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On multiple solutions to a nonlocal fractional $p(\cdot)$-Laplacian problem with concave–convex nonlinearities

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Abstract

The aim of this paper is to examine the existence of at least two distinct nontrivial solutions to a Schrödinger-type problem involving the nonlocal fractional $p(\cdot)$-Laplacian with concave–convex nonlinearities when, in general, the nonlinear term does not satisfy the Ambrosetti–Rabinowitz condition. The main tools for obtaining this result are the mountain pass theorem and a modified version of Ekeland's variational principle for an energy functional with the compactness condition of the Palais–Smale type, namely the Cerami condition. Also we discuss several existence results of a sequence of infinitely many solutions to our problem. To achieve these results, we employ the fountain theorem and the dual fountain theorem as main tools.

MSC: 35B38; 35D30; 35J20; 35J60; 35J66

Keywords: Fractional $p(x)$-Laplacian; Variable exponent Lebesgue–Sobolev spaces; Weak solution; Fountain theorem; Dual fountain theorem

1 Introduction

In the last years the study of problems involving differential equations and variational problems associated with the $p(\cdot)$-Laplacian operator has been paid to an increasing deal of attention because they can be viewed as a model for many physical phenomena which arise in several investigations related to elastic mechanics, electro-rheological fluid ("smart fluids"), image processing, *etc.* We refer the reader to [6, 16, 21, 32, 43, 49] and the references therein.

On the other hand, in the recent years the study of equations with nonstandard growth and related nonlocal equations has gained an increasing deal of attention due to both pure mathematical research aspects and real-world applications. This fact is justified by the occurrence of the aforementioned problems in many different applications such as conservation laws, ultra-materials and water waves, phase transitions, thin obstacle problem, optimization, flames propagation, stratified materials, anomalous diffusion, ultra-relativistic limits of quantum mechanics, crystal dislocation, soft thin films, minimal surfaces, semipermeable membranes and flame propagation, multiple scattering, mathematical finance, and so on. For a comprehensive introduction to the study of nonlocal problems, we refer to the work of Di Nezza, Palatucci, and Valdinoci [20], see [13, 25, 34, 36, 50] and the references therein for more details.

Therefore, a natural question is to understand if some results can be recovered when we change the local $p(\cdot)$-Laplacian, defined as $-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$, into the nonlocal fractional $p(\cdot)$-Laplacian. In this direction, several researchers have attempted to extend the study of the classical exponent variable case to include the fractional case (see for instance [4, 7, 8, 26, 27, 31, 38, 58]). In particular, as far as we are aware, Kaumann *et al.* [31] defined a new class of fractional Sobolev spaces with variable exponents that takes a fractional variable exponent operator into consideration. In particular, in [8] the authors discussed several fundamental properties related to the aforementioned function space and the related nonlocal operator and, using a direct variational method, the authors showed an application to a class of nonlocal fractional problems with several variable exponents. Precisely, as applications, they proved the existence of at least one solution for equations driven by the fractional $p(\cdot)$-Laplacian. Inspired by these recent works, further fundamental embeddings for the fractional Sobolev spaces with variable exponents and their applications—such as *a priori* bounds and multiplicity of solutions of problems driven by the fractional $p(\cdot)$-Laplacian—have been provided by Ho and Kim [26]. Also they obtained the existence of many solutions for a class of critical nonlocal problems with variable exponents; see [27]. We refer the interested reader to [4, 5, 58] for the existence results to Kirchhoff-type problems driven by a $p(\cdot)$-fractional operator.

This paper is devoted to the study of the existence of nontrivial solutions for the following Schrödinger-type problem involving the nonlocal fractional $p(\cdot)$-Laplacian:

$$-\mathfrak{L}_{\mathcal{K}}z + V(x)|z|^{p(x,x)-2}z = \lambda a(x)|z|^{r(x)-2}z + f(x,z) \quad \text{in } \mathbb{R}^N, \tag{P}$$

where $N \geq 2$, $\lambda > 0$ is a parameter, $p : \mathbb{R}^N \times \mathbb{R}^N \to (1,\infty)$ is a continuous function satisfying $p(x) := p(x,x)$ for all $x \in \mathbb{R}^N$, $r : \mathbb{R}^N \to (1,\infty)$ is continuous, V and a are suitable potential functions in $(0,\infty)$, and $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfies a Carathéodory condition. Here, $\mathfrak{L}_{\mathcal{K}}$ stands for the following pointwise-defined nonlocal operator:

$$\mathfrak{L}_{\mathcal{K}}z(x) = 2\int_{\mathbb{R}^N} \bigl|z(x) - z(y)\bigr|^{p(x,y)-2}\bigl(z(x) - z(y)\bigr)\mathcal{K}(x,y)\,dy \quad \text{for all } x \in \mathbb{R}^N,$$

where $p \in C(\mathbb{R}^N \times \mathbb{R}^N)$ is uniformly continuous such that p is symmetric, i.e., $p(x,y) = p(y,x)$ for all $x, y \in \mathbb{R}^N$; $0 < s < 1$; $1 < \inf_{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N} p(x,y) \leq \sup_{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N} p(x,y) < \frac{N}{s}$; and $\mathcal{K} : \mathbb{R}^N \times \mathbb{R}^N \to (0,+\infty)$ is a kernel function such that the following conditions are fulfilled:

($\mathcal{K}1$) $m\mathcal{K} \in L^1(\mathbb{R}^N \times \mathbb{R}^N)$, where $m(x,y) = \min\{|x-y|^{p(x,y)}, 1\}$;

($\mathcal{K}2$) There exists a constant $\theta_0 > 0$ such that $\mathcal{K}(x,y)|x-y|^{N+sp(x,y)} \geq \theta_0$ for almost all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$ and $x \neq y$;

($\mathcal{K}3$) $\mathcal{K}(x,y) = \mathcal{K}(y,x)$ for all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$.

With the choice $\mathcal{K}(x,y) = |x-y|^{-N-sp(x,y)}$, the operator $\mathfrak{L}_{\mathcal{K}}$ becomes the fractional $p(\cdot)$-Laplacian operator $(-\Delta)^s_{p(\cdot)}$ defined as

$$(-\Delta)^s_{p(x)}z(x) = \text{P.V.} \int_{\mathbb{R}^N} \frac{|z(x) - z(y)|^{p(x,y)-2}(z(x) - z(y))}{|x-y|^{N+sp(x,y)}}\,dy, \quad x \in \mathbb{R}^N.$$

The first purpose of the present paper is to establish the existence of at least two distinct nontrivial solutions for Schrödinger-type problems involving the nonlocal fractional $p(\cdot)$-Laplacian in case where the nonlinear term is concave–convex. The primary tools for obtaining this result are the mountain pass theorem (see [3]) and a variant of Ekeland's variational principle (see [6]) for an energy functional. We assume that this energy functional satisfies a Palais–Smale-type compactness condition, namely the Cerami condition. This kind of nonlinearity has been extensively studied since the seminal work of Ambrosetti, Brezis, and Cerami [2]. For elliptic equations with the concave–convex nonlinearity, we refer the reader also to [12, 14, 15, 19, 28, 53–55] and the references therein. Precisely, the existence of multiple solutions for an elliptic problem of a nonhomogeneous fractional p-Kirchhoff-type, involving concave–convex nonlinearities, has been studied in [55]. By means of variational techniques and Ekeland's variational principle, the authors in [28] obtained the existence of two nontrivial nonnegative solutions and infinitely many solutions for the following degenerated $p(x)$-Laplacian equations involving concave–convex type nonlinearities with two parameters:

$$\begin{cases} -\operatorname{div}(w(x)|\nabla z|^{p(x)-2}\nabla z) = \lambda a(x)|z|^{r(x)-2}z + \mu b(x)|z|^{q(x)-2}z & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with a smooth boundary $\partial\Omega, p, q, r \in C(\overline{\Omega}, (1,\infty))$ with $r(x) < p(x) < q(x)$ for all $x \in \Omega$, w, a, b are measurable functions on Ω that are positives a.e. in Ω, and λ, μ are real parameters. Very recently, Biswas and Tiwari [11] investigated an elliptic problem involving nonlocal operator with variable exponents and concave–convex nonlinearity in a bounded domain with Dirichlet boundary condition. Biswas and Tiwari assumed the condition by Ambrosetti and Rabinowitz [3] (see [1, 24] for elliptic equations with variable exponents) and then employed the mountain pass theorem and Ekeland's variational principle to obtain the multiplicity result.

