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On initial inverse problem for nonlinear couple heat with Kirchhoff type

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Abstract

The main objective of the paper is to study the final model for the Kirchhoff-type parabolic system. Such type problems have many applications in physical and biological phenomena. Under some smoothness of the final Cauchy data, we prove that the problem has a unique mild solution. The main tool is Banach's fixed point theorem. We also consider the non-well-posed problem in the Hadamard sense. Finally, we apply truncation method to regularize our problem. The paper is motivated by the work of Tuan, Nam, and Nhat [Comput. Math. Appl. 77(1):15–33, 2019].

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1 Introduction

In this paper, we consider the following Kirchhoff-type problem for parabolic equation systems:

$$\begin{cases} \frac{\partial u}{\partial t} = \mathcal{L}(\|\nabla u\|_{L^2}, \|\nabla v\|_{L^2})\Delta u, & (x,t) \in \Omega \times (0,T), \\ \frac{\partial v}{\partial t} = \mathcal{L}(\|\nabla u\|_{L^2}, \|\nabla v\|_{L^2})\Delta v, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = v(x,t) = 0, & x \in \partial\Omega, t \in (0,T], \end{cases} \tag{1.1}$$

with the following terminal condition

$$u(x,T) = f(x), \qquad v(x,T) = g(x), \quad x \in \Omega, \tag{1.2}$$

where $(f,g) \in L^2(\Omega) \times L^2(\Omega)$ is the Cauchy terminal data, and $\mathcal{L}$ is defined in Sect. 2. In recent years, partial differential equations concerning Kirchhoff terms have practical applications in continuum mechanics, phase transition phenomena, and population dynamics and attracted many authors; see, for example, [1–9].

The parabolic equations with nonlocal diffusion arise in a variety of physical and biological applications; see, for example, [10–15] and the references therein. To study interactions of two or more biological species, systems of parabolic equations have been proposed. For example, Almeida [16] studied the following system of two population densities u, v:

$$\begin{aligned}
&u_t - \mathcal{D}_1\big(\ell_1(u)(t), \ell_2(v)(t)\big)\Delta u + \lambda_1 |u|^{p-2}u = F(x,t), \quad (x,t) \in Q_T, \\
&v_t - \mathcal{D}_2\big(\ell_3(u)(t), \ell_4(v)(t)\big)\Delta v + \lambda_2 |v|^{p-2}v = G(x,t), \quad (x,t) \in Q_T, \\
&u(x,t) = 0, \qquad v(x,t) = 0, \quad (x,t) \in S_T, \\
&u(x,0) = u_0, \qquad v(x,0) = v_0, \quad x \in \Omega,
\end{aligned}$$

where the death in species u is proportional to $|u|^{p-2}u$ by the factor λ_1, the death in species v is proportional to $|v|^{p-2}v$ by the factor λ_2, and f_1, f_2 are the supplies of external sources. The author obtained the results on the existence, uniqueness, and long-time behavior of a smooth global solution of the system. Ferreira [17] also proved the well-posedness of the system of nonlocal reaction–diffusion equations with both homogeneous Dirichlet or Neumann boundary conditions.

Since model (1.1) is a system having a gradient element, perhaps, the techniques for this problem are more complex. To the best of our knowledge, there is no result concerning problem (1.1)–(1.2). The current main applications of backward in time parabolic equations are hydrological inversion and image processing. The parabolic equation with terminal conditions plays an important role in physics and engineering, especially with thermal conductivity dependent on both time and space. We refer the reader to some interesting papers [18–21]. Our paper is motivated by the recent results of Baleanu et al. [21], Nam et al. [19], and Tuan et al. [20]. The techniques of this paper are based on the previous paper [22].

The main tool in the paper is the Fourier series technique in H^s spaces, combined with Banach's fixed point theorem. The main and novel contributions of the paper are as follows:

- The first contribution result is the proof of the existence and uniqueness of a solution of our backward problem. To this end, we had to increase the smoothness properties of the input Cauchy data.
- The second result is showing that our backward problem is ill-posed in the Hadamard sense. Furthermore, we also regularize our inverse problem using the Fourier truncation method. We then obtain an error assessment between the regularized and exact solutions.

This paper is organized as follows. In Sect. 2, we introduce some preliminaries and the mild solution of problem (1.1)–(1.2). Using the Banach fixed point theorem, we show that our problem has a unique mild solution. In Sect. 3, we prove the ill-posedness of our problem. Applying the Fourier truncation method, we give a stability estimate of logarithmic type between the regularized and exact solutions.

2 Some preliminaries and the mild solution of problem (1.1)-(1.2)

In this section, we introduce some properties of the eigenvalues of the operator $-\Delta$; see, for example, [6]. We have the equality

$$-\Delta e_n(x) = -\lambda_n e_n(x), \quad x \in \Omega; \qquad e_n(x) = 0, \quad x \in \partial\Omega, n \in \mathbb{N}, \tag{2.1}$$

where $\{\lambda_n\}_{n=1}^{\infty}$ is the set of eigenvalues of $-\Delta$ satisfying

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \cdots \tag{2.2}$$

and $\lim_{n\to\infty} \lambda_n = \infty$. Let us recall the following Hilbert scale space for $\nu > 0$:

$$H^\nu(\Omega) = \left\{ \psi \in L^2(\Omega) : \sum_{n=1}^{\infty} \lambda_n^{2\nu} \langle \psi, e_n\rangle^2 < +\infty \right\}, \tag{2.3}$$

associated with the norm

$$\|u\|_{H^\nu(D)} = \left(\sum_{n=1}^{\infty} \lambda_n^2 \big|(u, e_n)\big|^2 \right)^{\frac{1}{2}}. \tag{2.4}$$

Let $\mathcal{L} \in C^1(\mathbb{R}^2)$ be a function such that:

- There exists two positive constants $\mathcal{M}_0$, $\mathcal{M}_1$ such that

$$\mathcal{M}_0 \le \mathcal{L}(z_1, z_2) \le \mathcal{M}_1 \quad \forall (z_1, z_2) \in \mathbb{R}^2. \tag{2.5}$$

- There exists a positive constant $K_l > 0$ such that

$$\big|\mathcal{L}(z_1, z_2) - \mathcal{L}(\bar{z}_1, \bar{z}_2)\big| \le K_l\big(|z_1 - \bar{z}_1| + |z_2 - \bar{z}_2|\big). \tag{2.6}$$

Due to conditions (2.5) and (2.6), we have the following lemma.

