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Reconstructing the right-hand side of the Rayleigh–Stokes problem with nonlocal in time condition

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Abstract

In this paper, the problem of finding the source function for the Rayleigh–Stokes equation is considered. According to Hadamard's definition, the sought solution of this problem is both unstable and independent of continuous data. By using the fractional Tikhonov method, we give the regularized solutions and then deal with a priori error estimate between the exact solution and its regularized solutions. Finally, the proposed regularized methods have been verified by simple numerical experiments to check error estimate between the sought solution and the regularized solution.

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Keywords: Rayleigh–Stokes problem; Inverse source problem; Fractional Tikhonov method; Ill-posed problem; Error estimate

1 Introduction

Equation (1.1) below arises in Newtonian fluids and magnetohydrodynamic flows in porous media [1], and initial value problems for fractional Rayleigh–Stokes were studied, for example, in [2–5]. In this study, we are interested in dealing with the Rayleigh–Stokes problem associated with fractional derivative as follows:

$$\begin{cases} \partial_t u(x,t) - (1+\tau\partial_t^\beta)\Delta u(x,t) = f(x)\varphi(t), & (x,t)\in\Omega\times(0,T), \\ u(x,t)=0, & x\in\partial\Omega, \\ u(x,0)=u_0(x), & x\in\Omega, \\ \int_0^T u(x,t)\,dt = \ell(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^d$ $(d=1,2,3)$ is a boundary domain with the boundary $\partial\Omega$ smooth enough, and $T>0$. $\tau>0$ is a constant, u_0 in $L^2(\Omega)$, the notations $\partial_t=\partial/\partial t$, and ∂_t^α is the Riemann–Liouville fractional derivative of order $\beta\in(0,1)$ defined by [6, 7]

$$\partial_t^\beta g(t) = \frac{d}{dt}\int_0^t \omega_{1-\beta}(t-z)g(z)\,dz, \qquad \omega_\beta(t)=\frac{t^{\beta-1}}{\Gamma(\beta)}.$$

The Rayleigh–Stokes introduced as above has much practical importance, see in [8, 9], and in describing the behavior of some non-Newtonian fluids [10]. The numerical solutions of the Rayleigh–Stokes problem with fractional derivatives have been considered and developed by Dehghan or Zaky, see [3–5, 11, 12].

According to our understanding, in recent times, the study of this problem begins to receive the attention of mathematicians, such as M. Kirane [13] and S. Tatar [14]. In [15], authors studied a Rayleigh–Stokes equation in the simple bounded domain by using the fractional Landweber method. Besides that, the study of problem (1.1) with random noise data also began to receive the attention of mathematicians. In [16], using the truncation method and some new techniques, the authors showed the regularized solution, and convergence rates were established. In [17], Triet et al. investigated an inverse source problem (1.1) by a general filter method for random noise, the results for the study of problem (1.1) were rare. However, articles about the survey of source functions for problem (1.1) were rarer than the results. See in [18], the authors investigated problem (1.1) by the Tikhonov regularization method, attached was a simple numerical calculation example to simulate research results in theoretical way. Besides, we also find relevant applications in a broad sense with problem (1.1), please see [19–25]. In most of these studies, mathematicians are interested in the final condition as follows: $u(x,T) = \ell(x)$.

Recently, a few papers mentioned the nonlocal condition $\int_0^T u(x,t)\,dt = \ell(x)$, for example, two papers [26, 27]. We repeat that if the source function $F(x,t) = \varphi(t)f(x)$ is given, then problem (1.1) is called the forward problem. The problem of determining the source function is understood as defining a function f when we know that $\int_0^T u(x,t)\,dt = \ell(x)$ and the function φ. It is worth pointing out that our article is one of the first results to study this problem with nonlocal in time condition. This work can be considered a development step of the results in the article [18]. In this paper, the couple functions (ℓ,φ) are approximated by $(\ell_\epsilon,\varphi_\epsilon)$ such that

$$\|\ell - \ell_\epsilon\|_{L^2(\Omega)} + \|\varphi - \varphi_\epsilon\|_{L^\infty(0,T)} \le \epsilon. \tag{1.2}$$

This paper is organized as follows. In Sect. 2, we introduce some preliminaries. The main results are given in Sect. 3 which presents the non-well-posedness of our problem (1.1). Next, in Sect. 4, we propose the fractional Tikhonov regularization method to find the regularized solution and the convergent rate. In Sect. 5, we present a simple numerical example to verify the results proved in our theory section. The conclusion is presented in Sect. 6.

2 Preliminaries

Definition 2.1 ([28]) Let $\{\lambda_j, e_j\}$ be the eigenvalues and corresponding eigenvectors of the Laplacian operator $-\Delta$ in Ω. The family of eigenvalues $\{\lambda_j\}_{j=1}^\infty$ satisfy $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_j \le \cdots$, where $\lambda_j \to \infty$ as $j \to \infty$:

$$\begin{cases} \Delta e_j(x) = -\lambda_j e_j(x), & x \in \Omega, \\ e_j(x) = 0, & x \in \partial\Omega. \end{cases}$$

Definition 2.2 For $\delta > 0$, define

$$\mathbb{H}^\delta(\Omega) := \left\{ w \in L^2(\Omega); \sum_{j=1}^{\infty} \lambda_j^\delta \left| \langle w, e_j \rangle \right|^2 < +\infty \right\} \tag{2.1}$$

equipped with the norm

$$\|v\|_{\mathbb{H}^\delta(\Omega)} = \left(\sum_{j=1}^{\infty} \lambda_j^\delta \left| \langle v, e_j \rangle \right|^2 \right)^{\frac{1}{2}}.$$

Based on [2], we can know that the solution of the Rayleigh–Stokes problem is as follows:

$$u(x,t) = \sum_{j=1}^{+\infty} \mathcal{C}_j(\beta,t)\langle u_0, e_j\rangle e_j(x) + \sum_{j=1}^{\infty} \left(\int_0^t \mathcal{C}_j(\beta, t-z)\varphi(z)\, dz \langle f, e_j\rangle \right) e_j(x), \tag{2.2}$$

where $F_j(z) = \varphi(z)\langle f, e_j\rangle$. Here, $\mathcal{C}_j(\beta,t)$ satisfies the following equation:

$$\begin{cases} \frac{d}{dt}\mathcal{C}_j(\beta,t) + \lambda_j(1 + \tau \partial_t^\beta)\mathcal{C}_j(\beta,t) = 0, & t \in (0,T), \\ \mathcal{C}_j(\beta, 0) = 1. \end{cases} \tag{2.3}$$

From the condition $\int_0^T u(x,t)\, dt = \ell(x)$ and $u_0 = 0$, we can check that

$$\ell(x) = \int_0^T u(x,t)\, dt = \int_0^T \sum_{j=1}^{\infty} \langle f, e_j\rangle \left(\int_0^t \mathcal{C}_j(\beta, t-z)\varphi(z)\, dz \right) dt e_j(x), \tag{2.4}$$

where we note that $F_j(z) = \varphi(z) f_j$. A simple calculation gives

$$f(x) = \sum_{j=1}^{\infty} f_j e_j(x) = \sum_{j=1}^{\infty} \frac{\langle \ell, e_j\rangle}{\int_0^T (\int_0^t \mathcal{C}_j(\beta, t-z)\varphi(z)\, dz)\, dt} e_j(x). \tag{2.5}$$

From the result of [2], we obtain

$$L\big(\mathcal{C}_j(\beta,t)\big) = \left[t + \gamma \lambda_j t^\beta + \lambda_j \right]^{-1}. \tag{2.6}$$

Lemma 2.3 *The function $\mathcal{C}_j(\beta,t), j = 1, 2, \ldots$, is equal to*

$$\mathcal{C}_j(\beta,t) = \int_0^\infty e^{-\xi t} \mathcal{K}_j(\beta, \xi)\, d\xi,$$

where $\mathcal{K}_j(\beta,\xi) = \frac{\tau}{\pi} \frac{\lambda_j \xi^\beta \sin \beta\pi}{(-\xi + \lambda_j \tau \xi^\beta \cos \beta\pi + \lambda_j)^2 + (\lambda_j \tau \xi^\beta \sin \beta\pi)^2}$.

Proof See the proof in [2]. □

Lemma 2.4 *Let us assume that $\beta \in (\frac{1}{2}, 1)$. For all $t \in [0, T]$, we have*

$$\mathcal{C}_j(\beta,t) \geq \lambda_j^{-1} \widetilde{C}(\tau, \beta, \lambda_1). \tag{2.7}$$

Besides, there exists $\mathcal{M}$ such that

$$\int_0^T \left|\mathcal{C}_j(\beta,t)\right|^2 dt \le \frac{1}{\lambda_j^2}\frac{\mathcal{M}^2 T^{2\beta-1}}{2\beta-1}, \tag{2.8}$$

where $\widetilde{C}(\tau,\beta,\lambda_1) = \tau\sin(\beta\pi)\int_0^{+\infty}\frac{e^{-\xi T}\xi^\beta\,d\xi}{\tau^2\xi^{2\beta}+\frac{\xi^2}{\lambda_1^2}+1}$.