As we known, the condition of Ambrosetti–Rabinowitz type in [3], that is, there exists a constant $\theta > p$ such that

$$\begin{aligned} &0 < \theta F(x,\tau) \le f(x,\tau)\tau, \\ &\quad \text{for all } \tau \in \mathbb{R}\setminus\{0\} \text{ and } x \in \mathbb{R}^N, \text{where } F(x,\tau) = \int_0^\tau f(x,t)\,dt, \end{aligned} \tag{1.1}$$

is essential in securing the boundedness of the Palais–Smale sequence of an energy functional. However, this condition is quite restrictive and removes several nonlinearities. For this reason, during the last few decades there have been extensive studies which tried to drop it; see [1, 16, 29, 30, 33, 34, 38–40, 42, 44].

In that sense, our first aim is to discuss the existence of two nontrivial distinct solutions to problem (P) for the case of a combined effect of concave–convex nonlinearities when the nonlinear growth f does not satisfy the condition of Ambrosetti–Rabinowitz type. The main point in the present paper is to discuss the existence of multiple solutions to (P) under a new and mild assumption for the convex term f that does not satisfy (1.1) and is different from those studied in [1, 16, 29, 30, 33, 34, 38–40, 42, 44]. In particular, we give some examples to demonstrate that this condition is not artificial. The main difficulty for obtaining the multiplicity result under this assumption on the convex term f is to verify the Cerami condition of the energy functional associated with (P). It is worth noting that we overcome it from the coercivity of the potential function V.

For recent developments in the context of concave–convex problems, we mention the work of Papageorgiou–Scapellato [45] where the authors studied nonlinear Robin problems driven by the p-Laplacian plus and indefinite potential in which the reaction exhibits the competing effects of a parametric concave (that is, $(p-1)$-sublinear) term and of a convex (that is, $(p-1)$-superlinear) term. In [45] the authors did not require the Ambrosetti–Rabinowitz condition and obtained a bifurcation-type theorem that describes the dependence of a set of positive solutions on the parameter $\lambda > 0$. In line with the contents of the paper [45], Papageorgiou and Scapellato [47] considered Robin problems driven by the (p,q)-Laplacian plus an indefinite potential term and did not require the Ambrosetti–Rabinowitz condition for the reaction. We mention that in [45] there is no parameter and the authors, in addition to constant sign solutions, produced nodal solutions. Finally, we cite a variant of the classical concave–convex problem studied in [46]. Precisely, Papageorgiou and Scapellato in [46] studied a nonlinear resonant boundary value problem where there is no parameter, the convex term is replaced by a resonant (that is, $(p-1)$-linear) term, and the concave contribution comes from the boundary condition.

The second main aim of this paper is to obtain several existence results of a sequence of infinitely many solutions to problem (P). First we are to discuss that multiple large energy solutions for problem (P) exist (see Theorem 3.12). The second is to establish that problem (P) possesses a sequence of infinitely many small energy solutions (see Theorem 3.16). The strategy of the proof for these consequences is based on the applications of variational tools such as the fountain theorem and the dual fountain theorem, which were initially built by the papers [9] and [10], respectively. Our study on such multiplicity results for nonlinear elliptic equations of variational type is particularly inspired by the contributions in recent works [18, 30, 38, 41, 48, 51] and the references therein. However, in some sense the proof of our consequence for multiple small energy solutions is different from that of the previous related works [10, 41, 51, 52]. To the best of our knowledge, while many authors are interested in the study of elliptic problems in both local and nonlocal cases, the present paper is the first endeavor to develop the existence results for the concave–convex-type problems driven by nonlocal fractional $p(\cdot)$-Laplacian.

This paper's outline is the following: we firstly present some necessary preliminary knowledge of function spaces. Next we give the variational framework associated with problem (P), and then we establish the results about at least two distinct nontrivial solutions to the nonlocal fractional $p(\cdot)$-Laplacian with concave–convex nonlinearities by applying the mountain pass theorem and a variant of Ekeland's variational principle for an energy functional with the Cerami condition. Finally, under suitable conditions on the convex term f, we carry out various existence results of infinitely many nontrivial solutions by employing the variational principle.

2 Preliminaries

In this section we present a natural functional framework associated with problem (P). We briefly recall some definitions and fundamental properties of the variable exponent Lebesgue spaces and a Lebesgue–Sobolev space of fractional type $W^{s,q(\cdot),p(\cdot,\cdot)}(\Omega)$ which will be used throughout the paper. For further details on these spaces, we refer the reader to [4, 7, 8, 26, 27, 31, 58].

Set

$$C_+(\overline{\Omega}) = \left\{\ell \in C(\overline{\Omega}) : \min_{x\in\overline{\Omega}} \ell(x) > 1\right\}.$$

For any $\ell \in C_+(\overline{\Omega})$, we define

$$\ell_+ = \sup_{x\in\Omega} \ell(x) \quad \text{and} \quad \ell_- = \inf_{x\in\Omega} \ell(x).$$

Let $M(\Omega)$ be the vector space of all measurable functions from Ω into $\mathbb{R}$. We identify two such functions which differ only on a Lebesgue-null set. Given $h \in C_+(\overline{\Omega})$, the anisotropic Lebesgue space $L^{h(\cdot)}(\Omega)$ is defined by

$$L^{h(\cdot)}(\Omega) = \left\{z \in M(\Omega) : \int_\Omega |z|^{h(x)}\,dx < \infty\right\}.$$

We equip this space with the so-called *Luxemburg norm* defined by

$$\|z\|_{L^{h(\cdot)}(\Omega)} = \inf\left[\vartheta > 0 : \int_\Omega \left|\frac{z(x)}{\vartheta}\right|^{h(x)} dx \le 1\right].$$

In the anisotropic Lebesgue spaces the following Hölder inequality holds.

Lemma 2.1 ([23, 35]) *The space $L^{h(\cdot)}(\mathbb{R}^N)$ is a separable, uniformly convex Banach space, and its conjugate space is $L^{h'(\cdot)}(\mathbb{R}^N)$ where $1/h(x) + 1/h'(x) = 1$. For any $z \in L^{h(\cdot)}(\mathbb{R}^N)$ and $\omega \in L^{h'(\cdot)}(\mathbb{R}^N)$, we have*

$$\left|\int_{\mathbb{R}^N} z\omega\,dx\right| \le \left(\frac{1}{h_-} + \frac{1}{(h')_-}\right)\|z\|_{L^{h(\cdot)}(\mathbb{R}^N)}\|\omega\|_{L^{h'(\cdot)}(\mathbb{R}^N)} \le 2\|z\|_{L^{h(\cdot)}(\mathbb{R}^N)}\|\omega\|_{L^{h'(\cdot)}(\mathbb{R}^N)}.$$

Lemma 2.2 ([23]) *Let us consider the modular function*

$$\psi(z) = \int_{\mathbb{R}^N} |z|^{h(x)}\,dx \quad \textit{for any } z \in L^{h(\cdot)}\big(\mathbb{R}^N\big).$$

Then we have

(1) $\psi(z) > 1$ $(= 1; < 1)$ *if and only if* $\|z\|_{L^{h(\cdot)}(\mathbb{R}^N)} > 1$ $(= 1; < 1)$, *respectively*;
(2) *If* $\|z\|_{L^{h(\cdot)}(\mathbb{R}^N)} > 1$, *then* $\|z\|^{h_-}_{L^{h(\cdot)}(\mathbb{R}^N)} \le \psi(z) \le \|z\|^{h_+}_{L^{h(\cdot)}(\mathbb{R}^N)}$;
(3) *If* $\|z\|_{L^{h(\cdot)}(\mathbb{R}^N)} < 1$, *then* $\|z\|^{h_+}_{L^{h(\cdot)}(\mathbb{R}^N)} \le \psi(z) \le \|z\|^{h_-}_{L^{h(\cdot)}(\mathbb{R}^N)}$.

Let Ω be a Lipschitz domain in $\mathbb{R}^N$. Let $s \in (0,1)$ and let $p \in C(\overline{\Omega}\times\overline{\Omega}, (1,\infty))$ be such that $p(x,y) = p(y,x)$ for all $x, y \in \overline{\Omega}$ and

$$1 < p^- := \inf_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} p(x,y) \le p^+ := \sup_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} p(x,y) < +\infty.$$

For $q \in C_+(\mathbb{R}^N)$, define

$$W^{s,q(\cdot),p(\cdot,\cdot)}\big(\mathbb{R}^N\big) := \left\{z \in L^{q(\cdot)}\big(\mathbb{R}^N\big) : \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|z(x)-z(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy < +\infty\right\},$$

and we set

$$[z]_{s,p(\cdot,\cdot)}\big(\mathbb{R}^N\big) := \inf\left\{\lambda > 0 : \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|z(x)-z(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{N+sp(x,y)}}\,dx\,dy < 1\right\}.$$

Then $W^{s,q(\cdot),p(\cdot,\cdot)}(\mathbb{R}^N)$ endowed with the norm

$$\|z\|_{s,q,p} := \|z\|_{L^{q(\cdot)}(\mathbb{R}^N)} + [z]_{s,p(\cdot,\cdot)}\big(\mathbb{R}^N\big)$$

is a separable reflexive Banach space (see [7, 8, 31]).