Lemma 2.1 *If u_1, u_2, v_1, v_2 belong to the space $H^1(\Omega)$, then*

$$\begin{aligned} &\mathcal{L}\big(\|\nabla u_1\|_{L^2(\Omega)}, \|\nabla u_2\|_{L^2(\Omega)}\big) - \mathcal{L}\big(\|\nabla v_1\|_{L^2(\Omega)}, \|\nabla v_2\|_{L^2(\Omega)}\big) \\ &\quad \le K_l\big(\big\|\nabla(u_1 - v_1)\big\|_{L^2(\Omega)} + \big\|\nabla(u_2 - v_2)\big\|_{L^2(\Omega)}\big). \end{aligned} \tag{2.7}$$

3 The existence and ill-posedness of our backward problem

Let us first investigate the formula of a mild solution of problem (1.1)–(1.2). Multiplying both sides of the main equation of problem (1.1)–(1.2) by e_n and integrating by parts, we get that

$$\begin{cases} \langle \frac{\partial u}{\partial t}, e_n\rangle = \langle \mathcal{L}(\|\nabla u\|_{L^2}, \|\nabla v\|_{L^2})\Delta u, e_n\rangle, \\ \langle \frac{\partial v}{\partial t}, e_n\rangle = \langle \mathcal{L}(\|\nabla u\|_{L^2}, \|\nabla v\|_{L^2})\Delta u e_n\rangle. \end{cases} \tag{3.1}$$

This equality immediately gives that

$$\begin{cases} \frac{d}{dt}\langle u, e_n\rangle = \lambda_n \langle \mathcal{L}(\|\nabla u\|_{L^2}, \|\nabla v\|_{L^2})\langle u, e_n\rangle, \\ \frac{d}{dt}\langle v, e_n\rangle = \lambda_n \langle \mathcal{L}(\|\nabla u\|_{L^2}, \|\nabla v\|_{L^2})\langle v, e_n\rangle. \end{cases} \tag{3.2}$$

It is easy to see that system (3.2) allows us to get that the equalities

$$u_n(t) = \exp\left(-\lambda_n \int_0^t \mathcal{L}\big(\big\|\nabla u(\cdot,\tau)\big\|_{L^2}, \big\|\nabla v(\cdot,\tau)\big\|_{L^2}\big)\, d\tau \right) u_n(0) \tag{3.3}$$

and

$$v_n(t) = \exp\left(-\lambda_n \int_0^t \mathcal{L}\left(\left\|\nabla u(\cdot,\tau)\right\|_{L^2}, \left\|\nabla v(\cdot,\tau)\right\|_{L^2}\right) d\tau\right) v_n(0). \tag{3.4}$$

Due to the terminal condition (1.2), we get that

$$u_n(T) = \langle f, e_n\rangle, \qquad v_n(T) = \langle g, e_n\rangle.$$

By a simple calculation we get the Fourier coefficients of u and v:

$$\begin{aligned} u_n(t) &= \exp\left(\lambda_n \int_t^T \mathcal{L}\left(\left\|\nabla u(\cdot,\tau)\right\|_{L^2}, \left\|\nabla v(\cdot,\tau)\right\|_{L^2}\right) d\tau\right) \langle f, e_n\rangle, \\ v_n(t) &= \exp\left(\lambda_n \int_t^T \mathcal{L}\left(\left\|\nabla u(\cdot,\tau)\right\|_{L^2}, \left\|\nabla v(\cdot,\tau)\right\|_{L^2}\right) d\tau\right) \langle g, e_n\rangle, \end{aligned} \tag{3.5}$$

which allow us to get that

$$u(x,t) = \sum_{n=1}^{\infty} \exp\left(\lambda_n \int_t^T \mathcal{L}\left(\left\|\nabla u(\cdot,s)\right\|_{L^2}, \left\|\nabla v(\cdot,s)\right\|_{L^2}\right) ds\right) \langle f, e_n\rangle e_n(x) \tag{3.6}$$

and

$$v(x,t) = \sum_{n=1}^{\infty} \exp\left(\lambda_n \int_t^T \mathcal{L}\left(\left\|\nabla u(\cdot,s)\right\|_{L^2}, \left\|\nabla v(\cdot,s)\right\|_{L^2}\right) ds\right) \langle g, e_n\rangle e_n(x). \tag{3.7}$$

Theorem 3.1 *Let the Cauchy terminal data $(f,g) \in L^2(\Omega) \times L^2(\Omega)$ be such that*

$$\sum_{n=1}^{\infty} \lambda_n^3 e^{2T\mathcal{M}_1\lambda_n} \langle f, e_n\rangle^2 = B_f, \qquad \sum_{n=1}^{\infty} \lambda_n^3 e^{2T\mathcal{M}_1\lambda_n} \langle g, e_n\rangle^2 = B_g \tag{3.8}$$

for two constants $B_f, B_g > 0$. Then problem (1.1) has a mild solution on the space

$$\left(L^\infty_\theta\left(0,T;H^1(\Omega)\right)\right)^2.$$

Proof To show the existence of a mild solution, we define the operator $\mathbf{Q}(u,v)(t) = (\mathcal{Q}_1(u,v)(t), \mathcal{Q}_2(u,v)(t))$ and show that $\mathbf{Q}$ has a fixed point in the space $(L^\infty_\theta(0,T;H^1(\Omega)))^2$. Here the operators $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are defined as follows:

$$\begin{cases} \mathcal{Q}_1(u,v)(t) = \sum_{n=1}^{\infty} \exp(\lambda_n \int_t^T \mathcal{L}(\|\nabla u(\cdot,s)\|_{L^2}, \|\nabla v(\cdot,s)\|_{L^2})\, ds) \langle f, e_n\rangle e_n(x), \\ \mathcal{Q}_2(u,v)(t) = \sum_{n=1}^{\infty} \exp(\lambda_n \int_t^T \mathcal{L}(\|\nabla u(\cdot,s)\|_{L^2}, \|\nabla v(\cdot,s)\|_{L^2})\, ds) \langle g, e_n\rangle e_n(x). \end{cases} \tag{3.9}$$

We will prove by induction that if $(u_1,v_1) \in (L^\infty(0,T;H^1(\Omega)))^2$ and $(u_2,v_2) \in (L^\infty(0,T;H^1(\Omega)))^2$, then

$$\begin{aligned} &\left\|\mathbf{Q}^m(u_1,v_1)(\cdot,t) - \mathbf{Q}^m(u_2,v_2)(\cdot,t)\right\|^2_{H^1(\Omega)} \\ &\quad \le \frac{(T-t)^m}{m!}\left(2C(B_f+B_g)K_l^2\right)^m \left\|(u_1,v_1)-(u_2,v_2)\right\|^2_{(L^\infty(0,T;H^1(\Omega)))^2}. \end{aligned} \tag{3.10}$$

For $m = 1$, using the inequality $(c+d)^2 \leq 2c^2 + 2d^2$, we have

$$\begin{aligned}
&\left\| \mathbf{Q}^m(u_1,v_1)(\cdot,t) - \mathbf{Q}^m(u_2,v_2)(\cdot,t) \right\|^2_{H^1(\Omega)} \\
&\quad \leq 2 \left\| \mathcal{Q}_1^m(u_1,v_1)(\cdot,t) - \mathcal{Q}_1^m(u_2,v_2)(\cdot,t) \right\|^2_{H^1(\Omega)} \\
&\qquad + 2 \left\| \mathcal{Q}_2^m(u_1,v_1)(\cdot,t) - \mathcal{Q}_2^m(u_2,v_2)(\cdot,t) \right\|^2_{H^1(\Omega)}.
\end{aligned} \tag{3.11}$$

