{\frac{3}{2}} + \frac{1}{2\sqrt{2}}|x_{n+1}+x_{n-1}|^{\frac{3}{2}}\right] + M\gamma' \\ &\quad - \frac{1}{2}(\Delta x)^\top P(\Delta x). \end{aligned} \tag{2.2}$$

The eigenvalues of P are $\underbrace{1,\dots,1}_{M-2}$, 0, 2, and the matrix P has M linearly independent eigenvectors. So we can construct the second new orthogonal direct sum decomposition $\mathbb{R}^M = L_0 \oplus L_1 \oplus L_2$, where

$$L_0 = \operatorname{span}\bigl\{\Delta x \in \mathbb{R}^M \mid P\Delta x = 0\bigr\}, \qquad L_1 = \operatorname{span}\bigl\{\Delta x \in \mathbb{R}^M \mid P\Delta x = \Delta x\bigr\},$$

and

$$L_2 = \operatorname{span}\bigl\{\Delta x \in \mathbb{R}^M \mid P\Delta x = 2\Delta x\bigr\}.$$

For difference cases, we have the following discussions.

- Case 1: $\Delta x \in L_0$. In this case, $(\Delta x)^\top P(\Delta x) = 0$ and $\Delta x_{n-1} = -\Delta x_n$. Then $x_1 = \dots = x_{n-1} = x_{n+1} = \dots = x_M$. Thus by (2.2) we have

$$\begin{aligned} J_1(x) &\le x_n^2 + x_{n-1}^2 + 2x_{n-1}x_n - \beta\|x\|^2 - M^4(M-1)\bigl(|x_1|^4 - |x_1|^{\frac{3}{2}}\bigr) \\ &\quad + M\gamma'. \end{aligned} \tag{2.3}$$

By the Cauchy–Schwarz inequality, for $\beta > 1$, from (2.3) we have

$$\begin{aligned} J_1(x) &\le x_n^2 + x_1^2 + \frac{2}{\beta-1}x_1^2 + \frac{\beta-1}{2}x_n^2 - \beta\|x\|^2 - M^4(M-1)\bigl[|x_1|^4 \\ &\quad - |x_1|^{\frac{3}{2}}\bigr] + M\gamma' \\ &\le \frac{\beta+1}{\beta-1}x_1^2 - \frac{\beta-1}{2}\|x\|^2 - M^4(M-1)\bigl[|x_1|^4 - |x_1|^{\frac{3}{2}}\bigr] + M\gamma'. \end{aligned} \tag{2.4}$$

Since

$$\lim_{x_1\to\infty}\left[\frac{\beta+1}{\beta-1}x_1^2 - M^4(M-1)|x_1|^4 + M^4(M-1)|x_1|^{\frac{3}{2}}\right] = -\infty,$$

there exists a constant M_1 such that $J_1(x) \le M_1$. So, when the eigenvalue of P is 0, the functional $J_1(x)$ is bounded from above on E_M.

• Case 2: $\Delta x \in L_2$.

Now $P(\Delta x) = \Delta x$ and $\Delta x_{n-1} = \Delta x_n = 0$. Therefore $x_n = x_{n+1} = x_{n-1}$, and

$$\begin{aligned} J_1(x) &\leq -\beta \|x\|^2 - M^4 \|x\|_4^4 + M^{-\frac{3}{2}} \|x\|_{\frac{3}{2}}^{\frac{3}{2}} + M\gamma' \\ &\leq -\beta \|x\|^2 - \|x\|^4 + \|x\|^{\frac{3}{2}} + M\gamma'. \end{aligned} \tag{2.5}$$

So there exists a constant M_2 such that $J_1(x) \leq M_2$.

• Case 3: $\Delta x \in L_3$.

Now $P(\Delta x) = 2\Delta x$, so $\Delta x_1 = \ldots = \Delta x_{n-2} = \Delta x_{n+2} = \ldots = \Delta x_M = 0$, $\Delta x_{n-1} = \Delta x_n$. Thus $2x_n = x_{n+1} + x_{n-1}$, and

$$J_1(x) \leq -\frac{1}{2} \|\Delta x\|^2 - \beta \|x\|^2 - \|x\|^4 + \|x\|^{\frac{3}{2}} + N\gamma' \leq M_2. \tag{2.6}$$

Take $M_3 = \max\{M_1, M_2\}$. Then by Cases 1–3 we get $J_1(x) \leq M_3$. Now the proof of Lemma 2.2 is complete. □