Lemma 2.3 ([26]) *Let Ω be a bounded Lipschitz domain, and let p, q, and s be as above. Assume furthermore that*

$$sp^+ < N \quad \text{and} \quad q(x) \geq p(x) \quad \text{for all } x \in \overline{\Omega}.$$

Then the following embedding holds:

$$W^{s,q(\cdot),p(\cdot,\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{r(\cdot)}(\Omega)$$

for any $r \in C_+(\overline{\Omega})$ such that $r(x) < p_s^*(x) := \frac{Np(x,x)}{N-sp(x,x)}$ for all $x \in \overline{\Omega}$.

For the sake of brevity, we write $p(x)$ in place of $p(x,x)$ for some cases, and hence $p \in C_+(\mathbb{R}^N)$. In addition, we write $W^{s,p(\cdot,\cdot)}(\mathbb{R}^N)$ in place of $W^{s,p(\cdot),p(\cdot,\cdot)}(\mathbb{R}^N)$. We recall the following embeddings (see [26, Theorem 3.5]).

Lemma 2.4 *Let $s \in (0,1)$. Let $p \in C_+(\mathbb{R}^N \times \mathbb{R}^N)$ be a uniformly continuous and symmetric function with $sp^+ < N$. Then it holds that*

(i) $W^{s,p(\cdot,\cdot)}(\mathbb{R}^N) \hookrightarrow L^{r(\cdot)}(\mathbb{R}^N)$ *for any uniform continuous function $r \in C_+(\mathbb{R}^N)$ fulfilling $p(x,x) \leq r(x)$ for all $x \in \mathbb{R}^N$ and $\inf_{x\in\mathbb{R}^N}(p_s^*(x) - r(x)) > 0$;*

(ii) $W^{s,p(\cdot,\cdot)}(\mathbb{R}^N) \hookrightarrow\hookrightarrow L^{r(\cdot)}_{\mathrm{loc}}(\mathbb{R}^N)$ *for any $r \in C_+(\mathbb{R}^N)$ with $r(x) < p_s^*(x)$ for all $x \in \mathbb{R}^N$.*

In the following, let $0 < s < 1$ and let $p \in C_+(\mathbb{R}^N \times \mathbb{R}^N)$ be a uniformly continuous and symmetric function such that $sp^+ < N$. Suppose that $\mathcal{K} : \mathbb{R}^N \times \mathbb{R}^N \to (0,\infty)$ is a function with conditions ($\mathcal{K}$1)–($\mathcal{K}$3). Let us denote with $W^{s,p(\cdot,\cdot)}_{\mathcal{K}}(\mathbb{R}^N)$ the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|z\|_{W^{s,p(\cdot,\cdot)}_{\mathcal{K}}(\mathbb{R}^N)} := \|z\|_{L^{p(\cdot)}(\mathbb{R}^N)} + |z|_{W^{s,p(\cdot,\cdot)}_{\mathcal{K}}(\mathbb{R}^N)},$$

where

$$|z|_{W^{s,p(\cdot,\cdot)}_{\mathcal{K}}(\mathbb{R}^N)} := \inf\left\{\lambda > 0 : \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{\lambda^{p(x,y)}}\big|z(x)-z(y)\big|^{p(x,y)}\mathcal{K}(x,y)\,dx\,dy < 1\right\}.$$

According to the basic idea in [23], we obtain the following result.

Lemma 2.5 *Denote*

$$\chi(z) = \int_{\mathbb{R}^N}|z|^{p(x)}\,dx + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\big|z(x)-z(y)\big|^{p(x,y)}\mathcal{K}(x,y)\,dx\,dy \quad \text{for any } z \in W^{s,p(\cdot,\cdot)}_{\mathcal{K}}\big(\mathbb{R}^N\big).$$

Then we have

(1) $\chi(z) > 1$ $(= 1; < 1)$ *if and only if* $\|z\|_{W^{s,p(\cdot,\cdot)}_{\mathcal{K}}(\mathbb{R}^N)} > 1$ $(= 1; < 1)$, *respectively*;

(2) *If* $\|z\|_{W^{s,p(\cdot,\cdot)}_{\mathcal{K}}(\mathbb{R}^N)} > 1$, *then* $\|z\|^{p^-}_{W^{s,p(\cdot,\cdot)}_{\mathcal{K}}(\mathbb{R}^N)} \le \chi(z) \le \|z\|^{p^+}_{W^{s,p(\cdot,\cdot)}_{\mathcal{K}}(\mathbb{R}^N)}$;

(3) *If* $\|z\|_{W^{s,p(\cdot,\cdot)}_{\mathcal{K}}(\mathbb{R}^N)} < 1$, *then* $\|z\|^{p^+}_{W^{s,p(\cdot,\cdot)}_{\mathcal{K}}(\mathbb{R}^N)} \le \chi(z) \le \|z\|^{p^-}_{W^{s,p(\cdot,\cdot)}_{\mathcal{K}}(\mathbb{R}^N)}$.

Throughout this paper, we denote $\mathcal{X} := W^{s,p(\cdot,\cdot)}_{\mathcal{K}}(\mathbb{R}^N)$, and let $\mathcal{X}^*$ be a dual space of $\mathcal{X}$. Furthermore, $\langle\cdot,\cdot\rangle$ denotes the pairing of $\mathcal{X}$ and its dual $\mathcal{X}^*$.

3 Main results

In this section, we show the multiplicity result of a weak solution to problem (P) by employing the variational principle.

Definition 3.1 We say that $z \in \mathcal{X}$ is a weak solution of problem (P) if

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \left|z(x)-z(y)\right|^{p(x,y)-2}\big(z(x)-z(y)\big)\big(\varphi(x)-\varphi(y)\big)\mathcal{K}(x,y)\,dx\,dy \\
&\qquad + \int_{\mathbb{R}^N} V(x)|z|^{p(x)-2}z\varphi\,dx \\
&\qquad = \lambda\int_{\mathbb{R}^N} a(x)|z|^{r(x)-2}z\varphi\,dx + \int_{\mathbb{R}^N} f(x,z)\varphi\,dx
\end{aligned}$$

for all $\varphi \in \mathcal{X}$.

Let us define the functional $\mathcal{A}:\mathcal{X}\to\mathbb{R}$ by

$$\mathcal{A}(z) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{p(x,y)}\left|z(x)-z(y)\right|^{p(x,y)}\mathcal{K}(x,y)\,dx\,dy + \int_{\mathbb{R}^N}\frac{V(x)}{p(x)}|z|^{p(x)}\,dx.$$

Then from [8, 31] it follows that $\mathcal{A} \in C^1(\mathcal{X},\mathbb{R})$, and its Fréchet derivative is given by

$$\begin{aligned}
\left\langle \mathcal{A}'(z),\varphi\right\rangle &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \left|z(x)-z(y)\right|^{p(x,y)-2}\big(z(x)-z(y)\big)\big(\varphi(x)-\varphi(y)\big)\mathcal{K}(x,y)\,dx\,dy \\
&\quad + \int_{\mathbb{R}^N} V(x)|z|^{p(x)-2}z\varphi\,dx.
\end{aligned}$$

Let $F(x,\tau) = \int_0^\tau f(x,s)\,ds$. Let us assume that

(H) $p,q,r \in C_+(\mathbb{R}^N)$ and $1 < r_- \le r_+ < p^- \le p^+ < q_- \le q_+ < p_s^*(x)$ for all $x \in \mathbb{R}^N$.

(V) $V \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$, $\operatorname{ess\,inf}_{x\in\mathbb{R}^N} V(x) > 0$, and $\lim_{|x|\to\infty} V(x) = +\infty$.

(A) $0 \le a \in L^{\frac{p(\cdot)}{p(\cdot)-r(\cdot)}}(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ with $|\{x\in\mathbb{R}^N : a(x)\ne 0\}| > 0$, where $|A|$ denotes the Lebesgue measure of a subset A of $\mathbb{R}^N$.

(F1) $f:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ satisfies the Carathéodory condition.

(F2) There exists $0 \le b \in L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ such that

$$\left|f(x,\tau)\right| \le b(x)|\tau|^{q(x)-1} \quad \text{for almost all } (x,\tau)\in\mathbb{R}^N\times\mathbb{R},$$

where $q \in C_+(\mathbb{R}^N)$ and $q(x) < p_s^*(x)$ for all $x \in \mathbb{R}^N$.

(F3) $\lim_{|\tau|\to\infty}\frac{F(x,\tau)}{|\tau|^{p^+}} = \infty$ uniformly for almost all $x \in \mathbb{R}^N$.