Applying Lemma 2.1 and the inequality $|e^r - e^q| \leq |r-q| \max(e^r, e^q)$ for $r, q \in \mathbb{R}$, we have

$$\begin{aligned}
&\left\| \mathcal{Q}_1(u_1,v_1)(\cdot,t) - \mathcal{Q}_1(u_2,v_2)(\cdot,t) \right\|^2_{H^1(\Omega)} \\
&\quad = \sum_{n=1}^{\infty} \lambda_n \biggl[\exp\biggl(\lambda_n \int_t^T \mathcal{L}\bigl(\|\nabla u_1(\cdot,s)\|_{L^2}, \|\nabla v_1(\cdot,s)\|_{L^2} \bigr)\, ds \biggr) \\
&\qquad - \exp\biggl(\lambda_n \int_t^T \mathcal{L}\bigl(\|\nabla u_2(\cdot,s)\|_{L^2}, \|\nabla v_2(\cdot,s)\|_{L^2} \bigr)\, ds \biggr) \biggr]^2 \langle f, e_n \rangle^2 \\
&\quad \leq \sum_{n=1}^{\infty} \lambda_n^3 \biggl[\int_t^T \mathcal{L}\bigl(\|\nabla u_1(\cdot,s)\|_{L^2}, \|\nabla v_1(\cdot,s)\|_{L^2} \bigr)\, ds \\
&\qquad - \int_t^T \mathcal{L}\bigl(\|\nabla u_2(\cdot,s)\|_{L^2}, \|\nabla v_2(\cdot,s)\|_{L^2} \bigr)\, ds \biggr]^2 e^{2(T-t)\mathcal{M}_1\lambda_n} \langle f, e_n \rangle^2 \\
&\quad \leq B_f K_l^2 \biggl(\int_t^T \|\nabla(u_1-u_2)(\cdot,s)\|_{L^2(\Omega)}\, ds + \int_t^T \|\nabla(v_1-v_2)(\cdot,s)\|_{L^2(\Omega)}\, ds \biggr)^2 \\
&\quad \leq C B_f K_l^2 \int_t^T \left\| (u_1,v_1)(\cdot,s) - (u_2,v_2)(\cdot,s) \right\|^2_{H^1(\Omega)}\, ds,
\end{aligned} \tag{3.12}$$

where in the above line, we applied the inequality $\|\nabla \psi\|_{L^2(\Omega)} \leq C\|\psi\|_{H^1(\Omega)}$. In a similar way, we get that

$$\begin{aligned}
&\left\| \mathcal{Q}_2(u_1,v_1)(\cdot,t) - \mathcal{Q}_2(u_2,v_2)(\cdot,t) \right\|^2_{H^1(\Omega)} \\
&\quad \leq C B_g K_l^2 \int_t^T \left\| (u_1,v_1)(\cdot,s) - (u_2,v_2)(\cdot,s) \right\|^2_{H^1(\Omega)}\, ds.
\end{aligned} \tag{3.13}$$

Combining (3.11), (3.12), and (3.13), we deduce that

$$\begin{aligned}
&\left\| \mathbf{Q}(u_1,v_1)(\cdot,t) - \mathbf{Q}(u_2,v_2)(\cdot,t) \right\|^2_{H^1(\Omega)} \\
&\quad \leq 2C(B_f + B_g) K_l^2 (T-t) \left\| (u_1,v_1) - (u_2,v_2) \right\|_{(L^\infty(0,T;H^1(\Omega)))^2}.
\end{aligned} \tag{3.14}$$

Let (3.10) hold for $m = p$. We will show that (3.10) holds for $m = p+1$. Indeed, we have

$$\begin{aligned}
&\left\| \mathcal{Q}_1^{p+1}(u_1,v_1)(\cdot,t) - \mathcal{Q}_1^{p+1}(u_2,v_2)(\cdot,t) \right\|^2_{H^1(\Omega)} \\
&\quad = \left\| \mathcal{Q}_1\bigl(\mathcal{Q}_1^p(u_1,v_1)(\cdot,t) \bigr) - \mathcal{Q}_1\bigl(\mathcal{Q}_1^p(u_2,v_2)(\cdot,t) \bigr) \right\|^2_{H^1(\Omega)} \\
&\quad \leq C B_f K_l^2 \int_t^T \left\| \mathcal{Q}_1^p(u_1,v_1)(\cdot,s) - \mathcal{Q}_1^p(u_2,v_2)(\cdot,s) \right\|^2_{H^1(\Omega)}\, ds
\end{aligned}$$

and

$$
\begin{aligned}
&\left\| \mathcal{Q}_2^{p+1}(u_1,v_1)(\cdot,t) - \mathcal{Q}_2^{p+1}(u_2,v_2)(\cdot,t)\right\|^2_{H^1(\Omega)} \\
&\quad = \left\| \mathcal{Q}_2\big(\mathcal{Q}_2^p(u_1,v_1)(\cdot,t)\big) - \mathcal{Q}_2\big(\mathcal{Q}_2^p(u_2,v_2)(\cdot,t)\big)\right\|^2_{H^1(\Omega)} \\
&\quad \le CB_gK_l^2 \int_t^T \left\| \mathcal{Q}_2^p(u_1,v_1)(\cdot,s) - \mathcal{Q}_2^p(u_2,v_2)(\cdot,s)\right\|^2_{H^1(\Omega)}\,ds.
\end{aligned}
$$

From two above observations we find that

$$
\begin{aligned}
&\left\| \mathbf{Q}^{p+1}(u_1,v_1)(\cdot,t) - \mathbf{Q}^{p+1}(u_2,v_2)(\cdot,t)\right\|^2_{H^1(\Omega)} \\
&\quad \le 2\left\| \mathcal{Q}_1^{p+1}(u_1,v_1)(\cdot,t) - \mathcal{Q}_1^{p+1}(u_2,v_2)(\cdot,t)\right\|^2_{H^1(\Omega)} \\
&\qquad + 2\left\| \mathcal{Q}_2^{p+1}(u_1,v_1)(\cdot,t) - \mathcal{Q}_2^{p+1}(u_2,v_2)(\cdot,t)\right\|^2_{H^1(\Omega)} \\
&\quad \le 2C(B_f+B_g)K_l^2 \int_t^T \left\| \mathbf{Q}^p(u_1,v_1)(\cdot,s) - \mathbf{Q}^p(u_2,v_2)(\cdot,s)\right\|^2_{H^1(\Omega)}\,ds.
\end{aligned}
\tag{3.15}
$$

By the induction assumption on (3.10), from (3.15) it follows that

$$
\begin{aligned}
&\left\| \mathbf{Q}^{p+1}(u_1,v_1)(\cdot,t) - \mathbf{Q}^{p+1}(u_2,v_2)(\cdot,t)\right\|^2_{H^1(\Omega)} \\
&\quad \le \big(2C(B_f+B_g)K_l^2\big)^{m+1} \left\|(u_1,v_1)-(u_2,v_2)\right\|^2_{(L^\infty(0,T;H^1(\Omega)))^2} \int_t^T \frac{(T-s)^m}{m!}\,ds \\
&\quad = \frac{(T-t)^{m+1}}{(m+1)!}\big(2C(B_f+B_g)K_l^2\big)^{m+1} \left\|(u_1,v_1)-(u_2,v_2)\right\|^2_{(L^\infty(0,T;H^1(\Omega)))^2}.
\end{aligned}
\tag{3.16}
$$

