Chen et al. *Advances in Difference Equations* (2021) 2021:431
https://doi.org/10.1186/s13662-021-03579-3

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH **Open Access**

Growth and fixed points of solutions and their arbitrary-order derivatives of higher-order linear differential equations in the unit disc

Yu Chen[1,2*], Guan-Tie Deng[1], Zhan-Mei Chen[3] and Wei-Wei Wang[1]

*Correspondence:
chenyugyi@sina.com
[1]School of Mathematical Sciences, Laboratory of Mathematics and Complex Systems of Ministry of Education, Beijing Normal University, Beijing, China
[2]School of Mathematics and Statistics, Jiangxi Normal University, Nanchang, China
Full list of author information is available at the end of the article

Abstract

In this paper, we investigate the growth and fixed points of solutions of higher-order linear differential equations in the unit disc. We extend the coefficient conditions to a type of one-constant-control coefficient comparison and obtain the same estimates of iterated order of solutions. We also obtain better estimates by providing a precise value of iterated order of solution instead of a range of that in the case of coefficient characteristic function comparison. Moreover, we utilize iteration to investigate and estimate the fixed points of solutions' arbitrary-order derivatives with higher-order equations $f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0$ and provide a concise method to judge if the items generated by the iteration do not vanish identically and ensure the iteration proceeds. Our results are an improvement over those by B. Belaïdi, T. B. Cao, G. W. Zhang and A. Chen.

MSC: 34M10; 30D35

Keywords: Linear differential equation; Unit disc; Iterated order; Fixed points; Analytic function

1 Introduction and main results

In this paper, we assume that the reader is familiar with the fundamental results and the standard notations of Nevanlinna's value distribution theory in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ (see [1–4]). As for the definition of the iterated order of meromorphic function, we know that for $r \in [0,1)$, $\exp_1 r = e^r$ and $\exp_{n+1} r = \exp(\exp_n r), n \in \mathbb{N}$, and for all r sufficiently large in $(0,1)$, $\log_1 r = \log r$ and $\log_{n+1} r = \log(\log_n r), n \in \mathbb{N}$. Moreover, we denote $\exp_0 r = r, \log_0 r = r, \exp_{-1} r = \log_1 r, \log_{-1} r = \exp_1 r$. Then, let us recall the following definitions for $n \in \mathbb{N}$.

Definition 1.1 (see [5]) Let f be a meromorphic function in $\mathbb{D}$. Then the iterated n-order of f is defined by

$$\sigma_n(f) = \varlimsup_{r\to 1^-} \frac{\log_n^+ T(r,f)}{-\log(1-r)},$$

where $\log_1^+ x = \log^+ x = \max\{\log x, 0\}, \log_{n+1}^+ x = \log^+(\log_n^+ x)$. For $n = 1$, $\sigma_1(f) = \sigma(f)$.

If f is analytic in $\mathbb{D}$, then the iterated n-order is defined by

$$\sigma_{M,n}(f) = \overline{\lim_{r\to 1^-}} \frac{\log_{n+1}^+ M(r,f)}{-\log(1-r)}.$$

For $n = 1$, $\sigma_{M,1}(f) = \sigma_M(f)$.

Remark 1.1 (see [5, 6]) It follows by M. Tsuji that if f is an analytic function in $\mathbb{D}$, then

$$\sigma_1(f) \le \sigma_{M,1}(f) \le \sigma_1(f) + 1,$$

which is best possible in the sense that there are analytic functions g and h such that $\sigma_{M,1}(g) = \sigma_1(g)$ and $\sigma_{M,1}(h) = \sigma_1(h) + 1$, see [7]. However, it follows by Proposition 2.2.2 in [3] that $\sigma_{M,n}(f) = \sigma_n(f)$ for $n \ge 2$.

Definition 1.2 (see [5]) Let f be a meromorphic function in $\mathbb{D}$. Then the iterated n-convergence exponent of the sequence of zeros in $\mathbb{D}$ of $f(z)$ is defined by

$$\lambda_n(f) = \overline{\lim_{r\to 1^-}} \frac{\log_n^+ N(r,\frac{1}{f})}{-\log(1-r)},$$

where $N(r,\frac{1}{f})$ is the integrated counting function of zeros of $f(z)$.

Similarly, the iterated n-convergence exponent of the sequence of distinct zeros in $\mathbb{D}$ of $f(z)$ is defined by

$$\bar{\lambda}_n(f) = \overline{\lim_{r\to 1^-}} \frac{\log_n^+ \overline{N}(r,\frac{1}{f})}{-\log(1-r)},$$

where $\overline{N}(r,\frac{1}{f})$ is the integrated counting function of distinct zeros of $f(z)$.

Definition 1.3 Let f be a meromorphic function in $\mathbb{D}$. Then the iterated convergence n-exponent of the sequence of fixed points in $\mathbb{D}$ of $f(z)$ is defined by

$$\lambda_n(f-z) = \overline{\lim_{r\to 1^-}} \frac{\log\log N(r,\frac{1}{f-z})}{-\log(1-r)};$$

and the iterated n-convergence exponent of the sequence of distinct fixed points in $\mathbb{D}$ of $f(z)$ is defined by

$$\bar{\lambda}_n(f-z) = \overline{\lim_{r\to 1^-}} \frac{\log\log \overline{N}(r,\frac{1}{f-z})}{-\log(1-r)}.$$

Definition 1.4 (see [8]) For a measurable set $E \subset [0,1)$, the upper and lower densities are defined as

$$\overline{\text{dens}}_{\mathbb{D}} E = \overline{\lim_{r\to 1^-}} \frac{m(E\cap[r,1))}{1-r} \quad \text{and} \quad \underline{\text{dens}}_{\mathbb{D}} E = \underline{\lim_{r\to 1^-}} \frac{m(E\cap[r,1))}{1-r},$$

respectively, where $m(F)$ is the Lebesgue measure of F.

The next Remark 1.2 follows by Sect. 7 in [8] or can be seen in [9].

Remark 1.2 (see [8, 9]) If a set E satisfies $\overline{\mathrm{dens}}_{\mathbb{D}}E > 0$, then $\int_E \frac{dr}{1-r} = +\infty$.

Remark 1.3 Remark 1.2 also holds with the definitions of the upper densities in [5, 10]; therefore, the results of this paper are still valid for the definitions of the upper densities in [5, 10].

The theory of complex linear differential equations in the unit disc has been developed since 1980s. In recent years, after J. Heittokangas' work in [4], there has been an increasing interest in studying the complex oscillation of linear differential equations in the unit disc, and many important results in the unit disc $\mathbb{D}$ analogous to those on the complex plane $\mathbb{C}$ have been obtained. G. G. Gundersen [11] studied the growth of solutions of one type of second-order linear differential equations on $\mathbb{C}$ (see the following Theorem A). After that, K. H. Kwon [12], Z. X. Chen and C. C. Yang [13], B. Belaïdi [14, 15], and the first author [16], extended the type of the coefficients with less control constants, removed infinitesimals, and obtained some improved results. T. B. Cao and H. X. Yi in [10], and T. B. Cao in [5] generalized the results of [13] and [14] on $\mathbb{C}$ to corresponding results in $\mathbb{D}$, respectively. Later, some improvements and extensions of the type of coefficients in the unit disc were investigated (see [17, 18]) and gave rise to the unit disc analogues of the results of [14, 16].

Theorem A (see [11]) *Let $A(z)$ and $B(z) \not\equiv 0$ be entire functions, and let α, β, θ_1 and θ_2 be real numbers with $\alpha > 0, \beta > 0$ and $\theta_1 < \theta_2$. If*

$$\left|B(z)\right| \geq \exp\left\{\left(1 + o(1)\right)\alpha|z|^{\beta}\right\}$$

and

$$\left|A(z)\right| \leq \exp\left\{o(1)|z|^{\beta}\right\}$$

as $|z| \to \infty$ with $\theta_1 \leq \arg z \leq \theta_2$, then every solution $f \not\equiv 0$ of the equation

$$f'' + A(z)f' + B(z) = 0$$

has infinite order.

T. B. Cao in [5] investigated the iterated order of solutions of higher-order equations and obtained the unit disc analogues of B. Belaïdi's results in [14]. As for the equation

$$A_k(z)f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0, \tag{1.1}$$

he obtained the following results.

Theorem B (see [5]) *Let H be a set of complex numbers satisfying $\overline{dens}_{\mathbb{D}}\{|z| : z \in H \subseteq \mathbb{D}\} > 0$, and let $A_0, A_1, \ldots, A_k$ be analytic functions in $\mathbb{D}$ such that for some real constants*

$0 \le \beta < \alpha$ *and* $\mu > 0$ *we have*

$$|A_0(z)| \ge \exp_n\left\{\alpha\left(\frac{1}{1-|z|}\right)^{\mu}\right\}$$

and

$$|A_i(z)| \le \exp_n\left\{\beta\left(\frac{1}{1-|z|}\right)^{\mu}\right\}, \quad i = 1, 2, \ldots, k,$$

as $|z| \to 1^-$ *for* $z \in H$. *Then every meromorphic (or analytic) solution* $f \not\equiv 0$ *of equation* (1.1) *satisfies* $\sigma_n(f) = \infty$ *and* $\sigma_{n+1}(f) \ge \mu$.