(F4) There are $\mu > p^+$, $M > 0$, and a function $\varrho \geq 0$ with $\varrho \in L^{\frac{p(\cdot)}{p(\cdot)-p^-}}(\mathcal{B}_1)$ on $\mathcal{B}_1 := \{x \in \mathbb{R}^N : p(x) > p^-\}$ and $\varrho(x) \equiv \tilde{\varrho}$ (constant function) on $\mathcal{B}_2 := \{x \in \mathbb{R}^N : p(x) = p^-\}$ such that $|\{x \in \mathbb{R}^N : \varrho(x) > 0\}| \neq 0$ and

$$\tau f(x,\tau) - \mu F(x,\tau) \geq -\varrho(x)|\tau|^{p^-} - \zeta(x)$$

for all $(x,\tau) \in \mathbb{R}^N \times \mathbb{R}$ with $|\tau| \geq M$ and for some $\zeta \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $\zeta(x) \geq 0$.

(F5) $F(x,\tau) = o(|\tau|^{p(x)})$ as $\tau \to 0$ uniformly for all $x \in \mathbb{R}^N$.

As mentioned in the introduction, assumption (F4) for the convex term f is different from that used in the works [1, 16, 29, 30, 33, 34, 38–40, 42, 44]. Hence we give some simple examples of functions that satisfy condition (F4).

Example 3.2 If $p(x) = 2$ for all $x \in \mathbb{R}^N$ and

$$f(x,\tau) = \rho(x)|\tau|\big(4\tau^3 - 2\tau\cos\tau - 4\sin\tau\big),$$

where $\rho(x) \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $0 < \inf_{x\in\mathbb{R}^N} \rho(x) \leq \sup_{x\in\mathbb{R}^N} \rho(x) < \infty$, then

$$F(x,\tau) = \rho(x)\left(\frac{4}{5}|\tau|^5 - 2\tau|\tau|\sin\tau\right).$$

We set $\tilde{\varrho} := \inf_{x\in\mathbb{R}^N} \rho(x)$ and $\zeta(x) := 2(\mu - 2)\rho(x)$ with $2 < \mu < \frac{15}{4}$ for all $x \in \mathbb{R}^N$. Then

$$\begin{aligned}
f(x,\tau)\tau - \mu F(x,\tau) &= \rho(x)\left(4|\tau|^5 - 2|\tau|^3\cos\tau - 4\tau|\tau|\sin\tau - \frac{4}{5}\mu|\tau|^5 + 2\mu\tau|\tau|\sin\tau\right) \\
&= \rho(x)\left(4|\tau|^5 - \frac{4}{5}\mu|\tau|^5 - 2|\tau|^3\cos\tau + (2\mu - 4)\tau|\tau|\sin\tau\right) \\
&\geq \rho(x)\left(4|\tau|^3 - \frac{4}{5}\mu|\tau|^3 - 2|\tau|\cos\tau - (2\mu - 4)\right)\tau^2 \\
&\geq \rho(x)\left(|\tau|^3 + \left(3 - \frac{4}{5}\mu\right)|\tau|^3 - 2|\tau| - (2\mu - 4)\right) \\
&\geq \rho(x)|\tau|^2 - (2\mu - 4)\rho(x) \\
&\geq -\tilde{\varrho}|\tau|^2 - \zeta(x)
\end{aligned}$$

for $|\tau| \geq r$, where $r > 1$ is chosen such that $(3 - \frac{4}{5}\mu)r^3 - 2r \geq 0$. Hence (F4) is fulfilled.

Example 3.3 If $p(x) = p > 1$ for all $x \in \mathbb{R}^N$ and

$$f(x,\tau) = \rho(x)\left(|\tau|^{p-2}\tau + \frac{2}{p}\sin\tau\right),$$

where ρ comes from the previous example, then

$$F(x,\tau) = \rho(x)\left(\frac{1}{p}|\tau|^p - \frac{2}{p}\cos\tau + \frac{2}{p}\right).$$

We set $\tilde{\varrho} := (\mu - 1)\sup_{x\in\mathbb{R}^N} \rho(x)$ and $\zeta(x) := \frac{4\mu}{p}\rho(x)$ with $p < \mu$ for all $x \in \mathbb{R}^N$. Then

$$\begin{aligned} f(x,\tau)\tau - \mu F(x,\tau) &= \rho(x)\left[|\tau|^p + \frac{2}{p}\tau\sin\tau - \frac{\mu}{p}|\tau|^p + \frac{2\mu}{p}\cos\tau - \frac{2\mu}{p}\right] \\ &\geq \rho(x)\left[\left(1 - \frac{\mu}{p}\right)|\tau|^p - \frac{2}{p}|\tau| - \frac{4\mu}{p}\right] \\ &= \rho(x)\left[(1-\mu)|\tau|^p + \frac{\mu(p-1)}{p}|\tau|^p - \frac{2}{p}|\tau|\right] - \frac{4\mu}{p}\rho(x) \\ &\geq \rho(x)(1-\mu)|\tau|^p - \frac{4\mu}{p}\rho(x) \\ &\geq -\tilde{\varrho}|\tau|^p - \zeta(x) \end{aligned}$$

for all $|\tau| \geq r$, where $r > 1$ is chosen such that $\mu(p-1)r^p - 2r \geq 0$. Hence (F4) is fulfilled.

Example 3.4 If $p \in C_+(\mathbb{R}^N)$ and

$$f(x,\tau) = \rho(x)|\tau|^{p(x)-1}\tau\left[\left(p(x)+3\right)\tau^2 - 2\left(p(x)+2\right)|\tau| + \left(p(x)+1\right)\right],$$

where $\rho(x) \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, then

$$F(x,\tau) = \rho(x)\left(|\tau|^{p(x)+3} - 2|\tau|^{p(x)+2} + |\tau|^{p(x)+1}\right).$$

We set $\varrho(x) := \rho(x) =: \zeta(x)$ and $p_- + 1 < \mu < p(x) + 2$ for all $x \in \mathbb{R}^N$. Then

$$\begin{aligned} &f(x,\tau)\tau - \mu F(x,\tau) \\ &\quad= \rho(x)\left[\left(p(x)+3-\mu\right)|\tau|^{p(x)+3} - 2\left(p(x)+2-\mu\right)|\tau|^{p(x)+2} + \left(p(x)+1-\mu\right)|\tau|^{p(x)+1}\right] \\ &\quad\geq \rho(x)\left[\left(p(x)+3-\mu\right)|\tau|^2 - 2\left(p(x)+2-\mu\right)|\tau| + \left(p^- + 1 - \mu\right)\right]|\tau|^{p(x)+1} \\ &\quad= \rho(x)\left[|\tau|^2 + \left(p(x)+2-\mu\right)\left(|\tau|^2 - 2|\tau|\right) + \left(p^- + 1 - \mu\right)\right]|\tau|^{p(x)+1} \\ &\quad\geq \rho(x)\left[|\tau|^2 - \left(\mu - p^- - 1\right)\right]|\tau|^{p^-} \\ &\quad\geq -\varrho(x)|\tau|^{p^-} - \zeta(x) \end{aligned}$$

for $|\tau| \geq r$, where $r > 1 + \sqrt{(\mu - p^- - 1)}$ is chosen such that $r^2 - 2r \geq 0$. Hence (F4) is fulfilled.

Let us define the functional $\Psi_\lambda : \mathcal{X} \to \mathbb{R}$ by

$$\Psi_\lambda(z) = \lambda \int_{\mathbb{R}^N} \frac{a(x)}{r(x)}|z|^{r(x)}\,dx + \int_{\mathbb{R}^N} F(x,z)\,dx.$$

It is easy to check that $\Psi_\lambda \in C^1(\mathcal{X},\mathbb{R})$ and its Fréchet derivative is

$$\left\langle \Psi_\lambda'(z), \varphi\right\rangle = \lambda \int_{\mathbb{R}^N} a(x)|z|^{r(x)-2}z\varphi\,dx + \int_{\mathbb{R}^N} f(x,z)\varphi\,dx$$

for any $z, \varphi \in \mathcal{X}$. Next we define the functional $\mathcal{I}_\lambda : \mathcal{X} \to \mathbb{R}$ by

$$\mathcal{I}_\lambda(z) = \mathcal{A}(z) - \Psi_\lambda(z).$$

Then the functional $\mathcal{I}_\lambda \in C^1(\mathcal{X}, \mathbb{R})$ and its Fréchet derivative is

$$\begin{aligned}\langle \mathcal{I}'_\lambda(z), \varphi\rangle &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \left|z(x) - z(y)\right|^{p(x,y)-2}\big(z(x) - z(y)\big)\big(\varphi(x) - \varphi(y)\big)\mathcal{K}(x,y)\,dx\,dy \\ &\quad + \int_{\mathbb{R}^N} V(x)|z|^{p(x)-2}z\varphi\,dx - \lambda\int_{\mathbb{R}^N} a(x)|z|^{r(x)-2}z\varphi\,dx - \int_{\mathbb{R}^N} f(x,z)\varphi\,dx\end{aligned}$$

for any $z, \varphi \in \mathcal{X}$.