Hence (3.10) holds for any positive integer m. As a consequence, we derive that

$$
\begin{aligned}
&\left\| \mathbf{Q}^m(u_1,v_1) - \mathbf{Q}^m(u_2,v_2)\right\|_{(L^\infty(0,T;H^1(\Omega)))^2} \\
&\quad \le \frac{T^m(2C(B_f+B_g)K_l^2)^m}{m!} \left\|(u_1,v_1)-(u_2,v_2)\right\|^2_{(L^\infty(0,T;H^1(\Omega)))^2}.
\end{aligned}
\tag{3.17}
$$

Since

$$
\frac{T^m(2C(B_f+B_g)K_l^2)^m}{m!} \to 0, \quad m\to\infty,
$$

there exists a positive constant m_0 such that the term $\frac{T^{m_0}(2C(B_f+B_g)K_l^2)^{m_0}}{(m_0)!} < 1$. Using the Banach fixed point theorem, we conclude that $\mathbf{Q}^{m_0}$ has a fixed point (u^*,v^*) on the space $(L^\infty(0,T;H^1(\Omega)))^2$. It is easy to get that (u^*,v^*) is also a solution of the equation $\mathbf{Q}(u,v) = (u,v)$. □

Theorem 3.2 *Problem* (1.1) *is ill-posed in the sense of Hadamard.*

Proof Let us illustrate by an example that the solution of problem (1.1) is not stable according to the input data. We take the input Cauchy data (f_m,g_m) with

$$
f_m(x) = g_m(x) = \frac{\phi_m(x)}{\lambda_m}
$$

for natural $m \geq 1$. It is easy to check that

$$\sum_{n=1}^{\infty} \lambda_n^3 e^{2T\mathcal{M}_1\lambda_n} \langle f, e_n\rangle^2 = \sum_{n=1}^{\infty} \lambda_n^3 e^{2T\mathcal{M}_1\lambda_n} \langle g, e_n\rangle^2 = \lambda_m^2 e^{2T\mathcal{M}_1\lambda_m}. \tag{3.18}$$

Under the Cauchy terminal data (f_m, g_m) as above, by Theorem (3.1) we get that problem (1.1) has a mild solution $(u_m, v_m) \in (L^\infty(0,T;H^1(\Omega)))^2$, which is given by

$$\begin{cases} u_m(x,t) = \sum_{n=1}^{\infty} \exp(\lambda_n \int_t^T \mathcal{L}(\|\nabla u_m(\cdot,s)\|_{L^2}, \|\nabla v_m(\cdot,s)\|_{L^2})\,ds)\langle f_m, e_n\rangle e_m(x), \\ v_m(x,t) = \sum_{n=1}^{\infty} \exp(\lambda_n \int_t^T \mathcal{L}(\|\nabla u_m(\cdot,s)\|_{L^2}, \|\nabla v_m(\cdot,s)\|_{L^2})\,ds)\langle g_m, e_n\rangle e_m(x). \end{cases} \tag{3.19}$$

Recalling the lower bound of $\mathcal{L}$, we derive that for any $0 \leq t \leq T$,

$$\begin{aligned} \|u_m(\cdot,t)\|_{L^2(\Omega)}^2 &= \frac{\exp(2\lambda_n \int_t^T \mathcal{L}(\|\nabla u_m(\cdot,s)\|_{L^2}, \|\nabla v_m(\cdot,s)\|_{L^2})\,ds)}{\lambda_m^2} \\ &\geq \frac{\exp(2\lambda_m \mathcal{M}_0(T-t))}{\lambda_m^2}. \end{aligned} \tag{3.20}$$

By a similar argument we also obtain that

$$\begin{aligned} \|v_m(\cdot,t)\|_{L^2(\Omega)}^2 &= \frac{\exp(2\lambda_n \int_t^T \mathcal{L}(\|\nabla u_m(\cdot,s)\|_{L^2}, \|\nabla v_m(\cdot,s)\|_{L^2})\,ds)}{\lambda_m^2} \\ &\geq \frac{\exp(2\lambda_m \mathcal{M}_0(T-t))}{\lambda_m^2}. \end{aligned} \tag{3.21}$$

From two recent observations we arrive at

$$\|(u_m, v_m)\|_{(L^\infty(0,T;H^1(\Omega)))^2} \geq \sup_{0\leq t\leq T} \frac{\exp(\lambda_m \mathcal{M}_0(T-t))}{\lambda_m} = \frac{\exp(\lambda_m \mathcal{M}_0 T)}{\lambda_m}.$$

Taking the limits as $m \to +\infty$, we get

$$\begin{aligned} &\lim_{m\to+\infty} \|(f_m, g_m)\|_{L^2(\Omega)\times L^2(\Omega)} = \lim_{m\to+\infty} \frac{1}{\lambda_m} = 0, \\ &\lim_{m\to+\infty} \|(u_m, v_m)\|_{(L^\infty(0,T;H^1(\Omega)))^2} = \lim_{m\to+\infty} \frac{\exp(\lambda_m \mathcal{M}_0 T)}{\lambda_m} = +\infty. \end{aligned}$$

Therefore, problem (1.1) is ill-posed in the sense of Hadamard. □

Remark 3.1 We expect to extend our model to noninteger derivatives according to papers [23–30].

4 Fourier truncation method and error estimate

Let us define the regularized solution by the Fourier truncation method as

$$\begin{cases} u^{N,\delta}(x,t) \\ \quad = \sum_{n=1}^{N(\delta)} \exp(\lambda_n \int_t^T \mathcal{L}(\|\nabla u^{N,\delta}(\cdot,s)\|_{L^2}, \|\nabla v^{N,\delta}(\cdot,s)\|_{L^2})\,ds)\langle f^\delta, e_n\rangle e_n(x), \\ v^{N,\delta}(x,t) \\ \quad = \sum_{n=1}^{N(\delta)} \exp(\lambda_n \int_t^T \mathcal{L}(\|\nabla u^{N,\delta}(\cdot,s)\|_{L^2}, \|\nabla v^{N,\delta}(\cdot,s)\|_{L^2})\,ds)\langle g^\delta, e_n\rangle e_n(x), \end{cases} \tag{4.1}$$

where $N := N(\delta)$ is a regularization parameter. Here the function $(f^\delta, g^\delta) \in L^2(\Omega) \times L^2(\Omega)$ satisfies

$$\|f^\delta - f\|_{L^2(\Omega)} + \|g^\delta - f\|_{L^2(\Omega)} \le \delta. \tag{4.2}$$

Theorem 4.1 *Let $f \in L^2(\Omega)$ be such that*

$$\sum_{n=1}^{\infty} \langle f, e_n\rangle^2 \lambda_n^{1+\delta} e^{2M_1 T\lambda_n} \le E$$

for some constants $E > 0$ and $\delta > 0$. Let us choose $N := N(\delta)$ such that

$$\lim_{\delta\to 0} N(\delta) = +\infty, \qquad \lim_{\delta\to 0} \delta^2 \lambda_{N(\delta)} e^{2TM_1\lambda_{N(\delta)}} = 0.$$