Proof We can see that in [29]. □

Lemma 2.5 *Let us assume that* $\beta\in(\frac{1}{2},1)$. *For all* $t\in[0,T]$, *we have*

$$\mathcal{C}_j(\beta,t) \ge \lambda_j^{-1}\widetilde{C}(\tau,\beta,\lambda_1). \tag{2.9}$$

Lemma 2.6 *Assume that there exist positive constants* $\mathcal{A}_0,\mathcal{A}_1$ *such that* $\mathcal{A}_0\le|\varphi(t)|\le\mathcal{A}_1$ $\forall t\in[0,T]$. *Let us choose* $\epsilon\in(0,\frac{\mathcal{A}_0}{4})$, *it gives*

$$4^{-1}\mathcal{A}_0 \le \left|\varphi_\epsilon(t)\right| \le \mathcal{A}_1 + 4^{-1}\mathcal{A}_0. \tag{2.10}$$

Proof From now on, for short, we denote $\mathcal{B}_0^1 = \mathcal{A}_1 + 4^{-1}\mathcal{A}_0$. For the proof of this lemma, readers can see document [30]. □

Lemma 2.7 *For constant* $C_1>0$, $\gamma>0$, $0<a<1$, $s\ge\lambda_1>0$, *it gives*

$$G(s) = \frac{s}{\gamma s^{a+1}+C_1^{a+1}} \le \left(\frac{1}{C_1}\right)^a a^{\frac{a}{a+1}}\gamma^{-\frac{1}{a+1}}. \tag{2.11}$$

Proof See document [31]. □

Lemma 2.8 *For constant* $\gamma>0$, $C_1>0$, $s\ge\lambda_1>0$, $0<a<1$, *we get*

$$G_1(s) = \frac{\gamma s^{a+1-\frac{\delta}{2}}}{\gamma s^{a+1}+C_1^{a+1}} \le \begin{cases} C_2\gamma^{\frac{\delta}{2a+2}}, & 0<\delta<2a+2,\\ C_3\gamma, & \delta\ge 2a+2,\end{cases} \tag{2.12}$$

where $C_2 = (\frac{2a+2-\delta}{\delta})^{-\frac{\delta}{2a+2}}C_1^{-\frac{\delta}{2}}$, $C_3 = \frac{1}{C_1^{a+1}\lambda_1^{\frac{\delta}{2}-a-1}}$.

Proof

(i) For $0<\delta<2a+2$, we have $s_0 = (\frac{2a+2-\delta}{\gamma\delta})^{\frac{1}{a+1}}C_1$ to make $G_1'(s_0)=0$. Then

$$G_1(s) \le G_1(s_0) = \frac{\gamma(\frac{2a+2-\delta}{\gamma\delta})^{1-\frac{\delta}{2(a+1)}}C_1^{a+1-\frac{\delta}{2}}}{(\frac{2a+2}{\delta})C_1^{a+1}} \le \frac{(\frac{2a+2-\delta}{\delta})^{-\frac{\delta}{2a+2}}}{C_1^{\frac{\delta}{2}}}\gamma^{\frac{\delta}{2(a+1)}}. \tag{2.13}$$

(ii) For $\delta\ge 2a+2$, then it gives

$$G_1(s) = \frac{\gamma s^{a+1-\frac{\delta}{2}}}{\gamma s^{a+1}+C_1^{a+1}} \le \frac{\gamma s^{a+1-\frac{\delta}{2}}}{C_1^{a+1}} \le \frac{\gamma}{C_1^{a+1}\lambda_1^{\frac{\delta}{2}-a-1}}. \tag{2.14}$$

□

Lemma 2.9 *For constants* $\gamma > 0$, $C_4 > 0$, $s \geq \lambda_1 > 0$, *and* $a \in (0,1)$, *we get*

$$G_2(s) = \frac{s}{\gamma s^{2a} + C_4^{2a}} \leq \frac{C_4^{1-2a}(2a-1)^{1-\frac{1}{2a}}}{2a} \gamma^{-\frac{1}{2a}}. \tag{2.15}$$

Proof Proof of this lemma can be found in document [31]. □

Lemma 2.10 *For constants* $\gamma > 0$, $C_5 > 0$, $s \geq \lambda_1 > 0$, *and* $a \in [\frac{1}{2}, 1)$, *we have*

$$G_3(s) = \frac{\gamma s^{2a-\frac{\delta}{2}}}{\gamma s^{2a} + C_5^{2a}} \leq \begin{cases} C_6 \gamma^{\frac{\delta}{4a}}, & 0 < \delta < 4a, \\ C_7 \gamma, & \delta \geq 4a, \end{cases} \tag{2.16}$$

where $C_6 = \delta^{\frac{\delta}{4a}} \frac{(4a-\delta)^{1-\frac{\delta}{4a}}}{4a} C_5^{-\frac{\delta}{4a}}$, $C_7 = \frac{1}{C_5^{2a} \lambda_1^{\frac{\delta}{2}-2a}}$.

Proof The proof is similar to that in Lemma 2.8. □

3 The non-well-posedness of problem (1.1)

Theorem 3.1 *Problem* (1.1) *is unstable.*

Proof Let $P : L^2(\Omega) \to L^2(\Omega)$ be a linear operator as follows:

$$Pf(x) = \sum_{j=1}^{\infty} \left[\int_0^T \left(\int_0^t C_j(\beta, t-z) \varphi(z)\, dz \right) dt \right] \langle f, e_j \rangle e_j(x) = \int_\Omega q(x,\zeta) f(\zeta)\, d\zeta, \tag{3.1}$$

where $q(x,\zeta) = \sum_{j=1}^{\infty} [\int_0^T (\int_0^t C_j(\beta, t-z)\varphi(z)\, dz)\, dt] e_j(x) e_j(\zeta)$. Due to $q(x,\zeta) = q(\zeta, x)$, we know that P is a self-adjoint operator. Define the finite rank operators $P_{\mathcal{N}}$ by

$$P_{\mathcal{N}} f(x) = \sum_{j=1}^{\mathcal{N}} \left[\int_0^T \left(\int_0^t C_j(\beta, t-z) \varphi(z)\, dz \right) dt \right] \langle f, e_j \rangle e_j(x). \tag{3.2}$$

From (3.1) and (3.2), we have

$$\begin{aligned} \|P_{\mathcal{N}} f - Pf\|_{L^2(\Omega)}^2 &= \sum_{j=\mathcal{N}+1}^{\infty} \left[\int_0^T \left(\int_0^t C_j(\beta, t-z) \varphi(z)\, dz \right) dt \right]^2 \left| \langle f, e_j \rangle \right|^2 \\ &\leq \mathcal{A}_1^2 \sum_{j=\mathcal{N}+1}^{\infty} \frac{\mathcal{M}^2}{\lambda_j^2} \frac{T^{2\beta+2}}{2\beta-1} \left| \langle f, e_j \rangle \right|^2 \leq \mathcal{A}_1^2 \frac{\mathcal{M}^2}{\lambda_{\mathcal{N}}^2} \frac{T^{2\beta+2}}{2\beta-1} \sum_{j=\mathcal{N}+1}^{\infty} \left| \langle f, e_j \rangle \right|^2. \end{aligned} \tag{3.3}$$

From (3.3), we can know that

$$\|P_{\mathcal{N}} f - Pf\|_{L^2(\Omega)} \leq \frac{1}{\lambda_{\mathcal{N}}} \frac{\mathcal{M} \mathcal{A}_1 T^{\beta+1}}{\sqrt{2\beta-1}} \|f\|_{L^2(\Omega)}. \tag{3.4}$$

Hence, we can deduce that

$$\lim_{\mathcal{N} \to \infty} \|P_{\mathcal{N}} - P\|_{L(L^2(\Omega); L^2(\Omega))} = 0.$$

So, we get immediately that P is a compact operator. From (3.1), the inverse source problem can be formulated as an operator equation $Pf(x) = \ell(x)$, and by Kirsch [32], it is unstable. Next, we propose an example, with input final data $\ell^k = \frac{e_k}{\sqrt{\lambda_k}}$. By (2.5), f^k depending on ℓ^k is

$$f^k(x) = \sum_{j=1}^{\infty} \frac{\langle \frac{e_k}{\sqrt{\lambda_k}}, e_j\rangle}{\int_0^T (\int_0^t \mathcal{C}_j(\beta, t-z)\varphi(z)\,dz)\,dt} = \frac{e_k}{\sqrt{\lambda_k}\int_0^T (\int_0^t \mathcal{C}_k(\beta, t-z)\varphi(z)\,dz)\,dt}. \tag{3.5}$$

If we choose $\ell = 0$, then $f = 0$, an error in L^2-norm between ℓ^k and ℓ is

$$\|\ell^k - \ell\|_{L^2(\Omega)} = \frac{1}{\sqrt{\lambda_k}} \quad \rightarrow \quad \lim_{k\to+\infty}\|\ell^k - \ell\|_{L^2(\Omega)} = \lim_{k\to+\infty}\left(\frac{1}{\sqrt{\lambda_k}}\right) = 0. \tag{3.6}$$