Theorem C (see [5]) *Let* H *be a set of complex numbers satisfying* $\overline{\mathrm{dens}}_{\mathbb{D}}\{|z| : z \in H \subseteq \mathbb{D}\} > 0$, *and let* $A_0, A_1, \ldots, A_k$ *be analytic functions in* $\mathbb{D}$ *such that for some real constants* $0 \le \beta < \alpha$ *and* $\mu > 0$ *we have*

$$T(r, A_0) \ge \exp_{n-1}\left\{\alpha\left(\frac{1}{1-|z|}\right)^{\mu}\right\}$$

and

$$T(r, A_i) \le \exp_{n-1}\left\{\beta\left(\frac{1}{1-|z|}\right)^{\mu}\right\}, \quad i = 1, 2, \ldots, k,$$

as $|z| \to 1^-$ *for* $z \in H$. *Then every meromorphic (or analytic) solution* $f \not\equiv 0$ *of equation* (1.1) *satisfies* $\sigma_n(f) = \infty$ *and* $\sigma_{n+1}(f) \ge \mu$.

And as for the equation

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0, \tag{1.2}$$

T. B. Cao obtained more accurate results as follows.

Theorem D (see [5]) *Let* H *be a set of complex numbers satisfying* $\overline{\mathrm{dens}}_{\mathbb{D}}\{|z| : z \in H \subseteq \mathbb{D}\} > 0$, *and let* $A_0, A_1, \ldots, A_{k-1}$ *be analytic functions in* $\mathbb{D}$ *such that*

$$\max\{\sigma_{M,n}(A_i) : i = 1, 2, \ldots, k-1\} \le \sigma_{M,n}(A_0) = \sigma < \infty,$$

and for some constants $0 \le \beta < \alpha$ *we have, for all* $\varepsilon > 0$ *sufficiently small,*

$$|A_0(z)| \ge \exp_n\left\{\alpha\left(\frac{1}{1-|z|}\right)^{\sigma-\varepsilon}\right\}$$

and

$$|A_i(z)| \le \exp_n\left\{\beta\left(\frac{1}{1-|z|}\right)^{\sigma-\varepsilon}\right\}, \quad i = 1, 2, \ldots, k-1,$$

as $|z| \to 1^-$ *for* $z \in H$. *Then every solution* $f \not\equiv 0$ *of equation* (1.2) *satisfies* $\sigma_n(f) = \infty$ *and* $\sigma_{n+1}(f) = \sigma_{M,n}(A_0)$.

Theorem E (see [5]) *Let H be a set of complex numbers satisfying $\overline{\text{dens}}_{\mathbb{D}}\{|z| : z \in H \subseteq \mathbb{D}\} > 0$, and let $A_0, A_1, \dots, A_{k-1}$ be analytic functions in $\mathbb{D}$ such that*

$$\max\{\sigma_n(A_i) : i = 1, 2, \dots, k-1\} \le \sigma_n(A_0) = \sigma < \infty,$$

and for some constants $0 \le \beta < \alpha$ we have, for all $\varepsilon > 0$ sufficiently small,

$$T(r, A_0) \ge \exp_{n-1}\left\{\alpha\left(\frac{1}{1-|z|}\right)^{\sigma-\varepsilon}\right\}$$

and

$$T(r, A_i) \le \exp_{n-1}\left\{\beta\left(\frac{1}{1-|z|}\right)^{\sigma-\varepsilon}\right\}, \quad i = 1, 2, \dots, k-1,$$

as $|z| \to 1^-$ for $z \in H$. Then every solution $f \not\equiv 0$ of equation (1.2) satisfies $\sigma_n(f) = \infty$ and $\alpha_{M,n} \ge \sigma_{n+1}(f) \ge \sigma_n(A_0)$, where $\alpha_{M,n} = \max\{\sigma_{M,n}(A_j) : j = 0, 1, \dots, k-1\}$.

The first aim of this paper is to extend the results of B. Belaïdi, T. B. Cao, and the authors by optimizing the conditions of coefficients with less control constants to contrast coefficients, and obtaining better estimates of the growth of the solutions. We will decrease the control constants of the coefficients' modulus or characteristic functions to one or two constants and obtain the same results. Furthermore, we will improve the result of Theorem E to obtain more accurate estimate of the growth of solution. Note that Theorem E only gave a range of $\sigma_{n+1}(f)$ between $\alpha_{M,n}$ and $\sigma_n(A_0)$, while by Remark 1.1, it is best possible that $\sigma_{M,n}(A_0) \neq \sigma_n(A_0)$ as $n = 1$. This means even if A_0, as a dominant coefficient, satisfies $\alpha_{M,1} = \max\{\sigma_M(A_j) : j = 1, 2, \dots, k-1\} = \sigma_M(A_0)$ and $\max\{\sigma(A_i) : i = 1, 2, \dots, k-1\} \le \sigma(A_0)$, we still cannot conclude that $\sigma_2(f)$ is equal to $\sigma(A_0)$ or $\sigma_M(A_0)$ by Theorem E. Thus it is natural to pose the question: Can we get a precise value of $\sigma_{n+1}(f)$ in the case of contrasting coefficients by their characteristic functions? We will solve this problem in Theorems 1.3, 1.4 and Corollary 1.1.

In addition, T. B. Cao also investigated the fixed points of homogeneous linear differential equations in $\mathbb{D}$ and obtained the following results in [5].

Theorem F (see [5]) *Under the hypothesis of one of Theorem D and E, if $A_1(z) + zA_0(z) \not\equiv 0$, then every solution $f \not\equiv 0$ of (1.2) satisfies $\overline{\lambda}_{n+1}(f - z) = \sigma_{n+1}(f)$.*

After that, when $A_j(z) \equiv 0 (j = 1, \dots, k-1)$, G. W. Zhang and A. Chen [19] studied the fixed points of the i-order $(1 \le i \le k)$ derivatives of solutions of the equation

$$f^{(k)} + A(z)f = 0 \tag{1.3}$$

and obtained the results as follows.

Theorem G (see [19]) *Let H be a set of complex numbers satisfying $\overline{\text{dens}}_{\mathbb{D}}\{|z| : z \in H \subseteq \mathbb{D}\} > 0$, and let $A(z)$ be an analytic function in $\mathbb{D}$ such that $\sigma_{M,n}(A) = \sigma < \infty$, and for constant α we have, for all $\varepsilon > 0$ sufficiently small,*

$$|A(z)| \ge \exp_n\left\{\alpha\left(\frac{1}{1-|z|}\right)^{\sigma-\varepsilon}\right\}$$

as $|z| \to 1^-$ *for* $z \in H$. *Assume that* f *is a nontrivial solution of equation* (1.3). *Then*

$$\overline{\lambda}_n\big(f^{(i)} - z\big) = \lambda_n\big(f^{(i)} - z\big) = \sigma_n(f) = \infty,$$
$$\overline{\lambda}_{n+1}\big(f^{(i)} - z\big) = \lambda_{n+1}\big(f^{(i)} - z\big) = \sigma_{n+1}(f) = \sigma.$$

Theorem H (see [19]) *Let* H *be a set of complex numbers satisfying* $\overline{\mathrm{dens}}_{\mathbb{D}}\{|z| : z \in H \subseteq \mathbb{D}\} > 0$, *and let* $A(z)$ *be an analytic function in* $\mathbb{D}$ *such that* $\sigma_n(A) = \sigma < \infty$, *and for constant* α *we have, for all* $\varepsilon > 0$ *sufficiently small,*

$$T\big(r, A(z)\big) \geq \exp_{n-1}\left\{\alpha\left(\frac{1}{1-|z|}\right)^{\sigma-\varepsilon}\right\}$$

as $|z| \to 1^-$ *for* $z \in H$. *Assume that* f *is a nontrivial solution of equation* (1.3). *Then*

$$\overline{\lambda}_n\big(f^{(i)} - z\big) = \lambda_n\big(f^{(i)} - z\big) = \sigma_n(f) = \infty,$$
$$\sigma_{M,n}(A) \geq \overline{\lambda}_{n+1}\big(f^{(i)} - z\big) = \lambda_{n+1}\big(f^{(i)} - z\big) = \sigma_{n+1}(f) \geq \sigma.$$

However, if the equation has two or more than two coefficients, there arises some difficulties to consider: whether applying the general iterative method, by substituting $g^{(i)} = f^{(i)} - z$ into the corresponding equations (see the proof of Theorem 1.9), would result in vanishing $A_{0,j}$ and F_j for any $j \in \mathbb{N}$ (see (5.29)–(5.30)), which consequently would result in interruption of the iteration. Yet we have not seen any valid result on the fixed points of arbitrary-order derivatives of solutions of higher-order equation (1.2) in the unit disc for this. A similar problem has been found recently in an iterative process related to the solutions' arbitrary-order derivatives, where the corresponding items generated by iteration were supposed not to vanish identically (see, e.g., [20, 21]). Thus a natural question is: how to judge these possible vanishing items and whether the iteration can be carried out all the time.

The second aim of this paper is to investigate the fixed points of solutions and their arbitrary-order derivatives of general higher order equations (1.1) and (1.2). We tackle the above problem concisely, extend Theorems G and H to more general equation, and further improve them by obtaining precise estimates of the fixed points of arbitrary-order derivatives of solutions. At the same time, we improve Theorem F by removing the condition $A_1(z) + zA_0(z) \not\equiv 0$.

Theorem 1.1 *Let* H *be a set of complex numbers satisfying* $\overline{\mathrm{dens}}_{\mathbb{D}}\{|z| : z \in H \subseteq \mathbb{D}\} > 0$. *Let* $A_0, A_1, \dots, A_{k-1}$ *be analytic functions in the unit disc* $\mathbb{D}$ *such that* $\max\{\sigma_{M,n}(A_i) : i = 1, 2, \dots, k-1\} \leq \sigma_{M,n}(A_0) = \mu (0 < \mu < \infty)$, *and for a constant* $\alpha \geq 0$, *we have*

$$\varliminf_{|z|\to 1^-, z\in H} \big((1-|z|)^{\mu} \log_n\big|A_0(z)\big|\big) > \alpha \tag{1.4}$$

and

$$\big|A_i(z)\big| \leq \exp_n\left\{\alpha\left(\frac{1}{1-|z|}\right)^{\mu}\right\}, \quad i = 1, 2, \dots, k-1, \tag{1.5}$$

as $|z| \to 1^-$ *for* $z \in H$. *Then every solution* $f \not\equiv 0$ *of* (1.2) *satisfies* $\sigma_n(f) = \infty$ *and* $\sigma_{n+1}(f) = \sigma_{M,n}(A_0)$.