Lemma 2.3 *Let hypothesis* (F_2') *be in force. Then* J_1 *satisfies the PS condition.*

Proof Let $\{J_1(x^{(j)})\}$ be a bounded sequence from bellow, that is, there exists a positive constant M_4 such that

$$J_1\bigl(x^{(j)}\bigr) \geq -M_4, \quad \forall j \in N.$$

From (2.4)–(2.6) we have the following inequality:

$$J_1\bigl(x^{(j)}\bigr) \leq \begin{cases} \frac{\beta+1}{\beta-1} |x_1^{(j)}|^2 - \frac{\beta-1}{2} \|x^{(j)}\|^2 - M^4(M-1)[|x_1^{(j)}|^4 - |x_1^{(j)}|^{\frac{3}{2}}] & \\ \quad + M\gamma' & \text{when } \Delta x^{(j)} \in L_0, \\ -\beta \|x^{(j)}\|^2 - \|x^{(j)}\|^4 + \|x^{(j)}\|^{\frac{3}{2}} + M\gamma' & \text{otherwise,} \end{cases}$$

which implies

$$\begin{cases} \frac{\beta-1}{2} \|x^{(j)}\|^2 + M^4(M-1)[|x_1^{(j)}|^4 - |x_1^{(j)}|^{\frac{3}{2}}] - \frac{\beta+1}{\beta-1} |x_1^{(j)}|^2 & \\ \quad \leq M\gamma' + M_4 & \text{when } \Delta x^{(j)} \in L_0, \\ \beta \|x^{(j)}\|^2 + \|x^{(j)}\|^4 - \|x^{(j)}\|^{\frac{3}{2}} \leq M\gamma' + M_4 & \text{otherwise.} \end{cases} \tag{2.7}$$

From (2.7) it is not difficult to deduce that there exists a constant M_5 such that $\|x^{(j)}\| \leq M_5$, that is, $\{x^{(j)}\}$ is bounded in E_M. Since E_M is finite-dimensional, there exists a subsequence of $\{x^{(j)}\}$ (not labeled) convergent in E_M, so the PS condition is satisfied. □

Lemma 2.4 ([19, Theorem 5.3]; Linking theorem) *Let H be a real Hilbert space, $H = H_1 \oplus H_2$, where H_2 is a finite-dimensional subspace of H. Assume that $J \in \mathcal{C}^1(H)$ satisfies the PS condition and*

(A_1) *there exist constants $\sigma > 0$ and $\rho > 0$ such that $J|_{\partial B_\rho \cap H_1} \geq \sigma$;*

(A_2) *there are $e \in \partial B_1 \cap H_1$ and a constant $R_1 > \rho$ such that $J|_{\partial Q} \leq 0$,* *where* $Q = (\overline{B}_{R_1} \cap H_2) \oplus \{re \mid 0 < r < R_1\}$, B_ρ *denotes the open ball in* H *with radius* ρ *and centered at* 0, *and* ∂B_ρ *is its boundary. Then* J *possesses a critical value* $c \geq \sigma$, *where*

$$c = \inf_{h\in\Gamma} \max_{u\in Q} J\bigl(h(u)\bigr), \qquad \Gamma = \bigl\{h \in C(\overline{Q}, H) \mid h|_{\partial Q} = id\bigr\},$$

and id denotes the identity operator.

3 Proof of Theorem 1.2

Based upon Lemmas 2.2–2.4, we divide the proof into four steps.

Step 1. We show that (A_1) in the Linking theorem holds.

When $x \in H_1$ and $\Delta x \in L_0$, then by (F_1')

$$\begin{aligned} J_1(x) &\geq \frac{1}{2}\|\Delta x\|^2 - \alpha\|x\|^2 - M^4\Biggl[\sum_{j\neq n}|x_j|^4 + \frac{1}{16}|x_{n+1}+x_{n-1}|^4\Biggr] \\ &\quad + M^{-\frac{3}{2}}\Biggl[\sum_{j\neq n}|x_j|^{\frac{3}{2}} + \frac{1}{2\sqrt{2}}|x_{n+1}+x_{n-1}|^{\frac{3}{2}}\Biggr] \\ &= \bigl(x_n^2 + x_{n-1}^2 + 2x_{n-1}x_n\bigr) - \alpha\|x\|^2 - M^4\Biggl[\sum_{j\neq n}|x_j|^4 + \frac{1}{16}|x_{n+1} \\ &\quad + x_{n-1}|^4\Biggr] + M^{-\frac{3}{2}}\Biggl[\sum_{j\neq n}|x_j|^{\frac{3}{2}} + \frac{1}{2\sqrt{2}}|x_{n+1}+x_{n-1}|^{\frac{3}{2}}\Biggr]. \end{aligned} \tag{3.1}$$