And an error in L^2 norm between f^k and f is

$$\|f^k - f\|^2_{L^2(\Omega)} = \lambda_k^{-1}\left(\int_0^T \left(\int_0^t \mathcal{C}_k(\beta, t-z)\varphi(z)\,dz\right)dt\right)^{-2}. \tag{3.7}$$

From (3.7) and combining with Lemma 2.4, one has

$$\left|\int_0^T \left(\int_0^t \mathcal{C}_k(\beta, t-z)\varphi(z)\,dz\right)dt\right|^2 \le \frac{\mathcal{A}_1^2\mathcal{M}^2}{\lambda_k^2}\frac{T^{2\beta+2}}{2\beta-1}, \tag{3.8}$$

we have

$$\|f^k - f\|^2_{L^2(\Omega)} \ge \frac{\lambda_k}{\mathcal{M}^2\mathcal{A}_1^2}\left(\frac{2\beta-1}{T^{2\beta+2}}\right). \tag{3.9}$$

By choosing $\beta > \frac{1}{2}$, we get

$$\lim_{k\to+\infty}\|f^k - f\|_{L^2(\Omega)} \ge \lim_{k\to+\infty}\frac{\sqrt{\lambda_k}}{\mathcal{M}\mathcal{A}_1}\left(\frac{\sqrt{2\beta-1}}{T^{\beta+1}}\right) = +\infty. \tag{3.10}$$

Combining (3.6) and (3.10), this implies that problem (1.1) is non-well-posed. □

Next, we give the following theorem which shows the conditional stability of the function f.

Theorem 3.2 *Assume that $\|f\|_{\mathbb{H}^\delta(\Omega)} \le R$ for $R > 0$, then it gives*

$$\begin{aligned}&\|f\|_{L^2(\Omega)} \le \mathcal{D}(\delta, T)R^{\frac{1}{\delta+1}}\|\ell\|^{\frac{\delta}{\delta+1}}_{L^2(\Omega)},\\ &\text{where } \mathcal{D}(\delta, T) = \left[\mathcal{A}_0^{\frac{\delta}{\delta+1}}\left(\frac{T^2\widetilde{C}(\tau, \beta, \lambda_1)}{2}\right)^{\frac{\delta}{\delta+1}}\right]^{-1}.\end{aligned} \tag{3.11}$$

Proof From (2.5) and using the Hölder inequality, one has

$$\|f\|^2_{L^2(\Omega)} = \sum_{j=1}^{\infty}\left|\frac{\langle \ell, e_j\rangle}{\int_0^T (\int_0^t \mathcal{C}_j(\beta, t-z)\varphi(z)\,dz)\,dt}\right|^2$$

$$
\begin{aligned}
&= \sum_{j=1}^{\infty} \frac{|\langle \ell, e_j\rangle|^{\frac{2}{\delta+1}} |\langle \ell, e_j\rangle|^{\frac{2\delta}{\delta+1}}}{|\int_0^T (\int_0^t \mathcal{C}_j(\beta, t-z)\varphi(z)\,dz)\,dt|^2} \\
&\leq \left[\sum_{j=1}^{\infty} \frac{|\langle \ell, e_j\rangle|^2}{|\int_0^T (\int_0^t \mathcal{C}_j(\beta, t-z)\varphi(z)\,dz)\,dt|^{2\delta+2}}\right]^{\frac{1}{\delta+1}} \left[\sum_{j=1}^{\infty} \left|\langle \ell, e_j\rangle\right|^2\right]^{\frac{\delta}{\delta+1}} \\
&\leq \left[\sum_{j=1}^{\infty} \frac{|\langle f, e_j\rangle|^2}{|\int_0^T (\int_0^t \mathcal{C}_j(\beta, t-z)\varphi(z)\,dz)\,dt|^{2\delta}}\right]^{\frac{1}{\delta+1}} \|\ell\|_{L^2(\Omega)}^{\frac{2\delta}{\delta+1}}.
\end{aligned}
\tag{3.12}
$$

From Lemma 2.4, we can calculate that $\int_0^t \mathcal{C}_j(\beta, z)\,dz \geq \frac{t\widetilde{\mathcal{C}}(\tau,\beta,\lambda_1)}{\lambda_j}$, and this implies that $\int_0^T \frac{t\widetilde{\mathcal{C}}(\tau,\beta,\lambda_1)}{\lambda_j}\,dt = \frac{\widetilde{\mathcal{C}}(\tau,\beta,\lambda_1)}{\lambda_j}\int_0^T t\,dt = \frac{1}{\lambda_j}\frac{T^2\widetilde{\mathcal{C}}(\tau,\beta,\lambda_1)}{2}$. Using Lemma 2.4 gives

$$
\begin{aligned}
\sum_{j=1}^{\infty} \frac{|\langle f, e_j\rangle|^2}{|\int_0^T (\int_0^t \mathcal{C}_j(\beta, t-z)\varphi(z)\,dz)\,dt|^{2\delta}} &\leq \sum_{j=1}^{\infty} \frac{\lambda_j^{2\delta}|\langle f, e_j\rangle|^2}{\mathcal{A}_0^{2\delta}(\frac{T^2\widetilde{\mathcal{C}}(\tau,\beta,\lambda_1)}{2})^{2\delta}} \\
&= \frac{\|f\|_{\mathbb{H}^\delta(\Omega)}^2}{\mathcal{A}_0^{2\delta}(\frac{T^2\widetilde{\mathcal{C}}(\tau,\beta,\lambda_1)}{2})^{2\delta}}.
\end{aligned}
\tag{3.13}
$$

Combining (3.12) and (3.13), we have

$$
\|f\|_{L^2(\Omega)}^2 \leq \frac{\|f\|_{\mathbb{H}^\delta(\Omega)}^{\frac{2}{\delta+1}} \|\ell\|_{L^2(\Omega)}^{\frac{2\delta}{\delta+1}}}{\mathcal{A}_0^{\frac{2\delta}{\delta+1}}(\frac{T^2\widetilde{\mathcal{C}}(\tau,\beta,\lambda_1)}{2})^{\frac{2\delta}{\delta+1}}} \leq \left[\mathcal{D}(\delta, T)\right]^2 R^{\frac{2}{\delta+1}} \|\ell\|_{L^2(\Omega)}^{\frac{2\delta}{\delta+1}}. \tag{3.14}
$$

□

4 The fractional Tikhonov regularization method

In this section, we solve problem (1.1) by using the fractional Tikhonov method. The ideas of this method are based on the work of Hochstenbach in [33] or Yang in [31]. We use two kinds of fractional Tikhonov regularization methods to solve (1.1) as follows:

$$
\min_{f\in L^2(\Omega)} \left\{\|Pf - \ell\|_Y^2 + \left[\gamma(\epsilon)\right]\|f\|^2\right\}, \tag{4.1}
$$

in which $\|\cdot\|_Y$ is a weighted seminorm as $\|\sigma\|_Y = \|Y^{\frac{1}{2}}\sigma\|$ for any σ. We propose

$$
Y = \left(P^* P\right)^{\frac{a-1}{2}}. \tag{4.2}
$$

With the Tikhonov minimization problem (4.1) with Y defined by (4.2) given by

$$
\left(\left(P^* P\right)^{\frac{a+1}{2}} + \left[\gamma(\epsilon)\right]I\right)f^{[\gamma(\epsilon)]} = \left(P^* P\right)^{\frac{a-1}{2}} P^* \ell, \tag{4.3}
$$

the solution of (4.3) is uniquely determined for any $\gamma > 0$ and $a > 0$. It is obvious to see that the formula of $f^{[\gamma(\epsilon)]_1}$ is as follows:

$$
f^{[\gamma(\epsilon)]_1}(x) = \sum_{j=1}^{\infty} \frac{|\int_0^T (\int_0^t \mathcal{C}_j(\beta, t-z)\varphi(z)\,dz)\,dt|^a}{[\gamma(\epsilon)]_1 + |\int_0^T (\int_0^t \mathcal{C}_j(\beta, t-z)\varphi(z)\,dz)\,dt|^{a+1}} \langle \ell, e_j\rangle e_j(x). \tag{4.4}
$$

We have the fractional Tikhonov regularized solution

$$f_\epsilon^{[\gamma(\epsilon)]_1}(x) = \sum_{j=1}^{\infty} \frac{|\int_0^T (\int_0^t \mathcal{C}_j(\beta, t-z)\varphi_\epsilon(z)\,dz)\,dt|^a}{[\gamma(\epsilon)]_1 + |\int_0^T (\int_0^t \mathcal{C}_j(\beta, t-z)\varphi_\epsilon(z)\,dz)\,dt|^{a+1}} \langle \ell_\epsilon, e_j\rangle e_j(x). \tag{4.5}$$