Then we have the estimate

$$\begin{aligned}
&\big\|u^{N,\delta}(\cdot,t) - u(\cdot,t)\big\|^2_{H^1(\Omega)} + \big\|v^{N,\delta}(\cdot,t) - v(\cdot,t)\big\|^2_{H^1(\Omega)} \\
&\quad \le \big(6\lambda_{N(\delta)} \exp(2\mathcal{M}_1 T\lambda_{N(\delta)})\delta^2 + 6E\big(N(\delta)\big)^{-\gamma}\big) \exp\big(6K_l\lambda_1^{-\gamma} EC(T-t)\big).
\end{aligned} \tag{4.3}$$

Remark 4.1 It is obvious that $\lambda_N \sim N^{\frac{2}{d}}$. So we can choose a natural number N such that

$$\frac{1-\nu}{2TM_1} \ln\left(\frac{1}{\delta}\right) \le \lambda_N \le \frac{1-\nu}{MB_1} \ln\left(\frac{1}{\delta}\right), \quad 0 < \nu < 1.$$

Then the error $\|u^{N,\delta}(\cdot,t) - u(\cdot,t)\|^2_{H^1(\Omega)} + \|v^{N,\delta}(\cdot,t) - v(\cdot,t)\|^2_{H^1(\Omega)}$ is of logarithmic order

$$\max\left(\left[\ln\left(\frac{1}{\delta}\right)\right]^{-2}, \delta^{2\nu} \ln\left(\frac{1}{\delta}\right)\right).$$

Proof To show the existence of a mild solution, we define the operator $\mathbf{R}^m_\delta(u,v)(t) = (\mathcal{R}_{1,\delta}(u,v)(t), \mathcal{R}_{2,\delta}(u,v)(t))$ and show that $\mathbf{R}^m_\delta$ has a fixed point in the space $(L^\infty_\theta(0,T;H^1(\Omega)))^2$. Here the operators $\mathcal{R}_{1,\delta}$ and $\mathcal{R}_{2,\delta}$ are defined as follows:

$$\begin{cases}
\mathcal{R}_{1,\delta}(u,v)(t) \\
\quad = \sum_{n=1}^{N(\delta)} \exp(\lambda_n \int_t^T \mathcal{L}(\|\nabla u^{N,\delta}(\cdot,s)\|_{L^2}, \|\nabla v^{N,\delta}(\cdot,s)\|_{L^2})\,ds) \langle f^\delta, e_n\rangle e_n(x), \\
\mathcal{R}_{2,\delta}(u,v)(t) \\
\quad = \sum_{n=1}^{N(\delta)} \exp(\lambda_n \int_t^T \mathcal{L}(\|\nabla u^{N,\delta}(\cdot,s)\|_{L^2}, \|\nabla v^{N,\delta}(\cdot,s)\|_{L^2})\,ds) \langle g^\delta, e_n\rangle e_n(x).
\end{cases} \tag{4.4}$$

We will prove by induction that if $(u_1,v_1) \in (L^\infty(0,T;H^1(\Omega)))^2$ and $(u_2,v_2) \in (L^\infty(0,T;H^1(\Omega)))^2$, then

$$\big\|\mathbf{R}^m_\delta(u_1,v_1)(\cdot,t) - \mathbf{R}^m_\delta(u_2,v_2)(\cdot,t)\big\|^2_{H^1(\Omega)} \le \frac{(2\mathcal{D}(\delta,f,g))^m (T-t)^m}{m!}. \tag{4.5}$$

For $m = 1$, using the inequality $(c+d)^2 \le 2c^2 + 2d^2$, we get that

$$\begin{aligned}
&\big\|\mathbf{R}^m_\delta(u_1,v_1)(\cdot,t) - \mathbf{R}^m_\delta(u_2,v_2)(\cdot,t)\big\|^2_{H^1(\Omega)} \\
&\quad \le 2\big\|\mathcal{R}^m_{1,\delta}(u_1,v_1)(\cdot,t) - \mathcal{R}_{1,\delta}(u_2,v_2)(\cdot,t)\big\|^2_{H^1(\Omega)}
\end{aligned}$$

$$+ 2\left\| \mathcal{R}_{2,\delta}^{m}(u_1, v_1)(\cdot,t) - \mathcal{R}_{2,\delta}^{m}(u_2, v_2)(\cdot,t)\right\|_{H^1(\Omega)}^2. \tag{4.6}$$

By applying Lemma 2.1 and the inequality $|e^r - e^q| \le |r-q|\max(e^r, e^q)$ for $r, q \in \mathbb{R}$, we have

$$\begin{aligned}
&\left\| \mathcal{R}_{1,\delta}^{m}(u_1, v_1)(\cdot,t) - \mathcal{R}_{1,\delta}^{m}(u_2, v_2)(\cdot,t)\right\|_{H^1(\Omega)}^2 \\
&= \sum_{n=1}^{N(\delta)} \lambda_n \Bigg[\exp\left(\lambda_n \int_t^T \mathcal{L}\left(\|\nabla u_1(\cdot,s)\|_{L^2}, \|\nabla v_1(\cdot,s)\|_{L^2}\right) ds \right) \\
&\quad - \exp\left(\lambda_n \int_t^T \mathcal{L}\left(\|\nabla u_2(\cdot,s)\|_{L^2}, \|\nabla v_2(\cdot,s)\|_{L^2}\right) ds \right) \Bigg]^2 \langle f^\delta, e_n\rangle^2 \\
&\le \sum_{n=1}^{N(\delta)} \lambda_n^3 e^{2(T-t)\mathcal{M}_1\lambda_n} \Bigg[\int_t^T \mathcal{L}\left(\|\nabla u_1(\cdot,s)\|_{L^2}, \|\nabla v_1(\cdot,s)\|_{L^2}\right) ds \\
&\quad - \int_t^T \mathcal{L}\left(\|\nabla u_2(\cdot,s)\|_{L^2}, \|\nabla v_2(\cdot,s)\|_{L^2}\right) ds \Bigg]^2 \langle f^\delta, e_n\rangle^2 \\
&\le \left|N(\delta)\right|^3 e^{2TM_1 N(\delta)} K_l^2 \|f^\delta\|_{L^2(\Omega)}^2 \\
&\quad \times \left(\int_t^T \left\|\nabla(u_1 - u_2)(\cdot,s)\right\|_{L^2(\Omega)} ds + \int_t^T \left\|\nabla(v_1 - v_2)(\cdot,s)\right\|_{L^2(\Omega)} ds \right)^2 \\
&\le C\left|N(\delta)\right|^3 e^{2TM_1 N(\delta)} K_l^2 \|f^\delta\|_{L^2(\Omega)}^2 \int_t^T \left\|(u_1,v_1)(\cdot,s) - (u_2,v_2)(\cdot,s)\right\|_{H^1(\Omega)}^2 ds,
\end{aligned} \tag{4.7}$$

where in the last line, we used the inequality $\|\nabla\psi\|_{L^2(\Omega)} \le C\|\psi\|_{H^1(\Omega)}$. By a similar argument we obtain that

$$\begin{aligned}
&\left\| \mathcal{R}_{2,\delta}^{m}(u_1, v_1)(\cdot,t) - \mathcal{R}_{2,\delta}(u_2, v_2)(\cdot,t)\right\|_{H^1(\Omega)}^2 \\
&\le C\left|N(\delta)\right|^3 e^{2TM_1 N(\delta)} K_l^2 \|g^\delta\|_{L^2(\Omega)}^2 \int_t^T \left\|(u_1,v_1)(\cdot,s) - (u_2,v_2)(\cdot,s)\right\|_{H^1(\Omega)}^2 ds.
\end{aligned} \tag{4.8}$$