Under assumption (V), we can give the compact embedding.

Lemma 3.5 *If the potential function V satisfies assumption* (V), *then*

(1) *the embedding from $\mathcal{X} \hookrightarrow L^{p(\cdot)}(\mathbb{R}^N)$ is continuous and compact*;

(2) *for any measurable function $\ell : \mathbb{R}^N \to \mathbb{R}$ with $p(x) < \ell(x)$ for all $x \in \mathbb{R}^N$, there is a compact embedding $\mathcal{X} \hookrightarrow L^{\ell(\cdot)}(\mathbb{R}^N)$ if $\inf_{x\in\mathbb{R}^N}(p_s^*(x) - \ell(x)) > 0$.*

Proof In order to prove this lemma, we can adapt the proof of Lemma 2.6 in [1]. For the case that the potential function V is coercive, we obtain a similar result involving variable exponents of fractional type using Lemma 2.4. So, we omit the details of the proof. □

Next we give the following useful lemmas which are essential in obtaining the existence of at least two distinct nontrivial solutions to problem (P).

Definition 3.6 Let E be a real Banach space with dual space E^*, $\mathcal{I} \in C^1(E, \mathbb{R}^N)$. We say that $\mathcal{I}$ satisfies the Cerami condition ((C)-condition, for short) in E if any (C)-sequence $\{z_n\} \subset E$, i.e., $\{\mathcal{I}(z_n)\}$ is bounded and $\|\mathcal{I}'(z_n)\|_{E^*}(1 + \|z_n\|_E) \to 0$ as $n \to \infty$, has a convergent subsequence in E. We say that $\mathcal{I}$ satisfies the Cerami condition at level c ($(C)_c$-condition, for short) in E if any $(C)_c$-sequence $\{z_n\} \subset E$, i.e., $\mathcal{I}(z_n) \to c$ as $n \to \infty$ and $\|\mathcal{I}'(z_n)\|_{E^*}(1 + \|z_n\|_E) \to 0$ as $n \to \infty$, has a convergent subsequence in E.

Lemma 3.7 *Assume that* (H), (V), (A), *and* (F1)–(F4) *hold. Then the functional $\mathcal{I}_\lambda$ satisfies the (C)-condition for any $\lambda > 0$.*

Proof Let $\{z_n\}$ be a (C)-sequence in $\mathcal{X}$ for $\mathcal{I}_\lambda$, that is,

$$\sup_{n\in\mathbb{N}}\left|\mathcal{I}_\lambda(z_n)\right| \le \mathcal{M}_0 \quad \text{and} \quad \left\langle \mathcal{I}'_\lambda(z_n), z_n\right\rangle = o(1), \tag{3.1}$$

where $o(1) \to 0$ as $n \to \infty$, and $\mathcal{M}_0$ is a positive constant. From Lemma 4.2 in [8] and Lemma 3.3 in [33], it follows that $\mathcal{A}'$ and Ψ'_λ are of type (S_+). Since $\mathcal{X}$ is a reflexive Banach space, it is enough to ensure that the sequence $\{z_n\}$ is bounded in $\mathcal{X}$. We argue by contradiction. Assume that the sequence $\{z_n\}$ is unbounded in $\mathcal{X}$. So then we may suppose that $\|z_n\|_{\mathcal{X}} > 1$ and $\|z_n\|_{\mathcal{X}} \to \infty$ as $n \to \infty$. Let us denote the sequence $\{w_n\}$ with $w_n = z_n/\|z_n\|_{\mathcal{X}}$. Then, clearly, we have $\{w_n\} \subset \mathcal{X}$ and $\|w_n\|_{\mathcal{X}} = 1$. Hence, up to a subsequence, still denoted by $\{w_n\}$, we infer $w_n \rightharpoonup \omega$ in $\mathcal{X}$ as $n \to \infty$ and by Lemma 3.5

$$w_n(x) \to \omega(x) \quad \text{a.e. in } \mathbb{R}^N \quad \text{and} \quad w_n \to \omega \quad \text{in } L^{\ell(\cdot)}\big(\mathbb{R}^N\big) \text{ as } n \to \infty \tag{3.2}$$

for $p(x) \le \ell(x)$ with $\inf_{x\in\mathbb{R}^N}(p_s^*(x) - \ell(x)) > 0$. Notice that $V(x) \to +\infty$ as $|x| \to \infty$, then

$$\begin{aligned}&\int_{\mathbb{R}^N} V(x)\left(\frac{1}{p(x)} - \frac{1}{\mu}\right)|z_n|^{p(x)}\,dx - C_1\int_{|z_n|\le M}\left(|z_n|^{p(x)} + b(x)|z_n|^{q(x)}\right)dx\\&\quad\ge \frac{1}{2}\left(\frac{1}{p^+} - \frac{1}{\mu}\right)\int_{\mathbb{R}^N} V(x)|z_n|^{p(x)}\,dx - \mathcal{M}_1,\end{aligned}$$

where C_1 and $\mathcal{M}_1$ are positive constants. In fact we know that

$$\begin{aligned}&\left(\frac{1}{p^+} - \frac{1}{\mu}\right)\int_{\mathbb{R}^N} V(x)|z_n|^{p(x)}\,dx - C_1\int_{|z_n|\le M}|z_n|^{p(x)} + b(x)|z_n|^{q(x)}\,dx\\&\quad\ge \frac{1}{2}\left(\frac{1}{p^+} - \frac{1}{\mu}\right)\int_{\mathbb{R}^N} V(x)|z_n|^{p(x)}\,dx + \frac{1}{2}\left(\frac{1}{p^+} - \frac{1}{\mu}\right)\int_{|z_n|\le 1} V(x)|z_n|^{p(x)}\,dx\\&\qquad - C_1\int_{|z_n|\le 1}|z_n|^{p(x)} + b(x)|z_n|^{q(x)}\,dx - C_1\int_{1<|z_n|\le M}|z_n|^{p(x)} + b(x)|z_n|^{q(x)}\,dx\\&\quad\ge \frac{1}{2}\left(\frac{1}{p^+} - \frac{1}{\mu}\right)\int_{\mathbb{R}^N} V(x)|z_n|^{p(x)}\,dx + \frac{1}{2}\left(\frac{1}{p^+} - \frac{1}{\mu}\right)\int_{|z_n|\le 1} V(x)|z_n|^{p(x)}\,dx\\&\qquad - C_1\left(1 + \|b\|_{L^\infty(\mathbb{R}^N)}\right)\int_{|z_n|\le 1}|z_n|^{p(x)}\,dx - \widetilde{C}_1,\end{aligned}$$

where C_1 and $\widetilde{C}_1$ are positive constants. Let us set $\Sigma := \{x \in \mathbb{R}^N : |z_n(x)| > 1\}$. Since $|\Sigma|$ is finite ($|\cdot|$ is the Lebesgue measure in $\mathbb{R}^N$), $\Sigma = \widetilde{\Sigma} \cup N$ where $\widetilde{\Sigma}$ is a bounded set and N is of measure zero. Without loss of generality, suppose that there exists $B_r(0) \subseteq \mathbb{R}^N$ such that $\Sigma \subset B_r(0)$ where $B_r(0)$ is the open ball centered at 0 with radius r in the Euclidean space $\mathbb{R}^N$. Since $V(x) \to +\infty$ as $|x| \to \infty$, there is $r_0 > 0$ such that $|x| \ge r_0 > r$ implies $V(x) \ge 2C_1(1 + \|\sigma\|_{L^\infty(\mathbb{R}^N)})\frac{\mu p^+}{\mu - p^+}$. Consequently, we get