Refer to [34], another type of fractional Tikhonov regularized solution is given by the following formula:

$$f^{[\gamma(\epsilon)]_2}(x) = \sum_{j=1}^{\infty} \frac{|\int_0^T (\int_0^t \mathcal{C}_j(\beta, t-z)\varphi(z)\,dz)\,dt|^{2a-1}}{[\gamma(\epsilon)]_2 + |\int_0^T (\int_0^t \mathcal{C}_j(\beta, t-z)\varphi(z)\,dz)\,dt|^{2a}} \langle \ell, e_j\rangle e_j(x), \tag{4.6}$$

where $[\gamma(\epsilon)]_2$ is the regularized parameter, with $\frac{1}{2} \le a < 1$. For the noisy data, we get

$$f_\epsilon^{[\gamma(\epsilon)]_2}(x) = \sum_{j=1}^{\infty} \frac{|\int_0^T (\int_0^t \mathcal{C}_j(\beta, t-z)\varphi_\epsilon(z)\,dz)\,dt|^{2a-1}}{[\gamma(\epsilon)]_2 + |\int_0^T (\int_0^t \mathcal{C}_j(\beta, t-z)\varphi_\epsilon(z)\,dz)\,dt|^{2a}} \langle \ell_\epsilon, e_j\rangle e_j(x). \tag{4.7}$$

Putting $|\int_0^T (\int_0^t \mathcal{C}_j(\beta, t-z)\varphi(z)\,dz)\,dt| = \mathcal{D}_j(\beta, \varphi)$, we have

$$f^{[\gamma(\epsilon)]_1}(x) = \sum_{j=1}^{\infty} \frac{|\mathcal{D}_j(\beta, \varphi)|^a}{[\gamma(\epsilon)]_1 + |\mathcal{D}_j(\beta, \varphi)|^{a+1}} \langle \ell, e_j\rangle e_j(x) \tag{4.8}$$

and

$$f^{[\gamma(\epsilon)]_2}(x) = \sum_{j=1}^{\infty} \frac{|\mathcal{D}_j(\beta, \varphi)|^{2a-1}}{[\gamma(\epsilon)]_2 + |\mathcal{D}_j(\beta, \varphi)|^{2a}} \langle \ell, e_j\rangle e_j(x). \tag{4.9}$$

Next, we continue to investigate the convergence rates in two various cases.

4.1 The choices of regularization parameter $[\gamma(\epsilon)]_j$, $j = 1, 2$, and convergence results

4.1.1 An a priori parameter choice rule

Theorem 4.1 *Let the function f be as in formula (2.5), and assume that condition (1.2) holds. Suppose that a priori condition (3.11) holds. By choosing the parameter regularization as follows*:

$$\big\|f_\epsilon^{[\gamma(\epsilon)]_1} - f\big\|_{L^2(\Omega)} \quad \textit{is of order} \begin{cases} \epsilon^{\frac{\delta}{\delta+2}} & \textit{if } 0 < \delta < 2a+2, \\ \epsilon^{\frac{a+1}{a+2}} & \textit{if } \delta \ge 2a+2. \end{cases} \tag{4.10}$$

Proof We get

$$\big\|f_\epsilon^{[\gamma(\epsilon)]_1} - f\big\|_{L^2(\Omega)} \le \underbrace{\big\|f_\epsilon^{[\gamma(\epsilon)]_1} - f^{[\gamma(\epsilon)]_1}\big\|_{L^2(\Omega)}}_{\mathcal{K}_1} + \underbrace{\big\|f^{[\gamma(\epsilon)]_1} - f\big\|_{L^2(\Omega)}}_{\mathcal{K}_2}.$$

Next, we evaluate $\mathcal{K}_1$ for the error assessment:

$$f_\epsilon^{[\gamma(\epsilon)]_1}(x) - f^{[\gamma(\epsilon)]_1}(x)$$

$$= \sum_{j=1}^{\infty} \frac{|\mathcal{D}_j(\beta,\varphi_\epsilon)|^a}{[\gamma(\epsilon)]_1 + |\mathcal{D}_j(\beta,\varphi_\epsilon)|^{a+1}} \langle \ell_\epsilon - \ell, e_j\rangle e_j(x)$$
$$+ \sum_{j=1}^{\infty} \left[\frac{|\mathcal{D}_j(\beta,\varphi_\epsilon)|^a}{[\gamma(\epsilon)]_1 + |\mathcal{D}_j(\beta,\varphi_\epsilon)|^{a+1}} - \frac{|\mathcal{D}_j(\beta,\varphi)|^a}{[\gamma(\epsilon)]_1 + |\mathcal{D}_j(\beta,\varphi)|^{a+1}} \right] \langle \ell, e_j\rangle e_j(x). \tag{4.11}$$

Squaring the two sides, getting the standard in space $L^2(\Omega)$, applying a familiar inequality, we have

$$\begin{aligned} \mathcal{K}_1^2 &\le \frac{2|\mathcal{D}_j(\beta,\varphi_\epsilon)|^{2a}}{|[\gamma(\epsilon)]_1 + |\mathcal{D}_j(\beta,\varphi_\epsilon)|^{a+1}|^2} \|\ell - \ell_\epsilon\|^2_{L^2(\Omega)}\}\mathcal{S}_1^2 \\ &+ \frac{4[\gamma(\epsilon)]_1^2[|\mathcal{D}_j(\beta,\varphi_\epsilon)|^a - |\mathcal{D}_j(\beta,\varphi)|^a]^2}{|[\gamma(\epsilon)]_1 + |\mathcal{D}_j(\beta,\varphi_\epsilon)|^{a+1}|^2|[\gamma(\epsilon)]_1 + |\mathcal{D}_j(\beta,\varphi)|^{a+1}|^2} \left|\langle \ell, e_j\rangle\right|^2\}\mathcal{S}_2^2 \\ &+ \frac{4|\mathcal{D}_j(\beta,\varphi)|^{2a}|\mathcal{D}_j(\beta,\varphi_\epsilon)|^{2a}|\mathcal{D}_j(\beta,\varphi-\varphi_\epsilon)|^2}{|[\gamma(\epsilon)]_1 + |\mathcal{D}_j(\beta,\varphi_\epsilon)|^{a+1}|^2|[\gamma(\epsilon)]_1 + |\mathcal{D}_j(\beta,\varphi)|^{a+1}|^2} \left|\langle \ell, e_j\rangle\right|^2\}\mathcal{S}_3^2. \end{aligned} \tag{4.12}$$

Step 1: Estimate $\mathcal{S}_1$ by denoting $Q(\mathcal{M},T,\beta) = \frac{\mathcal{M}^2 T^{2\beta+2}}{2\beta-1}$, and $\mathcal{B}_0^1$ is defined in Lemma 2.6. Combining the Holder inequality, we have

$$\begin{aligned} \mathcal{S}_1^2 &\le 2\big[B_0^1\big]^{2a}\big[Q(\mathcal{M},T,\beta)\big]^{2a}\lambda_j^{-2a}\left|\frac{\lambda_j^{a+1}\|\ell_\epsilon - \ell\|_{L^2(\Omega)}}{[\gamma(\epsilon)]_1\lambda_j^{a+1} + |8^{-1}\mathcal{A}_0|^{a+1}|T^2\widetilde{C}(\tau,\beta,\lambda_1)|^{a+1}}\right|^2 \\ &\le 2\epsilon^2\big[B_0^1\big]^{2a}\big[Q(\mathcal{M},T,\beta)\big]^{2a}\lambda_j^2\big|\big[\gamma(\epsilon)\big]_1\lambda_j^{a+1} + \big(8^{-1}\mathcal{A}_0\big)^{a+1}\big|T^2\widetilde{C}(\tau,\beta,\lambda_1)\big|^{a+1}\big|^{-2} \\ &\le 2\epsilon^2\big[B_0^1\big]^{2a}\big[Q(\mathcal{M},T,\beta)\big]^{2a}\left(\frac{8}{\mathcal{A}_0 T^2\widetilde{C}(\tau,\beta,\lambda_1)}\right)^{2a} a^{\frac{2a}{a+1}}\big[\gamma(\epsilon)\big]^{-\frac{2}{a+1}}. \end{aligned} \tag{4.13}$$

Step 2: Estimate $\mathcal{S}_2$ as follows. Before going into evaluation $\mathcal{S}_2^2$, we have inequality for $a \in (0,1)$, $0 < y_0 < y_1$, then $|y_1^a - y_0^a| \le |y_1 - y_0|^a$, this implies that

$$\big|\mathcal{D}_j(\beta,\varphi_\epsilon)\big|^a - \big|\mathcal{D}_j(\beta,\varphi)\big|^a \le \big|\mathcal{D}_j(\beta,\varphi_\epsilon - \varphi)\big|^a \le \epsilon^a\big|\mathcal{D}_j(\beta)\big|^a.$$