Combining (4.6), (4.7), and (4.8), we find that

$$\begin{aligned}
&\left\| \mathbf{R}_\delta(u_1, v_1)(\cdot,t) - \mathbf{R}_\delta(u_2, v_2)(\cdot,t)\right\|_{H^1(\Omega)}^2 \\
&\le \mathcal{D}(\delta,f,g) \int_t^T \left\|(u_1,v_1)(\cdot,s) - (u_2,v_2)(\cdot,s)\right\|_{H^1(\Omega)}^2 ds \\
&\le \mathcal{D}(\delta,f,g)(T-t)\left\|(u_1,v_1) - (u_2,v_2)\right\|_{(L^\infty(0,T;H^1(\Omega)))^2},
\end{aligned} \tag{4.9}$$

where

$$\mathcal{D}(\delta,f,g) = 2C\left|N(\delta)\right|^3 e^{2TM_1 N(\delta)} K_l^2 \left(\|f^\delta\|_{L^2(\Omega)}^2 + \|g^\delta\|_{L^2(\Omega)}^2 \right).$$

This implies (4.5). Assume that (4.5) holds for $m = j$. We will check that (4.5) holds for $m = j+1$. Indeed, by similar arguments as before, we also get the following two bounds:

$$\begin{aligned}
&\left\| \mathcal{R}_{1,\delta}^{j+1}(u_1, v_1)(\cdot,t) - \mathcal{R}_{1,\delta}^{j+1}(u_2, v_2)(\cdot,t)\right\|_{H^1(\Omega)}^2 \\
&= \left\| \mathcal{R}_{1,\delta}\left(\mathcal{R}_{1,\delta}^{j}(u_1, v_1)(\cdot,t)\right) - \mathcal{R}_{1,\delta}\left(\mathcal{R}_{1,\delta}^{j}(u_2, v_2)(\cdot,t)\right)\right\|_{H^1(\Omega)}^2
\end{aligned}$$

$$\leq C\left|N(\delta)\right|^3 e^{2TM_1N(\delta)}K_l^2\left\|f^\delta\right\|^2_{L^2(\Omega)}\int_t^T\left\|\mathcal{R}^j_{1,\delta}(u_1,v_1)(\cdot,s)-\mathcal{R}^{j+1}_{1,\delta}(u_2,v_2)(\cdot,s)\right\|^2_{H^1(\Omega)}ds$$

and

$$\begin{aligned}
&\left\|\mathcal{R}^{j+1}_{2,\delta}(u_1,v_1)(\cdot,t)-\mathcal{R}^{j+1}_{2,\delta}(u_2,v_2)(\cdot,t)\right\|^2_{H^1(\Omega)}\\
&\quad=\left\|\mathcal{R}_{2,\delta}\left(\mathcal{R}^j_{2,\delta}(u_1,v_1)(\cdot,t)\right)-\mathcal{R}_{2,\delta}\left(\mathcal{R}^j_{2,\delta}(u_2,v_2)(\cdot,t)\right)\right\|^2_{H^1(\Omega)}\\
&\quad\leq C\left|N(\delta)\right|^3 e^{2TM_1N(\delta)}K_l^2\left\|g^\delta\right\|^2_{L^2(\Omega)}\int_t^T\left\|\mathcal{R}^j_{2,\delta}(u_1,v_1)(\cdot,s)-\mathcal{R}^{j+1}_{2,\delta}(u_2,v_2)(\cdot,s)\right\|^2_{H^1(\Omega)}ds.
\end{aligned}$$

From two above observation we find that

$$\begin{aligned}
&\left\|\mathbf{R}^{j+1}_\delta(u_1,v_1)(\cdot,t)-\mathbf{R}^{j+1}_\delta(u_2,v_2)(\cdot,t)\right\|^2_{H^1(\Omega)}\\
&\quad\leq 2\left\|\mathcal{R}^{j+1}_{1,\delta}(u_1,v_1)(\cdot,t)-\mathcal{R}^{j+1}_{1,\delta}(u_2,v_2)(\cdot,t)\right\|^2_{H^1(\Omega)}\\
&\qquad+2\left\|\mathcal{R}^{j+1}_{2,\delta}(u_1,v_1)(\cdot,t)-\mathcal{R}^{j+1}_{2,\delta}(u_2,v_2)(\cdot,t)\right\|^2_{H^1(\Omega)}\\
&\quad\leq 2\mathcal{D}(\delta,f,g)\int_t^T\left\|\mathbf{R}^{j+1}_\delta(u_1,v_1)(\cdot,s)-\mathbf{R}^{j+1}_\delta(u_2,v_2)(\cdot,s)\right\|^2_{H^1(\Omega)}ds.
\end{aligned}\tag{4.10}$$

Using the induction assumption of (3.10), from (3.15) it follows that

$$\begin{aligned}
&\left\|\mathbf{R}^{j+1}_\delta(u_1,v_1)(\cdot,t)-\mathbf{R}^{j+1}_\delta(u_2,v_2)(\cdot,t)\right\|^2_{H^1(\Omega)}\\
&\quad\leq\left(2\mathcal{D}(\delta,f,g)\right)^{j+1}\left\|(u_1,v_1)-(u_2,v_2)\right\|^2_{(L^\infty(0,T;H^1(\Omega)))^2}\int_t^T\frac{(T-s)^j}{j!}ds\\
&\quad=\frac{(T-t)^{j+1}}{(j+1)!}\left(2\mathcal{D}(\delta,f,g)\right)^{j+1}\left\|(u_1,v_1)-(u_2,v_2)\right\|^2_{(L^\infty(0,T;H^1(\Omega)))^2}.
\end{aligned}\tag{4.11}$$