$$\begin{aligned}&\left(\frac{1}{p^+} - \frac{1}{\mu}\right)\int_{\mathbb{R}^N} V(x)|z_n|^{p(x)}\,dx - C_1\int_{|z_n|\le M}|z_n|^{p(x)} + b(x)|z_n|^{q(x)}\,dx\\&\quad\ge \frac{1}{2}\left(\frac{1}{p^+} - \frac{1}{\mu}\right)\int_{\mathbb{R}^N} V(x)|z_n|^{p(x)}\,dx + \frac{1}{2}\left(\frac{1}{p^+} - \frac{1}{\mu}\right)\int_{\Sigma^c\cap B_{r_0}^c} V(x)|z_n|^{p(x)}\,dx\\&\qquad + \frac{1}{2}\left(\frac{1}{p^+} - \frac{1}{\mu}\right)\int_{\Sigma^c\cap B_{r_0}} V(x)|z_n|^{p(x)}\,dx - C_1\left(1 + \|b\|_{L^\infty(\mathbb{R}^N)}\right)\int_{\Sigma^c\cap B_{r_0}^c}|z_n|^{p(x)}\,dx\\&\qquad - C_1\left(1 + \|b\|_{L^\infty(\mathbb{R}^N)}\right)\int_{\Sigma^c\cap B_{r_0}}|z_n|^{p(x)}\,dx - \widetilde{C}_1\\&\quad\ge \frac{1}{2}\left(\frac{1}{p^+} - \frac{1}{\mu}\right)\int_{\mathbb{R}^N} V(x)|z_n|^{p(x)}\,dx + \frac{1}{2}\left(\frac{1}{p^+} - \frac{1}{\mu}\right)\int_{\Sigma^c\cap B_{r_0}^c} V(x)|z_n|^{p(x)}\,dx\\&\qquad - C_1\left(1 + \|b\|_{L^\infty(\mathbb{R}^N)}\right)\int_{\Sigma^c\cap B_{r_0}^c}|z_n|^{p(x)}\,dx - \mathcal{M}_1\\&\quad\ge \frac{1}{2}\left(\frac{1}{p^+} - \frac{1}{\mu}\right)\int_{\mathbb{R}^N} V(x)|z_n|^{p(x)}\,dx - \mathcal{M}_1,\end{aligned}$$

where $\mathcal{M}_1$ is a positive constant, as claimed. This fact, together with (F2) and (F4), leads to

$$
\begin{aligned}
&\mathcal{M}_0 + o(1) \\
&\geq \mathcal{I}_\lambda(z_n) - \frac{1}{\mu}\langle \mathcal{I}_\lambda'(z_n), z_n\rangle \\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{p(x,y)}\left|z_n(x)-z_n(y)\right|^{p(x,y)}\mathcal{K}(x,y)\,dx\,dy + \int_{\mathbb{R}^N}\frac{V(x)}{p(x)}|z_n|^{p(x)}\,dx \\
&\quad - \lambda\int_{\mathbb{R}^N}\frac{a(x)}{r(x)}|z_n|^{r(x)}\,dx - \int_{\mathbb{R}^N}F(x,z_n)\,dx \\
&\quad - \frac{1}{\mu}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left|z_n(x)-z_n(y)\right|^{p(x,y)}\mathcal{K}(x,y)\,dx\,dy - \frac{1}{\mu}\int_{\mathbb{R}^N}V(x)|z_n|^{p(x)}\,dx \\
&\quad + \frac{\lambda}{\mu}\int_{\mathbb{R}^N}a(x)|z_n|^{r(x)}\,dx + \frac{1}{\mu}\int_{\mathbb{R}^N}f(x,z_n)z_n\,dx \\
&\geq \left(\frac{1}{p^+}-\frac{1}{\mu}\right)\int_{\mathbb{R}^N\times\mathbb{R}^N}\left|z_n(x)-z_n(y)\right|^{p(x,y)}\mathcal{K}(x,y)\,dx\,dy \\
&\quad + \left(\frac{1}{p^+}-\frac{1}{\mu}\right)\int_{\mathbb{R}^N}V(x)|z_n|^{p(x)}\,dx - \lambda\left(\frac{1}{r_-}-\frac{1}{\mu}\right)\int_{\mathbb{R}^N}a(x)|z_n|^{r(x)}\,dx \\
&\quad - \int_{\mathbb{R}^N}F(x,z_n)\,dx + \frac{1}{\mu}\int_{\mathbb{R}^N}f(x,z_n)z_n\,dx \\
&\geq \left(\frac{1}{p^+}-\frac{1}{\mu}\right)\int_{\mathbb{R}^N\times\mathbb{R}^N}\left|z_n(x)-z_n(y)\right|^{p(x,y)}\mathcal{K}(x,y)\,dx\,dy \\
&\quad + \left(\frac{1}{p^+}-\frac{1}{\mu}\right)\int_{\mathbb{R}^N}V(x)|z_n|^{p(x)}\,dx - \lambda\left(\frac{1}{r_-}-\frac{1}{\mu}\right)\int_{\mathbb{R}^N}a(x)|z_n|^{r(x)}\,dx \\
&\quad + \int_{|z_n|>M}\left(\frac{1}{\mu}f(x,z_n)z_n - F(x,z_n)\right)dx - C_1\int_{|z_n|\leq M}\left(|z_n|^{p(x)} + b(x)|z_n|^{q(x)}\right)dx \\
&\geq \left(\frac{1}{p^+}-\frac{1}{\mu}\right)\int_{\mathbb{R}^N\times\mathbb{R}^N}\left|z_n(x)-z_n(y)\right|^{p(x,y)}\mathcal{K}(x,y)\,dx\,dy \\
&\quad + \frac{1}{2}\left(\frac{1}{p^+}-\frac{1}{\mu}\right)\int_{\mathbb{R}^N}V(x)|z_n|^{p(x)}\,dx - \lambda\left(\frac{1}{r_-}-\frac{1}{\mu}\right)\int_{\mathbb{R}^N}a(x)|z_n|^{r(x)}\,dx \\
&\quad - \frac{1}{\mu}\int_{\mathbb{R}^N}\left(\varrho(x)|z_n|^{p^-} + \zeta(x)\right)dx - \mathcal{M}_1 \\
&\geq \frac{1}{2}\left(\frac{1}{p^+}-\frac{1}{\mu}\right)\left(\int_{\mathbb{R}^N\times\mathbb{R}^N}\left|z_n(x)-z_n(y)\right|^{p(x,y)}\mathcal{K}(x,y)\,dx\,dy + \int_{\mathbb{R}^N}V(x)|z_n|^{p(x)}\,dx\right) \\
&\quad - \lambda\left(\frac{1}{r_-}-\frac{1}{\mu}\right)\int_{\mathbb{R}^N}a(x)|z_n|^{r(x)}\,dx - \frac{1}{\mu}\int_{\mathbb{R}^N}\varrho(x)|z_n|^{p^-}\,dx - \frac{1}{\mu}\|\zeta\|_{L^1(\mathbb{R}^N)} - \mathcal{M}_1 \\
&\geq \frac{1}{2}\left(\frac{1}{p^+}-\frac{1}{\mu}\right)\|z_n\|_{\mathcal{X}}^{p^-} - \left(\frac{\lambda}{r_-}-\frac{\lambda}{\mu}\right)\int_{\mathbb{R}^N}a(x)|z_n|^{r(x)}\,dx \\
&\quad - \frac{1}{\mu}\left(\int_{\mathcal{B}_1}\varrho(x)|z_n|^{p^-}\,dx + \int_{\mathcal{B}_2}\varrho(x)|z_n|^{p^-}\,dx\right) - \frac{1}{\mu}\|\zeta\|_{L^1(\mathbb{R}^N)} - \mathcal{M}_1 \\
&\geq \frac{1}{2}\left(\frac{1}{p^+}-\frac{1}{\mu}\right)\|z_n\|_{\mathcal{X}}^{p^-} - \left(\frac{\lambda}{r_-}-\frac{\lambda}{\mu}\right)\int_{\mathbb{R}^N}a(x)|z_n|^{r(x)}\,dx \\
&\quad - \frac{1}{\mu}\left(2\|\varrho\|_{L^{\frac{p(\cdot)}{p(\cdot)-p^-}}(\mathcal{B}_1)}\|z_n\|_{L^{p(\cdot)}(\mathcal{B}_1)}^{p^-} + \tilde{\varrho}\int_{\mathcal{B}_2}|z_n|^{p^-}\,dx\right) - \frac{1}{\mu}\|\zeta\|_{L^1(\mathbb{R}^N)} - \mathcal{M}_1
\end{aligned}
$$