From Lemma 2.10, we denote $Q_2^2 = \frac{4\mathcal{A}_1}{(8^{-1}\mathcal{A}_0)^{2a+2}}$, we have

$$\begin{aligned} \mathcal{S}_2^2 &\le \frac{4[\gamma(\epsilon)]_1^2[|\mathcal{D}_j(\beta,\varphi_\epsilon)|^a - |\mathcal{D}_j(\beta,\varphi)|^a]^2}{|[\gamma(\epsilon)]_1 + |\mathcal{D}_j(\beta,\varphi_\epsilon)|^{a+1}|^2 \times |[\gamma(\epsilon)]_1 + |\mathcal{D}_j(\beta,\varphi)|^{a+1}|^2}\left|\langle \ell, e_j\rangle\right|^2 \\ &\le \epsilon^{2a}Q_2^2\left(\frac{[\gamma(\epsilon)]_1\lambda_j^{-\frac{\delta}{2}}}{[\gamma(\epsilon)]_1 + |\mathcal{D}_j(\beta,\varphi_\epsilon)|^{a+1}}\right)^2\|f\|^2_{\mathbb{H}^\delta(\Omega)} \\ &\le \epsilon^{2a}Q_2^2\left(\frac{[\gamma(\epsilon)]_1\lambda_j^{a+1-\frac{\delta}{2}}}{[\gamma(\epsilon)]_1\lambda_j^{a+1} + |(8^{-1}\mathcal{A}_0)^{a+1}T^2\widetilde{C}(\beta,\tau,\lambda_1)|^{a+1}}\right)^2\|f\|^2_{\mathbb{H}^\delta(\Omega)} \\ &\le \epsilon^{2a}Q_2^2\sup_{j\ge1}\left(\frac{[\gamma(\epsilon)]_1\lambda_j^{a+1-\frac{\delta}{2}}}{[\gamma(\epsilon)]_1\lambda_j^{a+1} + |(8^{-1}\mathcal{A}_0)^{a+1}T^2\widetilde{C}(\beta,\tau,\lambda_1)|^{a+1}}\right)^2\|f\|^2_{\mathbb{H}^\delta(\Omega)} \\ &\le \epsilon^{2a}Q_2^2\begin{cases} C_2^2[\gamma(\epsilon)]_1^{\frac{2\delta}{2a+2}}R^2, & 0 < \delta < 2a+2, \\ C_3^2[\gamma(\epsilon)]_1^2 R^2, & \delta \ge 2a+2. \end{cases} \end{aligned} \tag{4.14}$$

Step 3: Applying Lemma 2.7, $\mathcal{S}_3^2$ can be bounded:

$$\begin{aligned}
\mathcal{S}_3^2 &\leq 4\big|\mathcal{D}_j(\beta,\varphi-\varphi_\epsilon)\big|^2 \\
&\quad\times \frac{|\mathcal{D}_j(\beta,\varphi_\epsilon)|^{2a}}{|[\gamma(\epsilon)]_1+|\mathcal{D}_j(\beta,\varphi_\epsilon)|^{a+1}|^2}\frac{|\mathcal{D}_j(\beta,\varphi)|^{2a+2}}{|[\gamma(\epsilon)]_1+|\mathcal{D}_j(\beta,\varphi)|^{a+1}|^2}\frac{|\langle\ell,e_j\rangle|^2}{|\mathcal{D}_j(\beta,\varphi)|^2} \\
&\leq 4\epsilon^2\frac{|\mathcal{D}_j(\beta,\varphi_\epsilon)|^{2a}}{|[\gamma(\epsilon)]_1+|\mathcal{D}_j(\beta,\varphi_\epsilon)|^{a+1}|^2}\lambda_j^{-\delta}\big|\mathcal{D}_j(\beta)\big|^2\frac{\lambda_j^\delta|\langle\ell,e_j\rangle|^2}{|\mathcal{D}_j(\beta,\varphi)|^2} \\
&\leq 4\epsilon^2\big[\mathcal{B}_0^1\big]^{2a}\left|\frac{[|\mathcal{D}_j(\beta)|^2]^{\frac{a+1}{2}}}{[\gamma(\epsilon)]_1+(8^{-1}\mathcal{A}_0)^{a+1}|\frac{T^2\widetilde{C}(\tau,\beta,\lambda_1)}{\lambda_j}|^{a+1}}\right|^2\lambda_j^\delta\big|\langle f,e_j\rangle\big|^2.
\end{aligned}\tag{4.15}$$

From the estimation of (3.3), denoting $Q_3=\frac{4}{\lambda_1^{2+\delta}}[\mathcal{B}_0^1]^{2a}(\frac{\mathcal{A}_1\mathcal{M}T^{2\beta+2}}{2\beta-1})^{2(a+1)}$, we get

$$\begin{aligned}
\mathcal{S}_3^2 &\leq \epsilon^2 4\big[\mathcal{B}_0^1\big]^{2a}\left(\frac{\mathcal{A}_1\mathcal{M}T^{2\beta+2}}{2\beta-1}\right)^{2(a+1)}\left(\frac{\lambda_j^{-a-1-\frac{\delta}{2}}}{[\gamma(\epsilon)]_1+|\frac{(8^{-1}\mathcal{A}_0)T^2\widetilde{C}(\tau,\beta,\lambda_1)}{\lambda_j}|^{a+1}}\right)^2\|f\|^2_{\mathbb{H}^\delta(\Omega)} \\
&\leq Q_3^2\epsilon^2\sup_{j\geq 1}\left(\frac{\lambda_j}{[\gamma(\epsilon)]_1\lambda_j^{a+1}+|(8^{-1}\mathcal{A}_0)T^2\widetilde{C}(\tau,\beta,\lambda_1)|^{a+1}}\right)^2\|f\|^2_{\mathbb{H}^\delta(\Omega)} \\
&\leq Q_3^2\epsilon^2\big[\gamma(\epsilon)\big]^{-\frac{2}{a+1}}R^2.
\end{aligned}\tag{4.16}$$

Combining (4.13), (4.14), and (4.15), we receive

$$\begin{aligned}
\mathcal{K}_1 &\leq \sqrt{2}\epsilon\big[\mathcal{B}_0^1\big]^a\big[Q(\mathcal{M},T,\beta)\big]^a\left(\frac{8}{\mathcal{A}_0T^2\widetilde{C}(\tau,\beta,\lambda_1)}\right)^a a^{\frac{a}{a+1}}\big[\gamma(\epsilon)\big]^{-\frac{1}{a+1}}+\epsilon Q_3R \\
&\quad+\epsilon^aQ_2\begin{cases} C_2[\gamma(\epsilon)]_1^{\frac{\delta}{2a+2}}R, & 0<\delta<2a+2, \\ C_3[\gamma(\epsilon)]_1R, & \delta\geq 2a+2.\end{cases}
\end{aligned}\tag{4.17}$$

And we show the error estimation for $\mathcal{K}_2$:

$$\mathcal{K}_2=\sum_{j=1}^\infty\left(\frac{|\mathcal{D}_j(\beta,\varphi)|^a}{[\gamma(\epsilon)]_1+|\mathcal{D}_j(\beta,\varphi)|^{a+1}}-\frac{1}{\mathcal{D}_j(\beta,\varphi)}\right)\langle\ell,e_j\rangle e_j(x).\tag{4.18}$$

Finally, we estimate $\mathcal{K}_2$. Squaring the two sides, using the Cauchy inequality, we get

$$\begin{aligned}
\mathcal{K}_2^2 &\leq \sum_{j=1}^\infty\left(\frac{[\gamma(\epsilon)]_1}{|[\gamma(\epsilon)]_1+|\mathcal{D}_j(\beta,\varphi)|^{a+1}|}\right)^2\frac{|\langle\ell,e_j\rangle|^2}{|\mathcal{D}_j(\beta,\varphi)|^2} \\
&\leq \sum_{j=1}^\infty\left|\frac{[\gamma(\epsilon)]_1\lambda_j^{-\frac{\delta}{2}}}{[\gamma(\epsilon)]_1+|\mathcal{D}_j(\beta,\varphi)|^{a+1}}\right|^2\lambda_j^\delta\big|\langle f,e_j\rangle\big|^2 \\
&\leq \sup_{j\geq 1}\left(\frac{[\gamma(\epsilon)]_1\lambda_j^{-\frac{\delta}{2}}}{[\gamma(\epsilon)]_1+|\frac{\mathcal{A}_0T^2\widetilde{C}(\tau,\beta,\lambda_1)}{2\lambda_j}|^{a+1}}\right)^2\|f\|^2_{\mathbb{H}^\delta(\Omega)} \\
&\leq \sup_{j\geq 1}\left(\frac{[\gamma(\epsilon)]_1\lambda_j^{a+1-\frac{\delta}{2}}}{[\gamma(\epsilon)]_1\lambda_j^{a+1}+|(2^{-1}\mathcal{A}_0)T^2\widetilde{C}(\tau,\beta,\lambda_1)|^{a+1}}\right)^2\|f\|^2_{\mathbb{H}^\delta(\Omega)}
\end{aligned}$$

$$\leq \begin{cases} C_2^2[\gamma(\epsilon)]_1^{\frac{2\delta}{2a+2}} R^2, & 0 < \delta < 2a+2, \\ C_3^2[\gamma(\epsilon)]_1^2 R^2, & \delta \geq 2a+2. \end{cases} \tag{4.19}$$

From estimation for $\mathcal{K}_1$ and $\mathcal{K}_2$, we conclude the following:

(i) *If* $0 < \delta < 2a+2$, *then*

$$\begin{aligned} &\left\| f_\epsilon^{[\gamma(\epsilon)]_1} - f \right\|_{L^2(\Omega)} \\ &\quad \leq \sqrt{2}\epsilon \left[B_0^1\right]^a \left[Q(\mathcal{M}, T, \beta)\right]^a \left(\frac{8}{\mathcal{A}_0 T^2 \widetilde{C}(\tau, \beta, \lambda_1)} \right)^a a^{\frac{a}{a+1}} \left[\gamma(\epsilon)\right]^{-\frac{1}{a+1}} \\ &\quad\quad + \epsilon R Q_3 + \left(\epsilon^a Q_2 + 1\right) C_2 \left[\gamma(\epsilon)\right]_1^{\frac{\delta}{2a+2}} R. \end{aligned} \tag{4.20}$$