Hence, (3.10) holds for any positive integer m. As a consequence, we conclude that (4.5) holds for any $m\in\mathbb{N}$. Since

$$\lim_{j\to+\infty}\frac{T^{j+1}}{(j+1)!}\left(2\mathcal{D}(\delta,f,g)\right)^{j+1}=0,$$

there exists a positive constant j_0 such that

$$\frac{T^{j+1}}{(j+1)!}\left(2\mathcal{D}(\delta,f,g)\right)^{j+1}<1.$$

Using the Banach fixed point theorem (see [31–33]), we conclude that $\mathbf{R}^{j_0}_\delta$ has a fixed point (u^+,v^+) on the space $(L^\infty(0,T;H^1(\Omega)))^2$. It is easy to get that (u^*,v^*) is also a solution of the nonlinear equation

$$\mathbf{R}^{j_0}_\delta\left(u^+,v^+\right)=\left(u^+,v^+\right).$$

From (3.6) and (4.1) we find that

$$\begin{aligned} &u^{N,\delta}(x,t) - u(x,t) \\ &\quad= \sum_{n=1}^{N(\delta)} \exp\left(\lambda_n \int_t^T \mathcal{L}\left(\left\|\nabla u^{N,\delta}(\cdot,s)\right\|_{L^2}, \left\|\nabla v^{N,\delta}(\cdot,s)\right\|_{L^2}\right) ds\right)\langle f^\delta, e_n\rangle e_n(x) \\ &\qquad - \sum_{n=1}^{\infty} \exp\left(\lambda_n \int_t^T \mathcal{L}\left(\left\|\nabla u(\cdot,s)\right\|_{L^2}, \left\|\nabla v(\cdot,s)\right\|_{L^2}\right) ds\right)\langle f, e_n\rangle e_n(x). \end{aligned} \tag{4.12}$$

By a simple calculation we find that

$$\begin{aligned} &u^{N,\delta}(x,t) - u(x,t) \\ &\quad= \sum_{n=1}^{N(\delta)} \exp\left(\lambda_n \int_t^T \mathcal{L}\left(\left\|\nabla u^{N,\delta}(\cdot,s)\right\|_{L^2}, \left\|\nabla v^{N,\delta}(\cdot,s)\right\|_{L^2}\right) ds\right)\langle f^\delta - f, e_n\rangle e_n(x) \\ &\qquad + \sum_{n=1}^{N(\delta)} \langle f, e_n\rangle e_n(x)\left[\exp\left(\lambda_n \int_t^T \mathcal{L}\left(\left\|\nabla u(\cdot,s)\right\|_{L^2}, \left\|\nabla v(\cdot,s)\right\|_{L^2}\right) ds\right)\right. \\ &\qquad \left.- \exp\left(\lambda_n \int_t^T \mathcal{L}\left(\left\|\nabla u^{N,\delta}(\cdot,s)\right\|_{L^2}, \left\|\nabla v^{N,\delta}(\cdot,s)\right\|_{L^2}\right) ds\right)\right] \\ &\qquad + \sum_{n>N(\delta)}^{\infty} \exp\left(\lambda_n \int_t^T \mathcal{L}\left(\left\|\nabla u(\cdot,s)\right\|_{L^2}, \left\|\nabla v(\cdot,s)\right\|_{L^2}\right) ds\right)\langle f, e_n\rangle e_n(x) \\ &\quad= \text{Error}_1 + \text{Error}_2 + \text{Error}_3. \end{aligned} \tag{4.13}$$

First of all, let us look at the first term. Using Parseval's equality and noting that $\mathcal{L}(z_1, z_2) \le \mathcal{M}_1$ for all $(z_1, z_2) \in \mathbb{R}^2$, Error_1 is bounded by

$$\begin{aligned} \|\text{Error}_1\|_{H^1(\Omega)}^2 &= \sum_{n=1}^{N(\delta)} \lambda_n \exp\left(2\lambda_n \int_t^T \mathcal{L}\left(\left\|\nabla u^{N,\delta}(\cdot,s)\right\|_{L^2}, \left\|\nabla v^{N,\delta}(\cdot,s)\right\|_{L^2}\right) ds\right)\langle f^\delta - f, e_n\rangle^2 \\ &\le \sum_{n=1}^{N(\delta)} \lambda_{N(\delta)} \exp(2\mathcal{M}_1 T\lambda_{N(\delta)}) \sum_{n=1}^{N(\delta)} \langle f^\delta - f, e_n\rangle^2 \\ &\le \lambda_{N(\delta)} \exp(2\mathcal{M}_1 T\lambda_{N(\delta)}) \left\|f^\delta - f\right\|^2 \le \lambda_{N(\delta)} \exp(2\mathcal{M}_1 T\lambda_{N(\delta)})\delta^2, \end{aligned} \tag{4.14}$$

which allows us to derive that

$$\|\text{Error}_1\|_{H^1(\Omega)} \le \sqrt{\lambda_{N(\delta)}} \exp(\mathcal{M}_1 T\lambda_{N(\delta)})\delta. \tag{4.15}$$

Next, we treat Error_2. Using Parseval's equality, by the inequality $|e^c - e^d| \le |c - d| \max(e^c, e^d)$, we get

$$\begin{aligned} \|\text{Error}_2\|_{H^1(\Omega)}^2 &\le \sum_{n=1}^{N(\delta)} \langle f, e_n\rangle^2 \lambda_n e^{2M_1 T\lambda_n} \\ &\quad \times \left(\int_t^T \mathcal{L}\left(\left\|\nabla u^{N,\delta}(\cdot,s)\right\|_{L^2}, \left\|\nabla v^{N,\delta}(\cdot,s)\right\|_{L^2}\right) ds\right. \end{aligned}$$

$$
\begin{aligned}
&- \int_t^T \mathcal{L}\big(\|\nabla u(\cdot,s)\|_{L^2}, \|\nabla v(\cdot,s)\|_{L^2}\big)\,ds\bigg)^2 \\
&\leq \bigg(\int_t^T \mathcal{L}\big(\|\nabla u^{N,\delta}(\cdot,s)\|_{L^2}, \|\nabla v^{N,\delta}(\cdot,s)\|_{L^2}\big)\,ds \\
&- \int_t^T \mathcal{L}\big(\|\nabla u(\cdot,s)\|_{L^2}, \|\nabla v(\cdot,s)\|_{L^2}\big)\,ds\bigg)^2 E,
\end{aligned}
\tag{4.16}
$$

where we used that

$$
\begin{aligned}
&\sum_{n=1}^{N(\delta)} \langle f, e_n\rangle^2 \lambda_n e^{2M_1T\lambda_n} \leq \sum_{n=1}^{\infty} \langle f, e_n\rangle^2 \lambda_n e^{2M_1T\lambda_n} \leq \lambda_1^{-\gamma} E, \\
&\exp\bigg(2\lambda_n \int_t^T \mathcal{L}\big(\|\nabla u^{N,\delta}(\cdot,s)\|_{L^2}, \|\nabla v^{N,\delta}(\cdot,s)\|_{L^2}\big)\,ds\bigg) \leq \exp\bigg(2\lambda_n \int_t^T M_1\,ds\bigg) \\
&\qquad \leq e^{2M_1T\lambda_n},
\end{aligned}
\tag{4.17}
$$

and

$$
\begin{aligned}
\exp\bigg(\lambda_n \int_t^T \mathcal{L}\big(\|\nabla u^{N,\delta}(\cdot,s)\|_{L^2}, \|\nabla v^{N,\delta}(\cdot,s)\|_{L^2}\big)\,ds\bigg)\big) &\leq \exp\bigg(2\lambda_n \int_t^T M_1\,ds\bigg) \\
&\leq e^{2M_1T\lambda_n}.
\end{aligned}
\tag{4.18}
$$