$$
\begin{aligned}
&\geq \frac{1}{2}\left(\frac{1}{p^+}-\frac{1}{\mu}\right)\|z_n\|_{\mathcal{X}}^{p^-}-\left(\frac{\lambda}{r_-}-\frac{\lambda}{\mu}\right)\int_{\mathbb{R}^N} a(x)|z_n|^{r(x)}\,dx \\
&\quad -\frac{1}{\mu}\bigl(2\|\varrho\|_{L^{\frac{p(\cdot)}{p(\cdot)-p^-}}(\mathcal{B}_1)}\|z_n\|_{L^{p(\cdot)}(\mathbb{R}^N)}^{p^-}+\tilde{\varrho}\|z_n\|_{L^{p(\cdot)}(\mathcal{B}_2)}^{p^-}\bigr)-\frac{1}{\mu}\|\zeta\|_{L^1(\mathbb{R}^N)}-\mathcal{M}_1 \\
&\geq \frac{1}{2}\left(\frac{1}{p^+}-\frac{1}{\mu}\right)\|z_n\|_{\mathcal{X}}^{p^-}-\left(\frac{\lambda}{r_-}-\frac{\lambda}{\mu}\right)\int_{\mathbb{R}^N} a(x)|z_n|^{r(x)}\,dx \\
&\quad -\frac{1}{\mu}\bigl(2\|\varrho\|_{L^{\frac{p(\cdot)}{p(\cdot)-p^-}}(\mathcal{B}_1)}+\tilde{\varrho}\bigr)\|z_n\|_{L^{p(\cdot)}(\mathbb{R}^N)}^{p^-}-\frac{1}{\mu}\|\zeta\|_{L^1(\mathbb{R}^N)}-\mathcal{M}_1 \\
&\geq \frac{1}{2}\left(\frac{1}{p^+}-\frac{1}{\mu}\right)\|z_n\|_{\mathcal{X}}^{p^-}-\left(\frac{\lambda}{r_-}-\frac{\lambda}{\mu}\right)\|a\|_{L^{\frac{p(\cdot)}{p(\cdot)-r(\cdot)}}(\mathbb{R}^N)}\max\bigl\{\|z_n\|_{L^{p(\cdot)}(\mathbb{R}^N)}^{r_+},\|z_n\|_{L^{p(\cdot)}(\mathbb{R}^N)}^{r_-}\bigr\} \\
&\quad -\frac{1}{\mu}\bigl(2\|\varrho\|_{L^{\frac{p(\cdot)}{p(\cdot)-p^-}}(\mathcal{B}_1)}+\tilde{\varrho}\bigr)\|z_n\|_{L^{p(\cdot)}(\mathbb{R}^N)}^{p^-}-\frac{1}{\mu}\|\zeta\|_{L^1(\mathbb{R}^N)}-\mathcal{M}_1
\end{aligned}
$$

for sufficiently large n because $\int_{\mathcal{B}_2}|z_n|^{p^-}\,dx \le \int_{\mathbb{R}^N}|z_n|^{p(x)}\,dx$. This fact implies

$$
\begin{aligned}
\frac{1}{2} &\leq \frac{p^+(2\|\varrho\|_{L^{\frac{p(\cdot)}{p(\cdot)-p^-}}(\mathcal{B}_1)}+\tilde{\varrho})}{\mu-p^+}\limsup_{n\to\infty}\|w_n\|_{L^{p(\cdot)}(\mathbb{R}^N)}^{p^-} \\
&= \frac{p^+(2\|\varrho\|_{L^{\frac{p(\cdot)}{p(\cdot)-p^-}}(\mathcal{B}_1)}+\tilde{\varrho})}{\mu-p^+}\|\omega\|_{L^{p(\cdot)}(\mathbb{R}^N)}^{p^-}.
\end{aligned}
\tag{3.3}
$$

Hence, from (3.3), it follows that $\omega \neq 0$. However, to obtain the boundedness of $\{z_n\}$, we should prove that $\omega(x) = 0$ for almost all $x \in \mathbb{R}^N$. Set $\Omega_1 = \{x \in \mathbb{R}^N : \omega(x) \neq 0\}$. By virtue of relation (3.1), one has

$$
\begin{aligned}
\mathcal{I}_\lambda(z_n) &= \int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{1}{p(x,y)}\bigl|z_n(x)-z_n(y)\bigr|^{p(x,y)}\mathcal{K}(x,y)\,dx\,dy+\int_{\mathbb{R}^N}\frac{V(x)}{p(x)}|z_n|^{p(x)}\,dx \\
&\quad -\lambda\int_{\mathbb{R}^N}\frac{a(x)}{r(x)}|z_n|^{r(x)}\,dx-\int_{\mathbb{R}^N}F(x,z_n)\,dx \\
&\geq \frac{1}{p^+}\|z_n\|_{\mathcal{X}}^{p^-}-\frac{\lambda}{r_-}\|a\|_{L^{\frac{p(\cdot)}{p(\cdot)-r(\cdot)}}(\mathbb{R}^N)}\max\bigl\{\|z_n\|_{L^{p(\cdot)}(\mathbb{R}^N)}^{r_+},\|z_n\|_{L^{p(\cdot)}(\mathbb{R}^N)}^{r_-}\bigr\} \\
&\quad -\int_{\mathbb{R}^N}F(x,z_n)\,dx.
\end{aligned}
\tag{3.4}
$$

Since $\mathcal{I}_\lambda(z_n) \le \mathcal{M}_0$ for all $n \in \mathbb{N}$ and $\|z_n\|_{\mathcal{X}} \to \infty$ as $n \to \infty$, we assert that

$$
\begin{aligned}
\int_{\mathbb{R}^N}F(x,z_n)\,dx &\geq \frac{1}{p^+}\|z_n\|_{\mathcal{X}}^{p^-}-\frac{\lambda}{r_-}\|a\|_{L^{\frac{p(\cdot)}{p(\cdot)-r(\cdot)}}(\mathbb{R}^N)}\max\bigl\{\|z_n\|_{L^{p(\cdot)}(\mathbb{R}^N)}^{r_+},\|z_n\|_{L^{p(\cdot)}(\mathbb{R}^N)}^{r_-}\bigr\}-\mathcal{M}_0 \\
&\to \infty
\end{aligned}
\tag{3.5}
$$

as $n \to \infty$. In addition,

$$
\begin{aligned}
\mathcal{I}_\lambda(z_n) &= \int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{1}{p(x,y)}\bigl|z_n(x)-z_n(y)\bigr|^{p(x,y)}\mathcal{K}(x,y)\,dx\,dy+\int_{\mathbb{R}^N}\frac{V(x)}{p(x)}|z_n|^{p(x)}\,dx \\
&\quad -\lambda\int_{\mathbb{R}^N}\frac{a(x)}{r(x)}|z_n|^{r(x)}\,dx-\int_{\mathbb{R}^N}F(x,z_n)\,dx
\end{aligned}
$$

$$\begin{aligned}&\le \frac{1}{p^-}\int_{\mathbb{R}^N\times\mathbb{R}^N}\left|z_n(x)-z_n(y)\right|^{p(x,y)}\mathcal{K}(x,y)\,dx\,dy\\&\quad+\frac{1}{p^-}\int_{\mathbb{R}^N}V(x)|z_n|^{p(x)}\,dx-\int_{\mathbb{R}^N}F(x,z_n)\,dx.\end{aligned}$$

Then

$$\begin{aligned}&\frac{1}{p^-}\int_{\mathbb{R}^N\times\mathbb{R}^N}\left|z_n(x)-z_n(y)\right|^{p(x,y)}\mathcal{K}(x,y)\,dx\,dy+\frac{1}{p^-}\int_{\mathbb{R}^N}V(x)|z_n|^{p(x)}\,dx\\&\quad\ge\int_{\mathbb{R}^N}F(x,z_n)\,dx+\mathcal{I}_\lambda(z_n).\end{aligned}\tag{3.6}$$

In accordance with assumption (F3), there is a positive constant $\tau_0>1$ such that $F(x,\tau)>|\tau|^{p^+}$ for all $x\in\mathbb{R}^N$ and $|\tau|>\tau_0$. From assumptions (F1) and (F2), it follows that there is $\mathcal{M}_2>0$ such that $|F(x,\tau)|\le\mathcal{M}_2$ for all $(x,\tau)\in\mathbb{R}^N\times[-\tau_0,\tau_0]$. Therefore, we can choose a real number $\mathcal{M}_3$ such that $F(x,\tau)\ge\mathcal{M}_3$ for all $(x,\tau)\in\mathbb{R}^N\times\mathbb{R}$, and thus

$$\frac{F(x,z_n)-\mathcal{M}_3}{\frac{1}{p^-}\int_{\mathbb{R}^N\times\mathbb{R}^N}|z_n(x)-z_n(y)|^{p(x,y)}\mathcal{K}(x,y)\,dx\,dy+\frac{1}{p^-}\int_{\mathbb{R}^N}V(x)|z_n|^{p(x)}\,dx}\ge 0,$$

for all $x\in\mathbb{R}^N$ and for all $n\in\mathbb{N}$. By convergence (3.2), we know that $|z_n(x)|=|w_n(x)|\|z_n\|_{\mathcal{X}}\to\infty$ as $n\to\infty$ for all $x\in\Omega_1$. Furthermore, from assumption (F3) it follows that for all $x\in\Omega_1$ we have

$$\begin{aligned}&\lim_{n\to\infty}\frac{F(x,z_n)}{\frac{1}{p^-}\int_{\mathbb{R}^N\times\mathbb{R}^N}|z_n(x)-z_n(y)|^{p(x,y)}\mathcal{K}(x,y)\,dx\,dy+\frac{1}{p^-}\int_{\mathbb{R}^N}V(x)|z_n|^{p(x)}\,dx}\\&\quad\ge\lim_{n\to\infty}\frac{F(x,z_n)}{\frac{1}{p^-}\|z_n\|_{\mathcal{X}}^{p^+}}=\lim_{n\to\infty}\frac{p^-F(x,z_n)}{|z_n(x)|^{p^+}}\left|w_n(x)\right|^{p^+}=\infty.\end{aligned}\tag{3.7}$$