(ii) *If* $\delta \geq 2a+2$, *then*

$$\begin{aligned} &\left\| f_\epsilon^{[\gamma(\epsilon)]_1} - f \right\|_{L^2(\Omega)} \\ &\quad \leq \sqrt{2}\epsilon \left[B_0^1\right]^a \left[Q(\mathcal{M}, T, \beta)\right]^a \left(\frac{8}{\mathcal{A}_0 T^2 \widetilde{C}(\tau, \beta, \lambda_1)} \right)^a a^{\frac{a}{a+1}} \left[\gamma(\epsilon)\right]^{-\frac{1}{a+1}} \\ &\quad\quad + \epsilon R Q_3 + \left(\epsilon^a Q_2 + 1\right) C_3 \left[\gamma(\epsilon)\right]_1 R. \end{aligned} \tag{4.21}$$

The regularization parameter $[\gamma(\epsilon)]_1$ by

$$\left[\gamma(\epsilon)\right]_1 = \begin{cases} \left(\frac{\epsilon}{R}\right)^{\frac{2a+2}{\delta+2}}, & 0 < \delta < 2a+2, \\ \left(\frac{\epsilon}{R}\right)^{\frac{a+1}{a+2}}, & \delta \geq 2a+2. \end{cases} \tag{4.22}$$

Hence, we conclude the following:

(i) *If* $0 < \delta < 2a+2$, *then*

$$\left\| f_\epsilon^{[\gamma(\epsilon)]_1} - f \right\|_{L^2(\Omega)} \leq \text{ is of order } \epsilon^{\frac{\delta}{\delta+2}}. \tag{4.23}$$

(ii) *If* $\delta \geq 2a+2$, *then*

$$\left\| f_\epsilon^{[\gamma(\epsilon)]_1} - f \right\|_{L^2(\Omega)} \leq \text{is of order } \epsilon^{\frac{a+1}{a+2}}. \tag{4.24}$$

Proof is completed. □

Theorem 4.2 *Let f be as* (2.5) *and $f_\epsilon^{[\gamma(\epsilon)]_2}$ be given by* (4.7). *Suppose that condition* (1.2) *holds. f satisfies condition* (3.11). *By choosing*

$$\left[\gamma(\epsilon)\right]_2 = \begin{cases} \left(\frac{\epsilon}{R}\right)^{\frac{4a}{\delta+2}}, & 0 < \delta < 4a, \\ \left(\frac{\epsilon}{R}\right)^{\frac{2a}{1+2a}}, & \delta \geq 4a, \end{cases} \tag{4.25}$$

we have

$$\left\| f_\epsilon^{[\gamma(\epsilon)]_2} - f \right\|_{L^2(\Omega)} \quad \textit{is of order} \begin{cases} \epsilon^{\frac{\delta}{\delta+2}} & \textit{if } 0 < \delta < 4a, \\ \epsilon^{\frac{2a}{1+2a}} & \textit{if } \delta \geq 4a. \end{cases}$$

Proof We have

$$\left\|f_\epsilon^{[\gamma(\epsilon)]_2} - f\right\|_{L^2(\Omega)} \le \left\|f_\epsilon^{[\gamma(\epsilon)]_2} - f^{[\gamma(\epsilon)]_2}\right\|_{L^2(\Omega)} + \left\|f^{[\gamma(\epsilon)]_2} - f\right\|_{L^2(\Omega)}. \tag{4.26}$$

First of all, we receive

$$\begin{aligned} & f_\epsilon^{[\gamma(\epsilon)]_2}(x) - f^{[\gamma(\epsilon)]_2}(x) \\ &\quad = \sum_{j=1}^{\infty} \frac{|\mathcal{D}_j(\beta,\varphi_\epsilon)|^{2a-1}}{[\gamma(\epsilon)]_2 + |\mathcal{D}_j(\beta,\varphi_\epsilon)|^{2a}} \langle \ell_\epsilon - \ell, e_j\rangle e_j(x) \\ &\qquad + \sum_{j=1}^{\infty} \left(\frac{|\mathcal{D}_j(\beta,\varphi_\epsilon)|^{2a-1}}{[\gamma(\epsilon)]_1 + |\mathcal{D}_j(\beta,\varphi_\epsilon)|^{2a}} - \frac{|\mathcal{D}_j(\beta,\varphi)|^{2a-1}}{[\gamma(\epsilon)]_1 + |\mathcal{D}_j(\beta,\varphi)|^{2a}} \right) \langle \ell, e_j\rangle e_j(x). \end{aligned} \tag{4.27}$$

Square the two sides, get the standard in $L^2(\Omega)$ space, it gives

$$\begin{aligned} & \left\|f_\epsilon^{[\gamma(\epsilon)]_2} - f^{[\gamma(\epsilon)]_2}\right\|_{L^2(\Omega)}^2 \\ &\quad \le \frac{2|\mathcal{D}_j(\beta,\varphi_\epsilon)|^{4a-2}}{|[\gamma(\epsilon)]_2 + |\mathcal{D}_j(\beta,\varphi_\epsilon)|^{2a}|^2} \|\ell_\epsilon - \ell\|_{L^2(\Omega)}^2 \}\mathcal{P}_1^2 \\ &\qquad + \frac{4[\gamma(\epsilon)]_2^2[|\mathcal{D}_j(\beta,\varphi_\epsilon)|^{2a-1} - |\mathcal{D}_j(\beta,\varphi)|^{2a-1}]^2}{|[\gamma(\epsilon)]_2 + |\mathcal{D}_j(\beta,\varphi_\epsilon)|^{2a}|^2 \times |[\gamma(\epsilon)]_2 + |\mathcal{D}_j(\beta,\varphi)|^{2a}|^2} \left|\langle \ell, e_j\rangle\right|^2 \}\mathcal{P}_2^2 \\ &\qquad + \frac{4|\mathcal{D}_j(\beta,\varphi)|^{4a-2}|\mathcal{D}_j(\beta,\varphi_\epsilon)|^{4a-2}|\mathcal{D}_j(\beta,\varphi-\varphi_\epsilon)|^2}{|[\gamma(\epsilon)]_2 + |\mathcal{D}_j(\beta,\varphi_\epsilon)|^{2a}|^2 \times |[\gamma(\epsilon)]_2 + |\mathcal{D}_j(\beta,\varphi)|^{2a}|^2} \left|\langle \ell, e_j\rangle\right|^2 \}\mathcal{P}_3^2. \end{aligned} \tag{4.28}$$

Step 1. Estimate $\mathcal{P}_1^2$ by denoting $Q_4^2 = 2[\mathcal{B}_0^1]^{4a-2}[Q(\mathcal{M},T,\beta)]^{2a-1}|(8^{-1}\mathcal{A}_0)T^2\widetilde{C}(\tau,\beta,\lambda_1)|^{4a-2}\frac{(2a-1)^{2-a}}{(2a)^2}$, using Lemma 2.9, we get

$$\begin{aligned} \mathcal{P}_1^2 &\le 2\epsilon^2\left|\mathcal{D}_j(\beta,\varphi_\epsilon)\right|^{4a-2}\left|\left[\gamma(\epsilon)\right]_2 + \left|\mathcal{D}_j(\beta,\varphi_\epsilon)\right|^{2a}\right|^{-2} \\ &\le \epsilon^2 Q_4^2 \lambda_j^{-4a+2}\left|\frac{\lambda_j^{2a}}{[\gamma(\epsilon)]_2\lambda_j^{2a} + (8^{-1}\mathcal{A}_0)^{2a}|T^2\widetilde{C}(\tau,\beta,\lambda_1)|^{2a}}\right|^2 \\ &\le \epsilon^2 Q_4^2\lambda_j^2\left|\left[\gamma(\epsilon)\right]_2\lambda_j^{2a} + \left(8^{-1}\mathcal{A}_0\right)^{2a}\left|T^2\widetilde{C}(\tau,\beta,\lambda_1)\right|^{2a}\right|^{-2} \\ &\le \epsilon^2 Q_4^2\left[\gamma(\epsilon)\right]_2^{-\frac{1}{a}}. \end{aligned} \tag{4.29}$$