Theorem 1.2 *Let* H *be a set of complex numbers satisfying* $\overline{\text{dens}}_{\mathbb{D}}\{|z| : z \in H \subseteq \mathbb{D}\} > 0$. *Let* $A_0, A_1, \ldots, A_{k-1}$ *be analytic functions in the unit disc* $\mathbb{D}$ *such that* $\max\{\sigma_{M,n}(A_i) : i = 1, 2, \ldots, k-1\} \le \sigma_{M,n}(A_0) = \mu (0 < \mu < \infty)$ *and*

$$\begin{aligned} &\varlimsup_{|z|\to 1^-, z\in H} \big((1-|z|)^{\mu} \log_n |A_i(z)|\big) \\ &\quad < \varliminf_{|z|\to 1^-, z\in H} \big((1-|z|)^{\mu} \log_n |A_0(z)|\big), \quad i = 1, 2, \ldots, k-1. \end{aligned} \tag{1.6}$$

Then every solution $f \not\equiv 0$ *of* (1.2) *satisfies* $\sigma_n(f) = \infty$ *and* $\sigma_{n+1}(f) = \sigma_{M,n}(A_0)$.

Theorem 1.3 *Let* H *be a set of complex numbers satisfying* $\overline{\text{dens}}_{\mathbb{D}}\{|z| : z \in H \subseteq \mathbb{D}\} > 0$. *Let* $A_0, A_1, \ldots, A_{k-1}$ *be analytic functions in the unit disc* $\mathbb{D}$ *such that* $\max\{\sigma_{M,n}(A_i) : i = 1, 2, \ldots, k-1\} \le \sigma_{M,n}(A_0) = \mu (0 < \mu < \infty)$, *and for a constant* $\alpha \ge 0$, *we have*

$$\varliminf_{|z|\to 1^-, z\in H} \big((1-|z|)^{\mu} \log_{n-1} T(r, A_0)\big) > \alpha \tag{1.7}$$

and

$$T(r, A_i) \le \exp_{n-1}\left\{\alpha \left(\frac{1}{1-|z|}\right)^{\mu}\right\}, \quad i = 1, 2, \ldots, k-1, \tag{1.8}$$

as $|z| \to 1^-$ *for* $z \in H$. *Then every solution* $f \not\equiv 0$ *of* (1.2) *satisfies* $\sigma_n(f) = \infty$ *and* $\sigma_{n+1}(f) = \sigma_{M,n}(A_0)$.

Theorem 1.4 *Let* H *be a set of complex numbers satisfying* $\overline{\text{dens}}_{\mathbb{D}}\{|z| : z \in H \subseteq \mathbb{D}\} > 0$. *Let* $A_0, A_1, \ldots, A_{k-1}$ *be analytic functions in the unit disc* $\mathbb{D}$ *such that* $\max\{\sigma_{M,n}(A_i) : i = 1, 2, \ldots, k-1\} \le \sigma_{M,n}(A_0) = \mu (0 < \mu < \infty)$ *and*

$$\begin{aligned} &\varlimsup_{|z|\to 1^-, z\in H} \big((1-|z|)^{\mu} \log_{n-1} T(r, A_i)\big) \\ &\quad < \varliminf_{|z|\to 1^-, z\in H} \big((1-|z|)^{\mu} \log_{n-1} T(r, A_0)\big), \quad i = 1, 2, \ldots, k-1. \end{aligned} \tag{1.9}$$

Then every solution $f \not\equiv 0$ *of* (1.2) *satisfies* $\sigma_n(f) = \infty$ *and* $\sigma_{n+1}(f) = \sigma_{M,n}(A_0)$.

By taking upper and lower limits after some identical transformations of inequalities, one can easily obtain the following corollary from Theorem 1.3 or Theorem 1.4.

Corollary 1.1 *Let* H *be a set of complex numbers satisfying* $\overline{\text{dens}}_{\mathbb{D}}\{|z| : z \in H \subseteq \mathbb{D}\} > 0$. *Let* $A_0, A_1, \ldots, A_{k-1}$ *be analytic functions in the unit disc* $\mathbb{D}$ *such that* $\max\{\sigma_{M,n}(A_i) : i = 1, 2, \ldots, k-1\} \le \sigma_{M,n}(A_0) = \mu (0 < \mu < \infty)$, *and for some constants* $0 \le \beta < \alpha$, *we have*

$$T(r, A_0) \ge \exp_{n-1}\left\{\alpha \left(\frac{1}{1-|z|}\right)^{\mu}\right\}$$

and

$$T(r,A_i) \le \exp_{n-1}\left\{\beta\left(\frac{1}{1-|z|}\right)^{\mu}\right\}, \quad i=1,2,\ldots,k-1,$$

as $|z| \to 1^-$ *for* $z \in H$. *Then every solution* $f \not\equiv 0$ *of* (1.2) *satisfies* $\sigma_n(f) = \infty$ *and* $\sigma_{n+1}(f) = \sigma_{M,n}(A_0)$.

Remark 1.4 Since for any given ε $(0 < \varepsilon < \mu)$ we can substitute μ by $\mu - \varepsilon$ in the proofs of Theorem 1.1 to Theorem 1.4 and ε is arbitrary, Theorem D is generalized to Theorem 1.1 and Theorem 1.2, and Theorem E to Theorem 1.3 and Theorem 1.4 with less control constants of coefficient conditions. Particularly, Theorem 1.3, Theorem 1.4 and Corollary 1.1 improve Theorem E further by providing a precise value of $\sigma_{n+1}(f) = \sigma_{M,n}(A_0)$ instead of a range $\alpha_{M,n} \ge \sigma_{n+1}(f) \ge \sigma_n(A_0)$.

For equation (1.1), we also generalize Theorem B to Theorem 1.5 and Theorem 1.6, Theorem C to Theorem 1.7 and Theorem 1.8 as follows.

Theorem 1.5 *Let* H *be a set of complex numbers satisfying* $\overline{\mathrm{dens}}_{\mathbb{D}}\{|z| : z \in H \subseteq \mathbb{D}\} > 0$. *Let* $A_0, A_1, \ldots, A_k$ *be analytic functions in the unit disc* $\mathbb{D}$, *and for some constants* $\alpha \ge 0$ *and* $\mu > 0$, *we have* (1.4) *and*

$$|A_i(z)| \le \exp_n\left\{\alpha\left(\frac{1}{1-|z|}\right)^{\mu}\right\}, \quad i=1,2,\ldots,k,$$

as $|z| \to 1^-$ *for* $z \in H$. *Then every meromorphic (or analytic) solution* $f \not\equiv 0$ *of* (1.1) *satisfies* $\sigma_n(f) = \infty$ *and* $\sigma_{n+1}(f) \ge \mu$.

Theorem 1.6 *Let* H *be a set of complex numbers satisfying* $\overline{\mathrm{dens}}_{\mathbb{D}}\{|z| : z \in H \subseteq \mathbb{D}\} > 0$. *Let* $A_0, A_1, \ldots, A_k$ *be analytic functions in the unit disc* $\mathbb{D}$, *and for a constant* $\mu > 0$, *we have*

$$\varlimsup_{|z|\to 1^-, z\in H}\left((1-|z|)^{\mu}\log_n|A_i(z)|\right) < \varliminf_{|z|\to 1^-, z\in H}\left((1-|z|)^{\mu}\log_n|A_0(z)|\right), \quad i=1,2,\ldots,k.$$

Then every meromorphic (or analytic) solution $f \not\equiv 0$ *of* (1.1) *satisfies* $\sigma_n(f) = \infty$ *and* $\sigma_{n+1}(f) \ge \mu$.

Theorem 1.7 *Let* H *be a set of complex numbers satisfying* $\overline{\mathrm{dens}}_{\mathbb{D}}\{|z| : z \in H \subseteq \mathbb{D}\} > 0$. *Let* $A_0, A_1, \ldots, A_k$ *be analytic functions in the unit disc* $\mathbb{D}$, *and for some constants* $\alpha \ge 0$ *and* $\mu > 0$, *we have* (1.7) *and*

$$T(r,A_i) \le \exp_{n-1}\left\{\alpha\left(\frac{1}{1-|z|}\right)^{\mu}\right\}, \quad i=1,2,\ldots,k,$$

as $|z| \to 1^-$ *for* $z \in H$. *Then every meromorphic (or analytic) solution* $f \not\equiv 0$ *of* (1.1) *satisfies* $\sigma_n(f) = \infty$ *and* $\sigma_{n+1}(f) \ge \mu$.

Theorem 1.8 *Let* H *be a set of complex numbers satisfying* $\overline{\mathrm{dens}}_{\mathbb{D}}\{|z| : z \in H \subseteq \mathbb{D}\} > 0$. *Let* $A_0, A_1, \ldots, A_k$ *be analytic functions in the unit disc* $\mathbb{D}$, *and for a constant* $\mu > 0$, *we have*

$$\varlimsup_{|z|\to 1^-, z\in H}\left((1-|z|)^{\mu}\log_{n-1}T(r,A_i)\right)$$

$$< \underline{\lim}_{|z|\to 1^-, z\in H} \left(\left(1-|z|\right)^{\mu} \log_{n-1} T(r,A_0) \right), \quad i = 1,2,\ldots,k.$$

Then every meromorphic (or analytic) solution $f \not\equiv 0$ *of* (1.1) *satisfies* $\sigma_n(f) = \infty$ *and* $\sigma_{n+1}(f) \geq \mu$.