Using Lemma (2.1), we get that

$$
\begin{aligned}
&\bigg|\int_t^T \mathcal{L}\big(\|\nabla u^{N,\delta}(\cdot,s)\|_{L^2}, \|\nabla v^{N,\delta}(\cdot,s)\|_{L^2}\big)\,ds - \int_t^T \mathcal{L}\big(\|\nabla u(\cdot,s)\|_{L^2}, \|\nabla v(\cdot,s)\|_{L^2}\big)\,ds\bigg| \\
&\quad \leq K_l \int_t^T \big(\|\nabla u^{N,\delta}(\cdot,s) - \nabla u(\cdot,s)\|_{L^2} + \|\nabla v^{N,\delta}(\cdot,s) - \nabla v(\cdot,s)\|_{L^2}\big),
\end{aligned}
\tag{4.19}
$$

and from the inequality $(c+d)^2 \leq 2c^2 + 2d^2$, $c, d \geq 0$, it follows that

$$
\begin{aligned}
&\bigg(\int_t^T \mathcal{L}\big(\|\nabla u^{N,\delta}(\cdot,s)\|_{L^2}, \|\nabla v^{N,\delta}(\cdot,s)\|_{L^2}\big)\,ds - \int_t^T \mathcal{L}\big(\|\nabla u(\cdot,s)\|_{L^2}, \|\nabla v(\cdot,s)\|_{L^2}\big)\,ds\bigg)^2 \\
&\quad \leq 2K_l\bigg(\int_t^T \|\nabla u^{N,\delta}(\cdot,s) - \nabla u(\cdot,s)\|_{L^2}\,ds + \int_t^T \|\nabla v^{N,\delta}(\cdot,s) - \nabla v(\cdot,s)\|_{L^2}\,ds\bigg).
\end{aligned}
\tag{4.20}
$$

Combining (4.16) and (4.20), we arrive at

$$
\begin{aligned}
\|\mathrm{Error}_2\|^2_{H^1(\Omega)} \leq 2K_l\lambda_1^{-\gamma} E\bigg(&\int_t^T \|\nabla u^{N,\delta}(\cdot,s) - \nabla u(\cdot,s)\|_{L^2}\,ds \\
&+ \int_t^T \|\nabla v^{N,\delta}(\cdot,s) - \nabla v(\cdot,s)\|_{L^2}\,ds\bigg).
\end{aligned}
\tag{4.21}
$$

The term Error_3 is bounded by

$$
\begin{aligned}
&\|\mathrm{Error}_3\|^2_{H^1(\Omega)} \\
&\quad = \sum_{n>N(\delta)}^{\infty} \lambda_n \exp\bigg(2\lambda_n \int_t^T \mathcal{L}\big(\|\nabla u(\cdot,s)\|_{L^2}, \|\nabla v(\cdot,s)\|_{L^2}\big)\,ds\bigg) \langle f, e_n\rangle^2
\end{aligned}
$$

$$\le \big(N(\delta)\big)^{-\gamma} \sum_{n>N(\delta)}^{\infty} \lambda_n^{1+\gamma} \exp\biggl(2\lambda_n \int_t^T \mathcal{L}\bigl(\|\nabla u(\cdot,s)\|_{L^2}, \|\nabla v(\cdot,s)\|_{L^2}\bigr)\,ds\biggr)\langle f, e_n\rangle^2$$
$$\le E\big(N(\delta)\big)^{-\gamma}. \tag{4.22}$$

Combining (4.14), (4.21), and (4.22), we find that

$$\begin{aligned}
&\bigl\|u^{N,\delta}(\cdot,t) - u(\cdot,t)\bigr\|_{H^1(\Omega)}^2 \\
&\quad\le 3\|\mathrm{Error}_1\|_{H^1(\Omega)}^2 + 3\|\mathrm{Error}_2\|_{H^1(\Omega)}^2 + 3\|\mathrm{Error}_3\|_{H^1(\Omega)}^2 \\
&\quad\le 3\lambda_{N(\delta)} \exp(2\mathcal{M}_1 T\lambda_{N(\delta)})\delta^2 + 3E\big(N(\delta)\big)^{-\gamma} \\
&\qquad + 6K_l\lambda_1^{-\gamma} E\biggl(\int_t^T \bigl\|\nabla u^{N,\delta}(\cdot,s) - \nabla u(\cdot,s)\bigr\|_{L^2}\,ds \\
&\qquad + \int_t^T \bigl\|\nabla v^{N,\delta}(\cdot,s) - \nabla v(\cdot,s)\bigr\|_{L^2}\,ds\biggr).
\end{aligned} \tag{4.23}$$

By a similar argument we also get that

$$\begin{aligned}
&\bigl\|v^{N,\delta}(\cdot,t) - v(\cdot,t)\bigr\|_{H^1(\Omega)}^2 \\
&\quad\le 3\lambda_{N(\delta)} \exp(2\mathcal{M}_1 T\lambda_{N(\delta)})\delta^2 + 3E\big(N(\delta)\big)^{-\gamma} \\
&\qquad + 6K_l\lambda_1^{-\gamma} E\biggl(\int_t^T \bigl\|\nabla u^{N,\delta}(\cdot,s) - \nabla u(\cdot,s)\bigr\|_{L^2}\,ds \\
&\qquad + \int_t^T \bigl\|\nabla v^{N,\delta}(\cdot,s) - \nabla v(\cdot,s)\bigr\|_{L^2}\,ds\biggr).
\end{aligned} \tag{4.24}$$

By the previous to equations, recalling that $\|\nabla\psi\|_{L^2(\Omega)} \le C\|\psi\|_{H^1(\Omega)}$, we obtain the estimate

$$\begin{aligned}
&\bigl\|u^{N,\delta}(\cdot,t) - u(\cdot,t)\bigr\|_{H^1(\Omega)}^2 + \bigl\|v^{N,\delta}(\cdot,t) - v(\cdot,t)\bigr\|_{H^1(\Omega)}^2 \\
&\quad\le 6\lambda_{N(\delta)} \exp(2\mathcal{M}_1 T\lambda_{N(\delta)})\delta^2 + 6E\big(N(\delta)\big)^{-\gamma} \\
&\qquad + 6K_l\lambda_1^{-\gamma} EC\int_t^T \bigl(\bigl\|u^{N,\delta}(\cdot,s) - u(\cdot,s)\bigr\|_{H^1(\Omega)}^2 + \bigl\|v^{N,\delta}(\cdot,s) - v(\cdot,s)\bigr\|_{H^1(\Omega)}^2\bigr)\,ds.
\end{aligned} \tag{4.25}$$

By applying Grönwall's inequality we deduce that

$$\begin{aligned}
&\bigl\|u^{N,\delta}(\cdot,t) - u(\cdot,t)\bigr\|_{H^1(\Omega)}^2 + \bigl\|v^{N,\delta}(\cdot,t) - v(\cdot,t)\bigr\|_{H^1(\Omega)}^2 \\
&\quad\le \bigl(6\lambda_{N(\delta)} \exp(2\mathcal{M}_1 T\lambda_{N(\delta)})\delta^2 + 6E\big(N(\delta)\big)^{-\gamma}\bigr)\exp\bigl(6K_l\lambda_1^{-\gamma} EC(T-t)\bigr).
\end{aligned} \tag{4.26}$$

□

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Authors' contributions
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