Step 2. Estimate $\mathcal{P}_2^2$ by noting $a \in [\frac{1}{2}, 1)$ and $Q_5^2 = 4\mathcal{A}_1^2(4^{-1}\mathcal{A}_0)^{4a-2}\lambda_1^{-4a}[Q(\mathcal{M},T,\beta)]^{2a}$, using Lemma 2.9, we get

$$\begin{aligned} \mathcal{P}_2^2 &\le \frac{4[\gamma(\epsilon)]_2^2[|\mathcal{D}_j(\beta,\varphi_\epsilon)|^{2a-1} - |\mathcal{D}_j(\beta,\varphi)|^{2a-1}]^2}{|[\gamma(\epsilon)]_2 + |\mathcal{D}_j(\beta,\varphi_\epsilon)|^{2a}|^2 \times |[\gamma(\epsilon)]_2 + |\mathcal{D}_j(\beta,\varphi)|^{2a}|^2}\left|\langle \ell, e_j\rangle\right|^2 \\ &\le 4\left[\gamma(\epsilon)\right]_2^2 \frac{|\mathcal{D}_j(\beta,\varphi_\epsilon|^{4a-2}}{|[\gamma(\epsilon)]_2 + |\mathcal{D}_j(\beta,\varphi_\epsilon)|^{2a}|^2}\frac{|\mathcal{D}_j(\beta,\varphi)|^2}{\lambda_j^\delta}\frac{|\langle \ell, e_j\rangle|^2}{|\mathcal{D}_j(\beta,\varphi)|^2} \\ &\le 4\left(4^{-1}\mathcal{A}_0\right)^{4a-2}\mathcal{A}_1^2\frac{[\gamma(\epsilon)]_2^2|\mathcal{D}_j(\beta)|^{4a}}{|[\gamma(\epsilon)]_2 + |\mathcal{D}_j(\beta,\varphi_\epsilon)|^{2a}|^2}\lambda_j^{-\delta}\|f\|_{\mathbb{H}^\delta(\Omega)}^2 \end{aligned} \tag{4.30}$$

From (4.30), it gives

$$\begin{aligned}
\mathcal{P}_2^2 &\le Q_5^2\left(\frac{[\gamma(\epsilon)]_2\lambda_j^{2a-\frac{\delta}{2}}}{[\gamma(\epsilon)]_2\lambda_j^{2a}+|(8^{-1}\mathcal{A}_0)T^2\widetilde{C}(\tau,\beta,\lambda_1)|^{2a}}\right)^2\|f\|^2_{\mathbb{H}^\delta(\Omega)}\\
&\le Q_5^2\sup_{j\ge1}\left(\frac{[\gamma(\epsilon)]_2\lambda_j^{2a-\frac{\delta}{2}}}{[\gamma(\epsilon)]_2\lambda_j^{2a}+(8^{-1}\mathcal{A}_0)|T^2\widetilde{C}(\tau,\beta,\lambda_1)]^{2a}}\right)^2\|f\|^2_{\mathbb{H}^\delta(\Omega)}\\
&\le Q_5^2\begin{cases}C_6^2[\gamma(\epsilon)]^{\frac{\delta}{2a}}R^2, & 0<\delta<4a,\\ C_7^2[\gamma(\epsilon)]^2R^2, & \delta\ge 4a.\end{cases}
\end{aligned}\tag{4.31}$$

Step 3. Before estimating $\mathcal{P}_3^2$, denoting $Q_6^2=\frac{4(8^{-1}\mathcal{A}_0)^{4a-2}}{\lambda_1^{2a+\delta+2}}[Q(\mathcal{M},T,\beta)]^{2a}|(8^{-1}\mathcal{A}_0)T^2\widetilde{C}(\tau,\beta,\lambda_1)|^{4a-2}\frac{(2a-1)^{2-a}}{(2a)^2}$, we get

$$\begin{aligned}
\mathcal{P}_3^2 &\le \epsilon^2\frac{4|\mathcal{D}_j(\beta,\varphi)|^{4a-2}|\mathcal{D}_j(\beta,\varphi_\epsilon)|^{4a-2}|\mathcal{D}_j(\beta,t,T)|^2}{|[\gamma(\epsilon)]_2+|\mathcal{D}_j(\beta,\varphi_\epsilon)|^{2a}|^2\times|[\gamma(\epsilon)]_2+|\mathcal{D}_j(\beta,\varphi)|^{2a}|^2}\left|\langle\ell,e_j\rangle\right|^2\\
&\le \epsilon^2 4\left|\frac{|\mathcal{D}_j(\beta,\varphi_\epsilon)|^{2a-1}}{[\gamma(\epsilon)]_2+|\mathcal{D}_j(\beta,\varphi_\epsilon)|^{2a}}\right|^2\left|\mathcal{D}_j(\beta)\right|^2\left|\langle f,e_j\rangle\right|^2\\
&\le \epsilon^2 4\mathcal{A}_1^{4a-2}\left(4^{-1}\mathcal{A}_0\right)^{4a-2}\left(\frac{\mathcal{M}^2T^{2\beta+2}}{2\beta-1}\right)^{4a}\frac{1}{\lambda_j^{8a+\delta}}\lambda_j^\delta\left|\langle f,e_j\rangle\right|^2\\
&\le \epsilon^2 4\left(4^{-1}\mathcal{A}_0\right)^{4a-2}\frac{[Q(\mathcal{M},T,\beta)]^{2a}}{\lambda_j^{2a+\delta+2}}\lambda_j^\delta\left|\langle f,e_j\rangle\right|^2\\
&\quad\times\lambda_j^2\left|\left[\gamma(\epsilon)\right]_2\lambda_j^{2a}+\left|\left(8^{-1}\mathcal{A}_0\right)T^2\widetilde{C}(\tau,\beta,\lambda_1)\right|^{2a}\right|^{-2}\\
&\le \epsilon^2Q_6^2\left[\gamma(\epsilon)\right]_2^{-\frac{1}{a}}\|f\|^2_{\mathbb{H}^\delta(\Omega)}.
\end{aligned}\tag{4.32}$$

Combining (4.27) to (4.32), we conclude that

$$\begin{aligned}
\left\|f_\epsilon^{[\gamma(\epsilon)]_2}-f^{[\gamma(\epsilon)]_2}\right\|_{L^2(\Omega)} &\le \epsilon\left[\gamma(\epsilon)\right]_2^{-\frac{1}{2a}}\left(Q_4^2+Q_6^2\|f\|^2_{\mathbb{H}^\delta(\Omega)}\right)^{\frac12}\\
&\quad+Q_5\begin{cases}C_6[\gamma(\epsilon)]^{\frac{\delta}{4a}}R, & 0<\delta<4a,\\ C_7[\gamma(\epsilon)]R, & \delta\ge 4a.\end{cases}
\end{aligned}\tag{4.33}$$

Next, we estimate $\|f^{[\gamma(\epsilon)]_2}-f\|_{L^2(\Omega)}$:

$$\begin{aligned}
\left\|f^{[\gamma(\epsilon)]_2}-f\right\|_{L^2(\Omega)} &\le \sum_{j=1}^\infty\left|\frac{-[\gamma(\epsilon)]_2\lambda_j^{\frac{\delta}{2}}}{|[\gamma(\epsilon)]_2+|\mathcal{D}_j(\beta,\varphi)|^{2a}|}\right|^2\lambda_j^\delta\left|\langle f,e_j\rangle\right|^2\\
&\le \sum_{j=1}^\infty\left|\frac{[\gamma(\epsilon)]_2\lambda_j^{2a-\frac{\delta}{2}}}{[\gamma(\epsilon)]_2\lambda_j^{2a}+|\mathcal{A}_0T^2\widetilde{C}(\tau,\beta,\lambda_1)|^{2a}}\right|^2\lambda_j^\delta\left|\langle f,e_j\rangle\right|^2\\
&\le \sup_{j\ge1}\left|\frac{[\gamma(\epsilon)]_2\lambda_j^{2a-\frac{\delta}{2}}}{[\gamma(\epsilon)]_2\lambda_j^{2a}+|\mathcal{A}_0T^2\widetilde{C}(\tau,\beta,\lambda_1)|^{2a}}\right|^2\|f\|^2_{\mathbb{H}^\delta(\Omega)}
\end{aligned}$$

$$\leq \begin{cases} C_6^2[\gamma(\epsilon)]^{\frac{\delta}{2a}}R^2, & 0<\delta<4a, \\ C_7^2[\gamma(\epsilon)]^2R^2, & \delta\geq 4a. \end{cases} \tag{4.34}$$

Combining (4.33) to (4.34), we conclude that

$$\begin{aligned} \|f_\epsilon^{[\gamma(\epsilon)]_2}-f\|_{L^2(\Omega)} &\leq \epsilon\big[\gamma(\epsilon)\big]_2^{-\frac{1}{2a}}\big(Q_4^2+Q_6^2\|f\|^2_{\mathbb{H}^\delta(\Omega)}\big)^{\frac{1}{2}} \\ &\quad + (Q_5+1)\begin{cases} C_6[\gamma(\epsilon)]^{\frac{\delta}{4a}}R, & 0<\delta<4a, \\ C_7[\gamma(\epsilon)]R, & \delta\geq 4a. \end{cases} \end{aligned} \tag{4.35}$$

By choosing the parameter regularization

$$\big[\gamma(\epsilon)\big]_2 = \begin{cases} (\frac{\epsilon}{R})^{\frac{4a}{\delta+2}}, & 0<\delta<4a, \\ (\frac{\epsilon}{R})^{\frac{2a}{1+2a}}, & \delta\geq 4a. \end{cases} \tag{4.36}$$

From (4.35) and (4.36), we conclude the following:

(i) If $0<\delta<4a$, then

$$\|f^{[\gamma(\epsilon)]_2}-f\|_{L^2(\Omega)} \leq Q_7\epsilon^{\frac{\delta}{\delta+2}}R^{\frac{2}{\delta+2}}, \tag{4.37}$$

where $Q_7=(Q_5+1)C_6+(Q_4^2+Q_6^2\|f\|^2_{\mathbb{H}^\delta(\Omega)})^{\frac{1}{2}}$.