When $A_k(z) \equiv 1$, for any given ε $(0 < \varepsilon < \mu)$, by substituting μ with $\mu - \varepsilon$ in the proofs of Theorem 1.7, Theorem 1.8 and combining with Lemma 2.4, we obtain the extensions of Theorem E in the following corollaries. And of course, the corollaries still hold if we substitute $\mu - \varepsilon$ with μ in them.

Corollary 1.2 *Let* H *be a set of complex numbers satisfying* $\overline{\mathrm{dens}}_{\mathbb{D}}\{|z| : z \in H \subseteq \mathbb{D}\} > 0$. *Let* $A_0, A_1, \ldots, A_{k-1}$ *be analytic functions in the unit disc* $\mathbb{D}$ *such that*

$$\max\left\{\sigma_n(A_i) : i = 1,2,\ldots,k-1\right\} \leq \sigma_n(A_0) = \mu (0 < \mu < \infty),$$

and for a constant $\alpha \geq 0$ *and all* $\varepsilon > 0$ *sufficiently small, we have*

$$\underline{\lim}_{|z|\to 1^-, z\in H} \left(\left(1-|z|\right)^{\mu-\varepsilon} \log_{n-1} T(r,A_0) \right) > \alpha$$

and

$$T(r,A_i) \leq \exp_{n-1}\left\{ \alpha \left(\frac{1}{1-|z|} \right)^{\mu-\varepsilon} \right\}, \quad i = 1,2,\ldots,k-1,$$

as $|z| \to 1^-$ *for* $z \in H$. *Then every solution* $f \not\equiv 0$ *of* (1.2) *satisfies* $\sigma_n(f) = \infty$ *and* $\alpha_{M,n} \geq \sigma_{n+1}(f) \geq \sigma_n(A_0)$, *where* $\alpha_{M,n} = \max\{\sigma_{M,n}(A_j) : j = 0,1,\ldots,k-1\}$.

Corollary 1.3 *Let* H *be a set of complex numbers satisfying* $\overline{\mathrm{dens}}_{\mathbb{D}}\{|z| : z \in H \subseteq \mathbb{D}\} > 0$. *Let* $A_0, A_1, \ldots, A_{k-1}$ *be analytic functions in the unit disc* $\mathbb{D}$ *such that*

$$\max\left\{\sigma_n(A_i) : i = 1,2,\ldots,k-1\right\} \leq \sigma_n(A_0) = \mu (0 < \mu < \infty),$$

and for all $\varepsilon > 0$ *sufficiently small, we have*

$$\begin{aligned} &\overline{\lim}_{|z|\to 1^-, z\in H} \left(\left(1-|z|\right)^{\mu-\varepsilon} \log_{n-1} T(r,A_i) \right) \\ &< \underline{\lim}_{|z|\to 1^-, z\in H} \left(\left(1-|z|\right)^{\mu-\varepsilon} \log_{n-1} T(r,A_0) \right), \quad i = 1,2,\ldots,k-1. \end{aligned}$$

Then every solution $f \not\equiv 0$ *of* (1.2) *satisfies* $\sigma_n(f) = \infty$ *and* $\alpha_{M,n} \geq \sigma_{n+1}(f) \geq \sigma_n(A_0)$, *where* $\alpha_{M,n} = \max\{\sigma_{M,n}(A_j) : j = 0,1,\ldots,k-1\}$.

Moreover, we obtain some results of the fixed points of solutions and their arbitrary-order derivatives of general high-order linear differential equations (1.1) and (1.2) as follows.

Theorem 1.9 *Assume that the assumptions of Theorem 1.1 or Theorem 1.2 hold. Then every solution $f \not\equiv 0$ of (1.2) satisfies*

$$\overline{\lambda}_n\big(f^{(j)} - z\big) = \overline{\lambda}_n(f - z) = \sigma_n(f) = \infty,$$
$$\overline{\lambda}_{n+1}\big(f^{(j)} - z\big) = \overline{\lambda}_{n+1}(f - z) = \sigma_{n+1}(f) = \mu \quad (j = 1, 2, \dots).$$

Theorem 1.10 *Assume that the assumptions of one of Theorem 1.3, Theorem 1.4 and Corollary 1.1 hold. Then every solution $f \not\equiv 0$ of (1.2) satisfies*

$$\overline{\lambda}_n\big(f^{(j)} - z\big) = \overline{\lambda}_n(f - z) = \sigma_n(f) = \infty,$$
$$\overline{\lambda}_{n+1}\big(f^{(j)} - z\big) = \overline{\lambda}_{n+1}(f - z) = \sigma_{n+1}(f) = \mu \quad (j = 1, 2, \dots).$$

Theorem 1.11 *Assume that the assumptions of one of Theorem 1.5 to Theorem 1.8 hold. Then every meromorphic (or analytic) solution $f \not\equiv 0$ of (1.1) satisfies*

$$\overline{\lambda}_n\big(f^{(j)} - z\big) = \overline{\lambda}_n(f - z) = \sigma_n(f) = \infty,$$
$$\overline{\lambda}_{n+1}\big(f^{(j)} - z\big) = \overline{\lambda}_{n+1}(f - z) = \sigma_{n+1}(f) \geq \mu \quad (j = 1, 2, \dots).$$

Set $A_k(z) \equiv 1$. Then it is easy to know Corollary 1.4 follows by Theorem 1.11.

Corollary 1.4 *Assume that the assumptions of Corollary 1.2 or Corollary 1.3 hold. Then every solution $f \not\equiv 0$ of (1.2) satisfies*

$$\overline{\lambda}_n\big(f^{(j)} - z\big) = \overline{\lambda}_n(f - z) = \sigma_n(f) = \infty,$$
$$\overline{\lambda}_{n+1}\big(f^{(j)} - z\big) = \overline{\lambda}_{n+1}(f - z) = \sigma_{n+1}(f) \geq \mu \quad (j = 1, 2, \dots).$$

Remark 1.5 Obviously, Theorems F, G, and H are direct results of Theorem 1.9 to Theorem 1.11 and Corollary 1.4, which significantly improve the formers.

Remark 1.6 It is easy to know from the proofs that for any given ε $(0 < \varepsilon < \mu)$ all these results still hold if we substitute μ in (1.4)–(1.9) by $\mu - \varepsilon$. By taking both sides of inequalities upper or lower limits after some identical transformations, we can easily conclude that the conditions in the above results contain the corresponding conditions in Theorems B, C, D, and E, so these results extend those in [5] and [19].

2 Preliminary lemmas

Lemma 2.1 (see [7]) *Let k and j be integers satisfying $k > j \geq 0$, and let $\varepsilon > 0$ and $d \in (0,1)$. If f is meromorphic in $\mathbb{D}$ such that $f^{(j)}$ does not vanish identically, then*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq \left(\left(\frac{1}{1-|z|}\right)^{2+\varepsilon} \max\left\{\log\frac{1}{1-|z|}, T\big(s(|z|), f\big)\right\}\right)^{k-j}, \quad |z| \notin E,$$

where $E \subset [0,1)$ with finite logarithmic measure $\int_E \frac{dr}{1-r} < \infty$ and $s(|z|) = 1 - d(1-|z|)$. Moreover, if $\sigma_1(f) < \infty$, then

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq \left(\frac{1}{1-|z|}\right)^{(k-j)(\sigma_1(f)+2+\varepsilon)}, \quad |z| \notin E,$$

while if $\sigma_n(f) < \infty$ *for* $n \geq 2$, *then*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq \exp_{n-1}\left\{\left(\frac{1}{1-|z|}\right)^{\sigma_n(f)+\varepsilon}\right\}, \quad |z| \notin E.$$

Lemma 2.2 (see [22]) *Let* $g:(0,1)\to\mathbb{R}$ *and* $h:(0,1)\to\mathbb{R}$ *be monotone increasing functions such that* $g(r)\leq h(r)$ *holds outside of an exceptional set* $E\subset[0,1)$ *for which* $\int_E \frac{dr}{1-r}<\infty$. *Then there exists a constant* $d\in(0,1)$ *such that if* $s(r)=1-d(1-r)$, *then* $g(r)\leq h(s(r))$ *for all* $r\in[0,1)$.

Lemma 2.3 (see [23]) *Let* f *be a solution of* (1.2) *where the coefficients* $A_j(z)(j=0,\ldots,k-1)$ *are analytic functions in the disc* $D_R=\{z\in\mathbb{C}:|z|<R\}$, $0<R\leq\infty$, *let* $n_c\in\{1,\ldots,k\}$ *be the number of nonzero coefficients* $A_j(z), j=0,\ldots,k-1$, *and let* $\theta\in[0,2\pi)$ *and* $\varepsilon>0$. *If* $z_\theta=\nu e^{i\theta}\in D_R$ *is such that* $A_j(z_\theta)\neq 0$ *for some* $j=0,\ldots,k-1$, *then for all* $\nu<r<R$,

$$\left|f\left(re^{i\theta}\right)\right| \leq C\exp\left(n_c\int_\nu^r \max_{j=0,\ldots,k-1}\left|A_j\left(te^{i\theta}\right)\right|^{1/(k-j)}dt\right),$$

where $C>0$ *is a constant satisfying*

$$C\leq(1+\varepsilon)\max_{j=0,\ldots,k-1}\left(\frac{|f^{(j)}(z_\theta)|}{(n_c)^j\max\limits_{j=0,\ldots,k-1}|A_n(z_\theta)|^{j/(k-n)}}\right).$$

The next lemma follows by Lemma 2.3.

Lemma 2.4 *Let* $n\in\mathbb{N}$. *If the coefficients* $A_0(z),A_1(z),\ldots,A_{k-1}(z)$ *are analytic in* $\mathbb{D}$, *then all solutions of* (1.2) *satisfy* $\sigma_{M,n+1}(f)\leq\max\{\sigma_{M,n}(A_j):j=0,\ldots,k-1\}$.