Using the Cauchy–Schwarz inequality $2x_{n-1}x_n \leq \frac{1-\alpha}{2}x_{n-1}^2 + \frac{2}{1-\alpha}x_n^2$ and $0 < \alpha < 1$, from (3.1) we arrive at

$$J_1(x) \geq \frac{1-\alpha}{2}x_n^2 - \frac{1+\alpha}{1-\alpha}x_1^2 - (M-1)\alpha x_1^2 - M^5x_1^4 + x_1^{\frac{3}{2}}/\sqrt{M}. \tag{3.2}$$

On the other hand, when $\|x\| \leq \delta$, $(\sum_{j\neq n} x_j^2)^{1/2} \leq \delta$, so if we choose δ sufficiently small, then from (3.2) we conclude

$$\begin{aligned} J_1(x) &\geq \frac{1-\alpha}{2}x_n^2 + \frac{1}{2\sqrt{M}}x_1^{\frac{3}{2}} \\ &\geq \frac{1-\alpha}{2}x_n^2 + \frac{(1-\alpha)(M-1)}{2}x_1^2 = (1-\alpha)\|x\|^2. \end{aligned} \tag{3.3}$$

Taking $\sigma = (1-\alpha)\delta^2$, we have

$$J_1(x) \geq \sigma > 0, \quad \forall x \in H_1 \cap \partial B_\delta. \tag{3.4}$$

Thus condition (A_1) in Lemma 2.4 is satisfied.

Step 2. We show that (A_2) holds.

By Lemma 2.3, $J_1(x)$ meets the PS condition. Taking $e \in \partial B_1 \cap H_1$, for any $z \in H_2$, $r \in \mathbb{R}$. Let $x = re + z$. By (2.4)–(2.6)

$$J_1(x) \leq \begin{cases} \frac{\beta+1}{\beta-1}|x_1|^2 - \frac{\beta-1}{2}\|x\|^2 - M^4(M-1)[|x_1|^4 - |x_1|^{\frac{3}{2}}] + M\gamma' & \text{when } x \in H_1, \\ -\beta\|x\|^2 - \|x\|^4 + \|x\|^{\frac{3}{2}} + M\gamma' & \text{when } x \in H_2. \end{cases} \tag{3.5}$$

Observing that as $\|x\| \to \infty$, the right-hands in (3.5) are approaching to negative infinity, and thus there exists a big enough constant $R_2 > 0$ such that $J_1(x) \leq 0$ for all $x \in \partial Q$, where

$$Q = (\overline{B}_{R_2} \cap H_2) \oplus \{re \mid 0 < r < R_2\}.$$

Step 3. Existence of the first nontrivial N-periodic solution.

By Linking theorem (Lemma 2.4), J_1 has a critical value $c \geq \sigma > 0$, where

$$c = \inf_{h\in\Gamma} \max_{x\in Q} J_1\bigl(h(x)\bigr), \qquad \Gamma = \bigl\{h \in C(\overline{Q}, E_M) | h|_{\partial Q} = id\bigr\}.$$

Step 4. Existence of the second nontrivial N-periodic solution.

Inequalities (2.4)–(2.6) imply $\lim_{\|x\|\to\infty} J_1(x) = -\infty$. Let $c_0 = \sup_{x\in E_M} J_1(x)$. By the continuity of J_1 on E_M there exists $\bar{x} \in E_M$ such that $J_1(\bar{x}) = c_0$ and $\bar{x}$ is a critical point of J_1. Combining (3.4), we have $J_1(\bar{x}) = c_0 > 0$. Note that when $x_1 = \cdots = x_M = 0$, by (2.1) and condition (F_1') we get $J_1(x) = -\sum_{j=1}^{M} F(j, x_j) \leq 0$. So the critical point associated with the critical value c_0 of J_1, is a nontrivial M-periodic solution of system (1.1).

The rest of the proof of the other nontrivial M-periodic solution is similar to that of [18, Theorem 1.1], and we omit it.

By now the proof of Theorem 1.2 is complete, which means that discrete system (1.1) has at least two nontrivial M-periodic solutions.

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