(ii) If $\delta\geq 4a$, then

$$\|f^{[\gamma(\epsilon)]_2}-f\|_{L^2(\Omega)} \leq Q_8\epsilon^{\frac{2a}{2a+1}}R^{\frac{1}{2a+1}}, \tag{4.38}$$

where $Q_8=(Q_5+1)C_7+(Q_4^2+Q_6^2\|f\|^2_{\mathbb{H}^\delta(\Omega)})^{\frac{1}{2}}$. □

5 Simulation

In this section, we consider the problem as follows:

$$\begin{cases} \partial_t u(x,t)-(1+\tau\partial_t^\beta)\Delta u(x,t)=f(x)\varphi(t), & (x,t)\in\Omega\times(0,T), \\ u(x,t)=0, & x\in\partial\Omega, \\ u(x,0)=0, & x\in\Omega, \\ \int_0^T u(x,t)\,dt=\ell(x), & x\in\Omega. \end{cases} \tag{5.1}$$

The couple of $(\ell_\epsilon,\varphi_\epsilon)$ plays as observed data as follows:

$$\ell_\epsilon(\cdot)=\ell(\cdot)+\epsilon\big(2\,\mathrm{rand}(\cdot)-1\big), \qquad \varphi_\epsilon(\cdot)=\varphi(\cdot)+\frac{\epsilon\,\mathrm{rand}(\cdot)}{\sqrt{\pi}}. \tag{5.2}$$

In (5.1) with $u(x,t)=t^2\sin(x)$, we get $\ell(x)=\frac{T^3}{3}\sin(x)$ and $\varphi(t)=2t-1-2t\frac{t^{2-\beta}}{\Gamma(3-\beta)}$. Next, we can write the term $\mathcal{B}_j(\beta,t-z)$ as follows, see Lemma 2.3:

$$\mathcal{B}_j(\beta,t-z)=\int_0^\infty e^{-\xi(t-z)}K_j(\xi)\,d\xi=\lim_{M\to\infty}\int_0^M e^{-\xi(t-z)}K_j(\xi)\,d\xi. \tag{5.3}$$

Table 1 The error estimate for both FT1 and FT2

	Error estimate			
	$\epsilon_1 = 5 * 10^{-1}$	$\epsilon_2 = 5 * 10^{-2}$	$\epsilon_3 = 5 * 10^{-3}$	$\epsilon_4 = 5 * 10^{-4}$
FT1	0.93623968	0.882396582	0.672967434	0.539296586
FT2	0.991018439	0.881577282	0.590247118	0.507422364

From (5.3), we get

$$\begin{aligned} f(x) &= \sum_{j=1}^{N} \frac{\langle \ell, e_j\rangle e_j(x)}{\int_0^T (\int_0^t \mathcal{B}_j(\beta, t-z)\varphi(z)\,dz)\,dt} \\ &= \sum_{j=1}^{N} \frac{\langle \ell, e_j\rangle e_j(x)}{\lim_{M\to\infty} \int_0^T (\int_0^t (\int_0^M e^{-\xi(t-z)} K_j(\xi)\,d\xi)\varphi(z)\,dz)\,dt}, \end{aligned} \tag{5.4}$$

with M large enough. Using composite Simpson's rule for 2D, we have the observation of $f_\epsilon^{[\gamma(\epsilon)]_{1,2}} \in L^2(0,\pi)$. Trying to take $\|f\|_{\mathbb{H}^1(\Omega)} \le R$ with $R \approx 125.447$ leads to $[\gamma(\epsilon)]_1 = (\frac{\epsilon}{R})^{\frac{a_1+1}{a_1+2}}$ and $[\gamma(\epsilon)]_2 = (\frac{\epsilon}{R})^{\frac{2a_2}{1+2a_2}}$. Similarly, in the formula finding the methodological seriousization for fractional Tikhonov method type one and the regularized solution for fractional Tikhonov method type two, we just need replace ℓ with ℓ_ϵ and φ with φ_ϵ.

Step 1: As the discretization level, a uniform grid of mesh-point (x_i) is used to discrete the space interval

$$x_i = i\Delta x, \Delta x = \frac{1}{N}, \quad i = \overline{0, N}. \tag{5.5}$$

In this example, with $N = 121$, we take the following calculation steps.

Step 2: Set $f_\epsilon^{[\gamma(\epsilon)]}(x_j) = f_\epsilon^{\gamma,j}$ and $f(x_j) = f_j$, construct two vectors containing all discrete values of $f_\epsilon^{\gamma,j}$ and f denoted by $\Xi_\epsilon^{\gamma,j}$ and Ψ^j, respectively.

$$\Xi_\epsilon^{\gamma,j} = \left[f_\epsilon^{\gamma,0} f_\epsilon^{\gamma,1} \cdots f_\epsilon^{\gamma,N}\right] \in \mathbb{R}^{N+1}, \qquad \Psi = \left[f^0 f^1 \cdots f^{N-1} f^N\right] \in \mathbb{R}^{N+1}. \tag{5.6}$$

Step 3: Error estimate

$$\mathbb{E}_{rr} = \frac{(\sum_{j=1}^N |f_\epsilon^{[\gamma(\epsilon)]_{1,2}}(x_j) - f(x_j)|^2_{L^2(0,\pi)})^{\frac{1}{2}}}{(\sum_{j=1}^N |f(x_j)|^2_{L^2(0,\pi)})^{\frac{1}{2}}}. \tag{5.7}$$

From the results of the above calculations, Table 1 points out the relative error estimates for a regularized solution using the fractional Tikhonov method, see formula (4.8), and the fractional Tikhonov solution type two, see formula (4.9), respectively. In this table, the values are as follows: In the case of the regularization solution fractional Tikhonov type one, since $a \in (0,1)$, then we choose $a_1 = 0.65$; in case of the regularization solution fractional Tikhonov type two, because of $a_2 \in (\frac{1}{2}, 1)$, we choose $a_2 = 0.75$ and values $T = 1, \beta = 0.5, \tau = 1.2, \delta = 1$. Table 1 shows the relative error estimates between the exact solution and its regularized solution for both FT_1 and FT_2 with $\epsilon = 5 * 10^{-k}$, $k = 1,2,3,4$, respectively. Table 2 shows the error estimate with values β in the first column. Similarity, with different τ, this error can be found in Table 3. In general, it shows that with both fractional Tikhonov methods, the convergence rate is of almost the same level. From the

Table 2 With several different β values, we check the convergence rate between the sought solution and its approximation

β	$\epsilon_1 = 5 * 10^{-1}$		$\epsilon_2 = 5 * 10^{-2}$		$\epsilon_2 = 5 * 10^{-3}$	
	FT1	FT2	FT1	FT2	FT1	FT2
0.15	0.991598534	0.999785256	0.988619422	0.99664661	0.945522364	0.950280732
0.25	0.986505175	0.998811077	0.969395529	0.985012806	0.875451095	0.844481561
0.35	0.976662494	0.996670773	0.943285395	0.959946702	0.797661112	0.728825374
0.45	0.990847934	0.996606654	0.928212418	0.929867121	0.714487158	0.631034772
0.55	0.951093699	0.983236194	0.871209357	0.856203486	0.651612101	0.576545804
0.65	1.027188737	1.02169837	0.814338829	0.773675873	0.590714893	0.531025157
0.75	0.872317472	0.925135033	0.718266857	0.655089038	0.563014039	0.522375764
0.85	0.795948719	0.836312409	0.617989339	0.550282175	0.531663982	0.506525617
0.95	1.210791125	1.266609738	0.517328419	0.471143634	0.509131415	0.497055801

Table 3 With several different τ values, we check the convergence rate between the sought solution and its approximation

τ	$\epsilon_1 = 5 * 10^{-1}$		$\epsilon_2 = 5 * 10^{-2}$		$\epsilon_3 = 5 * 10^{-3}$	
	FT1	FT2	FT1	FT2	FT1	FT2
1.22	1.003848284	1.001944265	0.880044062	0.878952014	0.681262776	0.600737917
1.44	0.92715566	0.989440251	0.876786921	0.868227528	0.656775033	0.577016004
1.66	0.991484543	0.995604482	0.875417754	0.861799734	0.649778367	0.572823918
1.88	1.005412506	1.003185271	0.876032528	0.858939766	0.643017105	0.567850812

results obtained in the number table, we conclude that when ϵ tends to 0, the tensile test will converge the accuracy, although this convergence is relatively slow.

6 Conclusion

In this article, we consider problem (1.1) for the Rayleigh–Stokes problem. In this article, by using the fractional Tikhonov method, we establish an approximate solution. Then, we show the rate of convergence between the sought solution and the regularized one and provide a simple numerical experiment. In the future work, we may use the condition $\theta u(x,T) + \theta_2 \int_0^T u(x,t)\,dt = \ell(x)$ to study problem (1.1).

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