Lemma 2.5 (see [4]) *Let* f *be a meromorphic function in the unit disc, and let* $k\geq 1$ *be an integer. Then*

$$m\left(r,\frac{f^{(k)}}{f}\right)=S(r,f),$$

where $S(r,f)=O(\log^+T(r,f)+\log(\frac{1}{1-r}))$, *possibly outside a set* $E\subset[0,1)$ *with* $\int_E\frac{dr}{1-r}<+\infty$. *If* f *is of finite order of growth, then*

$$m\left(r,\frac{f^{(k)}}{f}\right)=O\left(\log\left(\frac{1}{1-r}\right)\right).$$

Lemma 2.6 (see [24]) *If* f *and* g *are meromorphic functions in* $\mathbb{D}$, $n\in\mathbb{N}$, *then we have*

(i) $\sigma_n(f)=\sigma_n(1/f), \sigma_n(a\cdot f)=\sigma_n(f)(a\in\mathbb{C}-\{0\})$;
(ii) $\sigma_n(f)=\sigma_n(f')$;
(iii) $\max\{\sigma_n(f+g),\sigma_n(f\cdot g)\}\leq\max\{\sigma_n(f),\sigma_n(g)\}$;
(iv) *if* $\sigma_n(f)<\sigma_n(g)$, *then* $\sigma_n(f+g)=\sigma_n(g), \sigma_n(f\cdot g)=\sigma_n(g)$.

Lemma 2.7 (see [25]) *Let* $A_0,A_1,\ldots,A_{k-1}$ *and* $F\not\equiv 0$ *be meromorphic functions in* $\mathbb{D}$, *and let* f *be a meromorphic solution of the differential equation*

$$f^{(k)}+A_{k-1}(z)f^{|(k-1)}+\cdots+A_1(z)f'+A_0(z)f=F(z) \tag{2.1}$$

such that

$$\max\{\sigma_p(F),\sigma_p(A_j)(j=0,1,\dots,k-1)\}<\sigma_p(f).$$

Then $\overline{\lambda}_p(f)=\lambda_p(f)=\sigma_p(f)$.

Using the same arguments as in the proof of Lemma 2.7 (see [25]), we easily obtain the following lemma.

Lemma 2.8 *Let* $A_0,A_1,\dots,A_{k-1}$ *and* $F(\not\equiv 0)$ *be finite iterated p-order meromorphic functions in* $\mathbb{D}$. *If f is a meromorphic solution with* $\sigma_p(f)=\infty$ *and* $\sigma_{p+1}(f)=\rho<\infty$ *of equation* (2.1), *then* $\overline{\lambda}_p(f)=\lambda_p(f)=\sigma_p(f)=\infty$ *and* $\overline{\lambda}_{p+1}(f)=\lambda_{p+1}(f)=\sigma_{p+1}(f)=\rho$.

Lemma 2.9 (see [26]) *Let* $A_0,A_1,\dots,A_{k-1}$ *and* $F(\not\equiv 0)$ *be finite iterated p-order analytic functions in* $\mathbb{D}$. *If f is a solution with* $\sigma_p(f)=\infty$ *and* $\sigma_{p+1}(f)=\rho<\infty$ *of the equation*

$$f^{(k)}+A_{k-1}f^{(k-1)}+\cdots+A_1f'+A_0f=F,$$

then $\overline{\lambda}_p(f)=\lambda_p(f)=\sigma_p(f)=\infty$ *and* $\overline{\lambda}_{p+1}(f)=\lambda_{p+1}(f)=\sigma_{p+1}(f)=\rho$.

3 Proofs of Theorems 1.1 to 1.4

Proof of Theorem 1.1 Suppose that $f\not\equiv 0$ is a solution of equation (1.2). From (1.2), we get

$$|A_0|\le\left|\frac{f^{(k)}}{f}\right|+|A_{k-1}|\left|\frac{f^{(k-1)}}{f}\right|+\cdots+|A_1|\left|\frac{f'}{f}\right|. \tag{3.1}$$

By the assumption of Theorem 1.1, there is a set H of complex numbers with $\overline{\mathrm{dens}}_{\mathbb{D}}\{|z|:z\in H\subseteq\mathbb{D}\}>0$ satisfying (1.4) and (1.5) as $|z|\to 1^-$ for $z\in H$. By (1.4), we know that there exists a real number γ such that

$$\varliminf_{|z|\to 1^-,z\in H}\left((1-|z|)^{\mu}\log_n\left|A_0(z)\right|\right)>\gamma>\alpha.$$

It is easy to know that

$$(1-|z|)^{\mu}\log_n\left|A_0(z)\right|>\gamma>\alpha\ge 0 \tag{3.2}$$

as $|z|\to 1^-$ for $z\in H$. By (1.5) and (3.2), we obtain

$$\begin{aligned}\left|A_i(z)\right|&\le\exp_n\left\{\alpha\left(\frac{1}{1-|z|}\right)^{\mu}\right\}\\&<\exp_n\left\{\gamma\left(\frac{1}{1-|z|}\right)^{\mu}\right\}<\left|A_0(z)\right|,\quad i=1,2,\dots,k-1,\end{aligned} \tag{3.3}$$

as $|z|\to 1^-$ for $z\in H$. By Lemma 2.1, there exists a set $E_1\subset[0,1)$ with $\int_{E_1}\frac{dr}{1-r}<\infty$ such that, for all $z\in\mathbb{D}$ satisfying $|z|\notin E_1$ and $i=1,2,\dots,k$, we have

$$\left|\frac{f^{(i)}(z)}{f(z)}\right|\le\left(\left(\frac{1}{1-|z|}\right)^{2+\varepsilon}\max\left\{\log\frac{1}{1-|z|},T\big(s(|z|),f\big)\right\}\right)^{i}, \tag{3.4}$$

where $s(|z|) = 1 - d(1 - |z|)$ and $d \in (0,1)$. Applying (3.3) and (3.4) to (3.1), we obtain

$$\begin{aligned} &\exp_n\left\{\gamma\left(\frac{1}{1-|z|}\right)^{\mu}\right\} \\ &\quad < \left|A_0(z)\right| \\ &\quad \le k\exp_n\left\{\alpha\left(\frac{1}{1-|z|}\right)^{\mu}\right\}\left(\left(\frac{1}{1-|z|}\right)^{2+\varepsilon}\max\left\{\log\frac{1}{1-|z|}, T\left(s\left(|z|\right),f\right)\right\}\right)^{k} \end{aligned}$$

as $|z| \to 1^-$ for $z \in H$ and $|z| \notin E_1$. It follows that

$$\begin{aligned} &\exp\left\{\left(\exp_{n-1}\left\{\gamma\left(\frac{1}{1-|z|}\right)^{\mu}\right\}\right)\left(1-o(1)\right)\right\} \\ &\quad \le k\left(\left(\frac{1}{1-|z|}\right)^{2+\varepsilon}\max\left\{\log\frac{1}{1-|z|}, T\left(s\left(|z|\right),f\right)\right\}\right)^{k} \end{aligned} \tag{3.5}$$

as $|z| \to 1^-$ for $z \in H$ and $|z| \notin E_1$. Obviously, $\int_{\{|z|:z\in H\}\setminus E_1} \frac{dr}{1-r} = \infty$. Hence, by (3.5) and combining with Lemma 2.2, we can easily obtain $\sigma_n(f) = \infty$ and

$$\sigma_{n+1}(f) = \overline{\lim_{r\to 1^-}} \frac{\log^+_{n+1} T(r,f)}{-\log(1-r)} \ge \mu.$$

By Lemma 2.4, we deduce that

$$\sigma_{n+1}(f) = \sigma_{M,n}(f) \le \max\left\{\sigma_{M,n}(A_j) : j = 1,2,\dots,k-1\right\} = \sigma_{M,n}(A_0) = \mu.$$

Therefore, we obtain $\sigma_{n+1}(f) = \sigma_{M,n}(A_0) = \mu$. □

Proof of Theorem 1.2 Set

$$\begin{aligned} \gamma_0 &= \underline{\lim}_{|z|\to 1^-, z\in H}\left(\left(1-|z|\right)^{\mu}\log_n\left|A_0(z)\right|\right), \\ \alpha_i &= \overline{\lim}_{|z|\to 1^-, z\in H}\left(\left(1-|z|\right)^{\mu}\log_n\left|A_i(z)\right|\right), \quad i = 1,2,\dots,k-1. \end{aligned}$$

By (1.6), there exist real numbers α, γ such that $\alpha_i < \alpha < \gamma < \gamma_0, i = 1,2,\dots,k-1$. It yields

$$\left(1-|z|\right)^{\mu}\log_n\left|A_i(z)\right| < \alpha < \gamma < \left(1-|z|\right)^{\mu}\log_n\left|A_0(z)\right|$$

as $|z| \to 1^-$ for $z \in H$. Hence, we have (3.3) as $|z| \to 1^-$ for $z \in H$. Using the same proof as in Theorem 1.1, we can get the conclusion of Theorem 1.2. □

Proof of Theorem 1.3 Suppose that $f \not\equiv 0$ is a solution of equation (1.2). From (1.2), we get

$$-A_0 = \frac{f^{(k)}}{f} + A_{k-1}\frac{f^{(k-1)}}{f} + \dots + A_1\frac{f'}{f}.$$

It follows that

$$m(r,A_0) \le \sum_{i=1}^{k-1} m(r,A_i) + \sum_{i=1}^{k} m\left(r,\frac{f^{(i)}}{f}\right) + O(1). \tag{3.6}$$

From the assumption of Theorem 1.3, there exists a set H of complex numbers with $\overline{\text{dens}}_{\mathbb{D}}\{|z| : z \in H \subseteq \mathbb{D}\} > 0$ satisfying (1.7) and (1.8) as $|z| \to 1^-$ for $z \in H$. By (1.7), we know that there exists a real number γ such that

$$\underline{\lim}_{|z|\to 1^-, z\in H} ((1-|z|)^{\mu} \log_{n-1} T(r, A_0) > \gamma > \alpha.$$

It is easy to know that

$$(1-|z|)^{\mu} \log_{n-1} T(r, A_0) > \gamma > \alpha \geq 0 \tag{3.7}$$

as $|z| \to 1^-$ for $z \in H$. By (1.8) and (3.7), we obtain

$$\begin{aligned} T(r, A_i) &\leq \exp_{n-1}\left\{\alpha\left(\frac{1}{1-|z|}\right)^{\mu}\right\} \\ &< \exp_{n-1}\left\{\gamma\left(\frac{1}{1-|z|}\right)^{\mu}\right\} < T(r, A_0), \quad i = 1, 2, \ldots, k-1, \end{aligned} \tag{3.8}$$

as $|z| \to 1^-$ for $z \in H$. With (3.6), (3.8), and Lemma 2.5, using a similar proof as in Theorem 1.3 of [5], we can easily obtain $\sigma_n(f) = \infty$ and

$$\sigma_{n+1}(f) = \overline{\lim}_{r\to 1^-} \frac{\log^+_{n+1} T(r, f)}{-\log(1-r)} \geq \mu.$$

By Lemma 2.4, we obtain $\sigma_{n+1}(f) \leq \sigma_{M,n}(A_0) = \mu$. Therefore, $\sigma_{n+1}(f) = \sigma_{M,n}(A_0) = \mu$. We complete the proof. □

Proof of Theorem 1.4 Set

$$\begin{aligned} \gamma_0 &= \underline{\lim}_{|z|\to 1^-, z\in H} \left((1-|z|)^{\mu} \log_{n-1} T(r, A_0)\right), \\ \alpha_i &= \overline{\lim}_{|z|\to 1^-, z\in H} \left((1-|z|)^{\mu} \log_{n-1} T(r, A_i)\right), \quad i = 1, 2, \ldots, k-1. \end{aligned}$$

By (1.9), there exist real numbers α, γ such that $\alpha_i < \alpha < \gamma < \gamma_0, i = 1, 2, \ldots, k-1$. It yields that

$$(1-|z|)^{\mu} \log_{n-1} T(r, A_i) < \alpha < \gamma < (1-|z|)^{\mu} \log_{n-1} T(r, A_0)$$

as $|z| \to 1^-$ for $z \in H$. Hence, we have (3.8) as $|z| \to 1^-$ for $z \in H$. Using the same proof as in Theorem 1.3, we can get the conclusion of Theorem 1.4. □

4 Proofs of Theorems 1.5 to 1.8

Proofs of Theorems 1.5 *and* 1.6 Using a similar discussion as in the proof of Theorem 1.1 or Theorem 1.2, we know that there exists a real number $\gamma > \alpha$ such that

$$|A_i(z)| \leq \exp_n\left\{\alpha\left(\frac{1}{1-|z|}\right)^{\mu}\right\}$$

$$< \exp_n\left\{\gamma\left(\frac{1}{1-|z|}\right)^{\mu}\right\} < \left|A_0(z)\right|, \quad i = 1, 2, \dots, k, \tag{4.1}$$

as $|z| \to 1^-$ for $z \in H$. Combining Theorem B with (4.1), we have the conclusion. □

Proofs of Theorems 1.7 *and* 1.8 Using a similar discussion as in the proof of Theorem 1.3 or Theorem 1.4, we know that there exists a real number $\gamma > \alpha$ such that

$$\begin{aligned} T(r, A_i) &\le \exp_{n-1}\left\{\alpha\left(\frac{1}{1-|z|}\right)^{\mu}\right\} \\ &< \exp_{n-1}\left\{\gamma\left(\frac{1}{1-|z|}\right)^{\mu}\right\} < T(r, A_0), \quad i = 1, 2, \dots, k, \end{aligned} \tag{4.2}$$

as $|z| \to 1^-$ for $z \in H$. Combining Theorem C with (4.2), we have the conclusion. □

5 Proofs of Theorems 1.9 to 1.11

Proof of Theorem 1.9 Suppose that $f \not\equiv 0$ is a solution of equation (1.2). By Theorem 1.1 or Theorem 1.2, we have

$$\sigma_n(f) = \infty, \qquad \sigma_{n+1}(f) = \mu. \tag{5.1}$$

Step 1. We consider the fixed points of $f(z)$. Set $g(z) = f(z) - z, z \in \mathbb{D}$. Then, by (5.1), we get

$$\sigma_n(g) = \sigma_n(f) = \infty, \qquad \sigma_{n+1}(g) = \sigma_{n+1}(f) = \mu, \qquad \overline{\lambda}_{n+1}(g) = \overline{\lambda}_{n+1}(f - z). \tag{5.2}$$

Substituting $f = g + z$ into (1.2), we get

$$g^{(k)} + A_{k-1}g^{(k-1)} + \cdots + A_1 g' + A_0 g = -A_1 - zA_0. \tag{5.3}$$

Next we prove that $-A_1 - zA_0 \not\equiv 0$. Suppose that $-A_1 - zA_0 \equiv 0$. Obviously, $A_0 \not\equiv 0$. Then $\frac{A_1}{A_0} = -z$. Thus we have $|\frac{A_1}{A_0}| \to 1$ as $|z| \to 1^-$ for $z \in H$. While by (3.3), we have

$$\left|\frac{A_1}{A_0}\right| < \frac{\exp_n\{\alpha(\frac{1}{1-|z|})^{\mu}\}}{\exp_n\{\gamma(\frac{1}{1-|z|})^{\mu}\}} = \frac{1}{\exp\{(\exp_{n-1}\{\gamma(\frac{1}{1-|z|})^{\mu}\})(1 - o(1))\}} \to 0 \tag{5.4}$$

as $|z| \to 1^-$ for $z \in H$. (5.4) yields $|\frac{A_1}{A_0}| \to 0$ as $|z| \to 1^-$ for $z \in H$. This is a contradiction. Hence, $-A_1 - zA_0 \not\equiv 0$. By Lemma 2.6, we have

$$\max\left\{\sigma_n(A_j)(j = 0, 1, \dots, k-1), \sigma_n(-A_1 - zA_0)\right\} < \infty.$$

Hence, we deduce by (5.2), (5.3) and Lemma 2.9 that $\overline{\lambda}_n(g) = \sigma_n(g) = \infty$, $\overline{\lambda}_{n+1}(g) = \sigma_{n+1}(g) = \mu$. Therefore, we obtain

$$\begin{aligned} &\overline{\lambda}_n(f - z) = \overline{\lambda}_n(g) = \sigma_n(g) = \sigma_n(f) = \infty, \\ &\overline{\lambda}_{n+1}(f - z) = \overline{\lambda}_{n+1}(g) = \sigma_{n+1}(g) = \sigma_{n+1}(f) = \mu. \end{aligned}$$

Step 2. We consider the fixed points of $f^{(j)}(z)(j = 1, 2, \ldots)$. For the proof, we use the principle of mathematical induction. Set $A_k(z) = 1$. Then $|A_k(z)| \le \exp_n\{\alpha(\frac{1}{1-|z|})^{\mu}\}$ and equation (1.2) becomes (1.1). Set $g_1(z) = f'(z) - z, z \in \mathbb{D}$. Then, by (5.1), we get

$$\begin{aligned} &\sigma_n(g_1) = \sigma_n(f') = \infty, \qquad \sigma_{n+1}(g_1) = \sigma_{n+1}(f') = \mu, \\ &\overline{\lambda}_{n+1}(g_1) = \overline{\lambda}_{n+1}(f' - z). \end{aligned} \tag{5.5}$$

Dividing both sides of (1.1) by A_0, we obtain

$$\frac{A_k}{A_0} f^{(k)} + \frac{A_{k-1}}{A_0} f^{(k-1)} + \cdots + \frac{A_1}{A_0} f' + f = 0. \tag{5.6}$$

Differentiating both sides of equation (5.6), we have

$$\begin{aligned} &\frac{A_k}{A_0} f^{(k+1)} + \left(\left(\frac{A_k}{A_0}\right)' + \frac{A_{k-1}}{A_0}\right) f^{(k)} + \cdots \\ &\quad + \left(\left(\frac{A_2}{A_0}\right)' + \frac{A_1}{A_0}\right) f'' + \left(\left(\frac{A_1}{A_0}\right)' + 1\right) f' = 0. \end{aligned} \tag{5.7}$$

Multiplying now (5.7) by A_0, we get

$$A_{k,1} f^{(k+1)} + A_{k-1,1} f^{(k)} + \cdots + A_{1,1} f'' + A_{0,1} f' = 0. \tag{5.8}$$

Substituting $f' = g_1 + z$ into (5.8), we get

$$A_{k,1} g_1^{(k)} + A_{k-1,1} g_1^{(k-1)} + \cdots + A_{1,1} g_1' + A_{0,1} g_1 = F_1, \tag{5.9}$$

where

$$A_{k,1} = 1, A_{i,1} = A_0\left(\left(\frac{A_{i+1}}{A_0}\right)' + \frac{A_i}{A_0}\right) \quad (i = 1, 2, \ldots, k-1), \tag{5.10}$$

$$A_{0,1} = A_0\left(\left(\frac{A_1}{A_0}\right)' + 1\right), \tag{5.11}$$

$$F_1 = -(A_{1,1} + zA_{0,1}). \tag{5.12}$$

Now we prove that $A_{0,1} \not\equiv 0$. Suppose that $A_{0,1} \equiv 0$. By $A_0 \not\equiv 0$, we get $(\frac{A_1}{A_0})' + 1 \equiv 0$, and then $\frac{A_1}{A_0} = -z + C_0$, where C_0 is an arbitrary constant. Thus, we have $A_1 + (z - C_0)A_0 = 0$. Then, by (1.2), we know that $f_0 = z - C_0$ is a solution of equation (1.2) and $\sigma_n(f_0) < \infty$. This contradicts (5.1). Hence $A_{0,1} \not\equiv 0$. Now we prove $F_1 \not\equiv 0$. Suppose that $F_1 \equiv 0$. By (5.12),

$$A_{1,1} + zA_{0,1} = 0. \tag{5.13}$$

If $f' = z$, then by (5.8) and (5.12) we know that f is a solution of (5.8). Hence equation (1.2) has a solution f_1 satisfying $f_1' = z$ and $\sigma_n(f_1) < \infty$. This contradicts (5.1). Hence $F_1 \not\equiv 0$. By (5.10)–(5.12) and Lemma 2.6, we have

$$\max\{\sigma_n(A_{i,1})\ (i = 0, 1, \ldots, k), \sigma_n(F_1)\} < \infty.$$

Hence, by (5.5), (5.9) and using Lemma 2.8 or Lemma 2.9, we have $\overline{\lambda}_n(g_1) = \sigma_n(g_1) = \infty, \overline{\lambda}_{n+1}(g_1) = \sigma_{n+1}(g_1) = \mu$. Therefore, we obtain

$$\begin{aligned}&\overline{\lambda}_n(f' - z) = \overline{\lambda}_n(g_1) = \sigma_n(g_1) = \sigma_n(f) = \infty,\\&\overline{\lambda}_{n+1}(f' - z) = \overline{\lambda}_{n+1}(g_1) = \sigma_{n+1}(g_1) = \sigma_{n+1}(f) = \mu.\end{aligned}$$

Set $g_2(z) = f''(z) - z, z \in \mathbb{D}$. Then, by (5.1), we get

$$\begin{aligned}&\sigma_n(g_2) = \sigma_n(f'') = \infty, \qquad \sigma_{n+1}(g_2) = \sigma_{n+1}(f'') = \mu,\\&\overline{\lambda}_{n+1}(g_2) = \overline{\lambda}_{n+1}(f'' - z).\end{aligned} \tag{5.14}$$

Dividing both sides of (5.8) by $A_{0,1}$, we obtain

$$\frac{A_{k,1}}{A_{0,1}} f^{(k+1)} + \frac{A_{k-1,1}}{A_{0,1}} f^{(k)} + \cdots + \frac{A_{1,1}}{A_{0,1}} f'' + f' = 0. \tag{5.15}$$

Differentiating both sides of equation (5.15), we have

$$\begin{aligned}&\frac{A_{k,1}}{A_{0,1}} f^{(k+2)} + \left(\left(\frac{A_{k,1}}{A_{0,1}}\right)' + \frac{A_{k-1,1}}{A_{0,1}}\right) f^{(k+1)} + \cdots\\&\quad + \left(\left(\frac{A_{2,1}}{A_{0,1}}\right)' + \frac{A_{1,1}}{A_{0,1}}\right) f''' + \left(\left(\frac{A_{1,1}}{A_{0,1}}\right)' + 1\right) f'' = 0.\end{aligned} \tag{5.16}$$

Multiplying now (5.16) by $A_{0,1}$, we get

$$A_{k,2} f^{(k+2)} + A_{k-1,2} f^{(k+1)} + \cdots + A_{1,2} f''' + A_{0,2} f'' = 0. \tag{5.17}$$

Substituting $f'' = g_2 + z$ into (5.17), we get

$$A_{k,2} g_2^{(k)} + A_{k-1,2} g_2^{(k-1)} + \cdots + A_{1,2} g_2' + A_{0,2} g_2 = F_2, \tag{5.18}$$

where

$$A_{k,2} = 1, \qquad A_{i,2} = A_{0,1}\left(\left(\frac{A_{i+1,1}}{A_{0,1}}\right)' + \frac{A_{i,1}}{A_{0,1}}\right) \quad (i = 1, 2, \ldots, k-1), \tag{5.19}$$

$$A_{0,2} = A_{0,1}\left(\left(\frac{A_{1,1}}{A_{0,1}}\right)' + 1\right), \tag{5.20}$$

$$F_2 = -(A_{1,2} + zA_{0,2}). \tag{5.21}$$

Now we prove that $A_{0,2} \not\equiv 0$. Suppose that $A_{0,2} \equiv 0$. By $A_{0,1} \not\equiv 0$, we get $(\frac{A_{1,1}}{A_{0,1}})' + 1 \equiv 0$, and then $\frac{A_{1,1}}{A_{0,1}} = -z + C$, where C is an arbitrary constant. Thus, we have

$$A_{1,1} + (z - C)A_{0,1} = 0. \tag{5.22}$$

If $f' = z - C$, then by (5.8) and (5.22) we know that f is a solution of (5.8). Hence equation (1.2) has a solution $f_{1,2}$ satisfying $f_{1,2}' = z - C$ and $\sigma_n(f_{1,2}) < \infty$. This contradicts (5.1). Hence

$A_{0,2} \not\equiv 0$. Next we prove $F_2 \not\equiv 0$. Suppose that $F_2 \equiv 0$. By (5.21),

$$A_{1,2} + zA_{0,2} = 0. \tag{5.23}$$

If $f'' = z$, then by (5.17) and (5.23) we know that f is a solution of (5.17). Hence equation (1.2) has a solution f_2 satisfying $f_2'' = z$ and $\sigma_n(f_2) < \infty$. This contradicts (5.1). Hence $F_2 \not\equiv 0$. By (5.19)–(5.21) and Lemma 2.6, we have

$$\max\{\sigma_n(A_{i,2})\ (i = 0, 1, \ldots, k), \sigma_n(F_2)\} < \infty.$$

Hence, we deduce by (5.14), (5.18), and Lemma 2.8 or Lemma 2.9 that $\overline{\lambda}_n(g_2) = \sigma_n(g_2) = \infty, \overline{\lambda}_{n+1}(g_2) = \sigma_{n+1}(g_2) = \mu$. Therefore, we obtain

$$\begin{aligned} &\overline{\lambda}_n(f'' - z) = \overline{\lambda}_n(g_2) = \sigma_n(g_2) = \sigma_n(f) = \infty, \\ &\overline{\lambda}_{n+1}(f'' - z) = \overline{\lambda}_{n+1}(g_2) = \sigma_{n+1}(g_2) = \sigma_{n+1}(f) = \mu. \end{aligned}$$

Suppose now that

$$\begin{aligned} &A_{0,s} \not\equiv 0, \\ &\overline{\lambda}_n(f^{(s)} - z) = \sigma_n(f) = \infty, \\ &\overline{\lambda}_{n+1}(f^{(s)} - z) = \sigma_{n+1}(f) = \mu \end{aligned} \tag{5.24}$$

for all $s = 0, 1, \ldots, j-1$, and we prove that (5.24) is true for $s = j$. Set $g_j(z) = f^{(j)}(z) - z, z \in \mathbb{D}$. Then, by (5.1), we get

$$\begin{aligned} &\sigma_n(g_j) = \sigma_n(f^{(j)}) = \infty, \\ &\sigma_{n+1}(g_j) = \sigma_{n+1}(f^{(j)}) = \mu, \qquad \overline{\lambda}_{n+1}(g_j) = \overline{\lambda}_{n+1}(f^{(j)} - z). \end{aligned} \tag{5.25}$$

By the same procedure as before, we can obtain

$$A_{k,j}f^{(k+j)} + A_{k-1,j}f^{(k+j-1)} + \cdots + A_{1,j}f^{(j-1)} + A_{0,j}f^{(j)} = 0 \tag{5.26}$$

and

$$A_{k,j}g_j^{(k)} + A_{k-1,j}g_j^{(k-1)} + \cdots + A_{1,j}g_j' + A_{0,j}g_j = F_j, \tag{5.27}$$

where

$$A_{k,j} = 1, A_{i,j} = A_{0,j-1}\left(\left(\frac{A_{i+1,j-1}}{A_{0,j-1}}\right)' + \frac{A_{i,j-1}}{A_{0,j-1}}\right) \quad (i = 1, 2, \ldots, k-1), \tag{5.28}$$

$$A_{0,j} = A_{0,j-1}\left(\left(\frac{A_{1,j-1}}{A_{0,j-1}}\right)' + 1\right), \qquad A_{0,0} = A_0, \qquad A_{1,0} = A_1, \tag{5.29}$$

$$F_j = -(A_{1,j} + zA_{0,j}) \tag{5.30}$$

satisfying $A_{0,j} \not\equiv 0, F_j \not\equiv 0$. By using Lemma 2.8 or Lemma 2.9 in (5.27), we have

$$\overline{\lambda}_n(g_j) = \lambda_n(g_j) = \sigma_n(g) = \infty, \qquad \overline{\lambda}_{n+1}(g_j) = \lambda_{n+1}(g_j) = \sigma_{n+1}(g) = \mu. \tag{5.31}$$

Then, by (5.25) and (5.31), we obtain

$$\begin{aligned}
&\overline{\lambda}_n\bigl(f^{(j)} - z\bigr) = \overline{\lambda}_n(g_j) = \sigma_n(g_j) = \sigma_n\bigl(f^{(j)}\bigr) = \infty, \\
&\overline{\lambda}_{n+1}\bigl(f^{(j)} - z\bigr) = \overline{\lambda}_{n+1}(g_j) = \sigma_{n+1}(g_j) = \sigma_{n+1}\bigl(f^{(j)}\bigr) = \mu, \quad j = 1, 2, \ldots.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\overline{\lambda}_n\bigl(f^{(j)} - z\bigr) = \overline{\lambda}_n(f - z) = \sigma_n(f) = \infty, \\
&\overline{\lambda}_{n+1}\bigl(f^{(j)} - z\bigr) = \overline{\lambda}_{n+1}(f - z) = \sigma_{n+1}(f) = \mu, \quad j = 1, 2, \ldots.
\end{aligned}$$

The proof is complete. □

Proof of Theorem 1.10 Suppose that $f \not\equiv 0$ is a solution of equation (1.2). By Theorem 1.3 or Theorem 1.4, we have (5.1). Now we prove that $-A_1 - zA_0 \not\equiv 0$. Suppose that $-A_1 - zA_0 \equiv 0$. It yields $A_1 = -zA_0$ and $A_0 = \frac{A_1}{-z}$. Hence

$$\begin{aligned}
&T(r, A_1) \le T(r, A_0) + T(r, -z) = T(r, A_0) + T(r, z), \\
&T(r, A_0) \le T(r, A_1) + T\left(r, \frac{-1}{z}\right) = T(r, A_1) + T(r, z) + O(1).
\end{aligned} \tag{5.32}$$

By (5.32), we obtain

$$1 - \frac{T(r, z) + O(1)}{T(r, A_0)} \le \frac{T(r, A_1)}{T(r, A_0)} \le 1 + \frac{T(r, z)}{T(r, A_0)}. \tag{5.33}$$

Using the same discussion as in the proof of Theorem 1.3, we have

$$T(r, A_1) \le \exp_{n-1}\left\{\alpha\left(\frac{1}{1 - |z|}\right)^{\mu}\right\} < \exp_{n-1}\left\{\gamma\left(\frac{1}{1 - |z|}\right)^{\mu}\right\} < T(r, A_0)$$

as $|z| \to 1^-$ for $z \in H$. It follows that

$$\frac{T(r, z)}{T(r, A_0)} \le \frac{T(r, z)}{\exp_{n-1}\{\gamma(\frac{1}{1-r})^{\mu}\}} \to 0, \tag{5.34}$$

and for $n = 1$,

$$\frac{T(r, A_1)}{T(r, A_0)} < \frac{\alpha}{\gamma} < 1, \tag{5.35}$$

and for $n \ge 2$,

$$\frac{T(r, A_1)}{T(r, A_0)} < \frac{\exp_{n-1}\{\alpha(\frac{1}{1-|z|})^{\mu}\}}{\exp_{n-1}\{\gamma(\frac{1}{1-|z|})^{\mu}\}} \to 0 \tag{5.36}$$

as $|z| \to 1^-$ for $z \in H$. By (5.33) and (5.34), we have $\frac{T(r,A_1)}{T(r,A_0)} \to 1$ as $|z| \to 1^-$ for $z \in H$. But, by (5.35) and (5.36), we can get $\frac{T(r,A_1)}{T(r,A_0)} \nrightarrow 1$ as $|z| \to 1^-$ for $z \in H$. This is a contradiction. Therefore, $-A_1 - zA_0 \not\equiv 0$. Set $A_k(z) = 1$. Then $T(r, A_k) \le \exp_{n-1}\{\alpha(\frac{1}{1-|z|})^\mu\}$. Obviously, $A_0 \not\equiv 0$. Arguing the same as in the proof of Theorem 1.9, we can complete the proof. □

Proof of Theorem 1.11 For any $A_k(z)$, using the same reasoning in the proofs of Theorem 1.9 and Theorem 1.10 and replacing $\sigma_{n+1}(f) = \mu$ with $\sigma_{n+1}(f) \ge \mu$, $\sigma_{n+1}(f^{(j)}) = \mu (j = 1, 2, \ldots, k)$ with $\sigma_{n+1}(f^{(j)}) \ge \mu (j = 1, 2, \ldots, k)$, we can complete the proof of Theorem 1.11. □

Acknowledgements
The authors would like to thank the editor and referees for their valuable and helpful comments and suggestions that improved the quality of the paper.

Funding
This work was supported by the National Natural Science Foundation of China (11971042 and 11561031) and Beijing Natural Science Foundation in China (1182008).

Availability of data and materials
Data sharing not applicable to this article as no data sets were generated or analyzed during the current study.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors drafted the manuscript, read and approved the final manuscript.

Author details
[1]School of Mathematical Sciences, Laboratory of Mathematics and Complex Systems of Ministry of Education, Beijing Normal University, Beijing, China. [2]School of Mathematics and Statistics, Jiangxi Normal University, Nanchang, China. [3]Mathematics teaching and Research Office, The Experiimental Er Long Lu Middle School Attached to Beijing Normal University, Beijing, China.

Publisher's Note

Received: 21 February 2020 Accepted: 1 September 2021 Published online: 27 September 2021

References

1. Hayman, W.K.: Meromorphic Functions. Oxford Mathematical Monographs. Clarendon Press, Oxford (1964)
2. Yang, L.: Value Distribution Theory. Springer, Berlin (1993). Translated and revised from the 1982 Chinese original. https://doi.org/10.1007/978-3-662-02915-2.
3. Laine, I.: Nevanlinna Theory and Complex Differential Equations. De Gruyter Studies in Mathematics, vol. 15. de Gruyter, Berlin (1993). https://doi.org/10.1515/9783110863147
4. Heittokangas, J.: On complex differential equations in the unit disc. Ann. Acad. Sci. Fenn. Math. Diss. 122, 54. Dissertation, University of Joensuu, Joensuu (2000)
5. Cao, T.B.: Growth of solutions of a class of complex differential equations. Ann. Pol. Math. **95**(2), 141–152 (2009)
6. Tsuji, M.: Potential Theory in Modern Function Theory. Chelsea, New York (1975). Reprinting of the 1959 original
7. Chyzhykov, I., Gundersen, G.G., Heittokangas, J.: Linear differential equations and logarithmic derivative estimates. Proc. Lond. Math. Soc. **86**(3), 735–754 (2003)
8. Heittokangas, J.: A survey on Blaschke-oscillatory differential equations, with updates. In: Blaschke Products and Their Applications. Fields Inst. Commun., vol. 65, pp. 43–98. Springer, New York (2013). https://doi.org/10.1007/978-1-4614-5341-3_3
9. Tu, J., Xuan, Z.X.: Complex linear differential equations with certain analytic coefficients of [p, q]-order in the unit disc. Adv. Differ. Equ. **2014**, 167 (2014)
10. Cao, T.B., Yi, H.X.: Complex oscillation of second-order linear differential equations with analytic coefficients in the unit disc. Chin. Ann. Math., Ser. A **28**(5), 719–732 (2007) (in Chinese)
11. Gundersen, G.G.: Finite order solutions of second order linear differential equations. Trans. Am. Math. Soc. **305**(1), 415–429 (1988)
12. Kwon, K.H.: On the growth of entire functions satisfying second order linear differential equations. Bull. Korean Math. Soc. **33**(3), 487–496 (1996)
13. Chen, Z.X., Yang, C.C.: Some further results on the zeros and growths of entire solutions of second order linear differential equations. Kodai Math. J. **22**(2), 273–285 (1999)
14. Belaïdi, B.: Estimation of the hyper-order of entire solutions of complex linear ordinary differential equations whose coefficients are entire functions. Electron. J. Qual. Theory Differ. Equ. **2002**(5), 1 (2002)

15. Belaïdi, B., Hamouda, S.: Growth of solutions of an n-th order linear differential equation with entire coefficients. Kodai Math. J. **25**(3), 240–245 (2002)
16. Chen, Y.: Estimates of the zeros and growths of meromorphic solutions of homogeneous and non-homogeneous second order linear differential equations. Math. Appl. **23**(1), 18–26 (2010)
17. Chen, Y., Deng, G.T.: Growth and fixed points of solutions of a second-order linear differential equation with analytic coefficients in the unit disc. J. Nat. Sci. Beijing Norm. Univ. **54**(6), 687–694 (2018) (in Chinese)
18. Chen, Y., Deng, G.T.: Growth and fixed points of solutions for second-order differential equations in unit disc. J. Nat. Sci. Beijing Norm. Univ. **55**(3), 299–308 (2019) (in Chinese)
19. Zhang, G., Chen, A.: Fixed points of the derivative and k-th power of solutions of complex linear differential equations in the unit disc. Electron. J. Qual. Theory Differ. Equ. **2009**(48), 1 (2009)
20. Latreuch, Z., Belaïdi, B.: On the zeros of solutions and their derivatives of second order non-homogeneous linear differential equations. Miskolc Math. Notes **16**(1), 237–248 (2015)
21. Gong, P., Xiao, L.P.: Oscillation of arbitrary-order derivatives of solutions to linear differential equations taking small functions in the unit disc. Electron. J. Differ. Equ. **2015**, 68 (2015)
22. Bank, S.B.: A general theorem concerning the growth of solutions of first-order algebraic differential equations. Compos. Math. **25**, 61–70 (1972)
23. Heittokangas, J., Korhonen, R., Rättyä, J.: Growth estimates for solutions of linear complex differential equations. Ann. Acad. Sci. Fenn., Math. **29**(1), 233–246 (2004)
24. Cao, T.B., Yi, H.X.: The growth of solutions of linear differential equations with coefficients of iterated order in the unit disc. J. Math. Anal. Appl. **319**(1), 278–294 (2006)
25. Cao, T.B., Deng, Z.S.: Solutions of non-homogeneous linear differential equations in the unit disc. Ann. Pol. Math. **97**(1), 51–61 (2010)
26. Belaïdi, B.: Oscillation of fast growing solutions of linear differential equations in the unit disc. Acta Univ. Sapientiae Math. **2**(1), 25–38 (2010)