Hence we infer that $|\Omega_1|=0$. Indeed, if $|\Omega_1|\neq 0$, then, from relations (3.5)–(3.7) and invoking the Fatou lemma, it follows that

$$\begin{aligned}1&=\liminf_{n\to\infty}\frac{\int_{\mathbb{R}^N}F(x,z_n)\,dx}{\int_{\mathbb{R}^N}F(x,z_n)\,dx+\mathcal{I}_\lambda(z_n)}\\&\ge\liminf_{n\to\infty}\int_{\mathbb{R}^N}\frac{F(x,z_n)}{\frac{1}{p^-}\int_{\mathbb{R}^N\times\mathbb{R}^N}|z_n(x)-z_n(y)|^{p(x,y)}\mathcal{K}(x,y)\,dx\,dy+\frac{1}{p^-}\int_{\mathbb{R}^N}V(x)|z_n|^{p(x)}\,dx}\,dx\\&\ge\liminf_{n\to\infty}\int_{\Omega_1}\frac{F(x,z_n)}{\frac{1}{p^-}\int_{\mathbb{R}^N\times\mathbb{R}^N}|z_n(x)-z_n(y)|^{p(x,y)}\mathcal{K}(x,y)\,dx\,dy+\frac{1}{p^-}\int_{\mathbb{R}^N}V(x)|z_n|^{p(x)}\,dx}\,dx\\&\quad-\limsup_{n\to\infty}\int_{\Omega_1}\frac{\mathcal{M}_3}{\frac{1}{p^-}\int_{\mathbb{R}^N\times\mathbb{R}^N}|z_n(x)-z_n(y)|^{p(x,y)}\mathcal{K}(x,y)\,dx\,dy+\frac{1}{p^-}\int_{\mathbb{R}^N}V(x)|z_n|^{p(x)}\,dx}\,dx\\&=\liminf_{n\to\infty}\int_{\Omega_1}\frac{F(x,z_n)-\mathcal{M}_3}{\frac{1}{p^-}\int_{\mathbb{R}^N\times\mathbb{R}^N}|z_n(x)-z_n(y)|^{p(x,y)}\mathcal{K}(x,y)\,dx\,dy+\frac{1}{p^-}\int_{\mathbb{R}^N}V(x)|z_n|^{p(x)}\,dx}\,dx\\&\ge\int_{\Omega_1}\liminf_{n\to\infty}\frac{F(x,z_n)-\mathcal{M}_3}{\frac{1}{p^-}\int_{\mathbb{R}^N\times\mathbb{R}^N}|z_n(x)-z_n(y)|^{p(x,y)}\mathcal{K}(x,y)\,dx\,dy+\frac{1}{p^-}\int_{\mathbb{R}^N}V(x)|z_n|^{p(x)}\,dx}\,dx\\&=\int_{\Omega_1}\liminf_{n\to\infty}\frac{F(x,z_n(x))}{\frac{1}{p^-}\int_{\mathbb{R}^N\times\mathbb{R}^N}|z_n(x)-z_n(y)|^{p(x,y)}\mathcal{K}(x,y)\,dx\,dy+\frac{1}{p^-}\int_{\mathbb{R}^N}V(x)|z_n|^{p(x)}\,dx}\,dx\end{aligned}$$

$$
\begin{aligned}
&-\int_{\Omega_1}\limsup_{n\to\infty}\frac{\mathcal{M}_3}{\frac{1}{p^-}\int_{\mathbb{R}^N\times\mathbb{R}^N}|z_n(x)-z_n(y)|^{p(x,y)}\mathcal{K}(x,y)\,dx\,dy+\frac{1}{p^-}\int_{\mathbb{R}^N}V(x)|z_n|^{p(x)}\,dx}\,dx\\
&=\infty.
\end{aligned}
$$

This is impossible. Thus we know $|\Omega_1| = 0$, and so $\omega(x) = 0$ for almost all $x \in \mathbb{R}^N$, as claimed. Therefore we conclude that $\{z_n\}$ is bounded in $\mathcal{X}$. The proof is complete. □

Lemma 3.8 *Assume conditions* (H), (V), (A), (F1)–(F3) *and* (F5). *Furthermore, suppose that*

(F6) $F(x,\tau) \geq 0$ *for all* $(x,\tau) \in \mathbb{R}^N \times \mathbb{R}^+$

holds. Then

1. *There is a positive constant* λ^* *such that for any* $\lambda \in (0,\lambda^*)$ *we can choose some constants* $R > 0$ *and* $0 < r < 1$ *that* $\mathcal{I}_\lambda(z) \geq R > 0$ *for all* $z \in \mathcal{X}$ *with* $\|z\|_{\mathcal{X}} = r$.
2. *There exists* $z \in C_c^\infty(\mathbb{R}^N)$, $z > 0$, *such that* $\mathcal{I}_\lambda(tz) \to -\infty$ *as* $t \to +\infty$.
3. *There exists* $w \in C_c^\infty(\mathbb{R}^N)$, $w > 0$, *such that* $\mathcal{I}_\lambda(tw) < 0$ *for all* $t \to 0^+$.

Proof Statement (1) is proved in [11, 37]. Thus, we first show statement (2). By assumptions (F2)–(F3) and (F5), for any $\mathcal{M} > 0$, there exist some constants $C_2 > 0$ and $C_3(\mathcal{M}) > 0$ such that

$$
F(x,\tau) \geq \mathcal{M}|\tau|^{p^+} - C_2|\tau|^{p(x)} - C_3(\mathcal{M})b(x) \tag{3.8}
$$

for all $(x,\tau) \in \mathbb{R}^N \times \mathbb{R}$ where b comes from (F2). Let us take $z \in C_c^\infty(\mathbb{R}^N)$. Then relation (3.8) implies that

$$
\begin{aligned}
\mathcal{I}_\lambda(tz) &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{1}{p(x,y)}\big|tz(x)-tz(y)\big|^{p(x,y)}\mathcal{K}(x,y)\,dx\,dy+\int_{\mathbb{R}^N}\frac{V(x)}{p(x)}|tz|^{p(x)}\,dx\\
&\quad-\lambda\int_{\mathbb{R}^N}\frac{a(x)}{r(x)}|tz|^{r(x)}\,dx-\int_{\mathbb{R}^N}F(x,tz)\,dx\\
&\leq t^{p^+}\bigg(\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{1}{p(x,y)}\big|z(x)-z(y)\big|^{p(x,y)}\mathcal{K}(x,y)\,dx\,dy+\int_{\mathbb{R}^N}\frac{V(x)}{p(x)}|z|^{p(x)}\,dx\\
&\quad-\mathcal{M}\int_{\mathbb{R}^N}|z|^{p^+}\,dx+C_2\int_{\mathbb{R}^N}|z|^{p(x)}\,dx\bigg)+C_3
\end{aligned}
$$

for $t > 1$ large enough and for a constant C_3. If $\mathcal{M}$ is sufficiently large, then we assert that $\mathcal{I}_\lambda(tz) \to -\infty$ as $t \to \infty$. Therefore the functional $\mathcal{I}_\lambda$ is unbounded from below.

Next, we have to show (3). Let us choose $w \in C_c^\infty(\mathbb{R}^N)$ such that $w > 0$. For $t > 0$ small enough, from (A) and (F5), it follows that

$$
\begin{aligned}
\mathcal{I}_\lambda(tw) &= \int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{1}{p(x,y)}\big|tw(x)-tw(y)\big|^{p(x,y)}\mathcal{K}(x,y)\,dx\,dy+\int_{\mathbb{R}^N}\frac{V(x)}{p(x)}|tw|^{p(x)}\,dx\\
&\quad-\lambda\int_{\mathbb{R}^N}\frac{a(x)}{r(x)}|tw|^{r(x)}\,dx-\int_{\mathbb{R}^N}F(x,tw)\,dx\\
&\leq t^{p^-}\bigg(\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{1}{p(x,y)}\big|w(x)-w(y)\big|^{p(x,y)}\mathcal{K}(x,y)\,dx\,dy+\int_{\mathbb{R}^N}\frac{V(x)}{p(x)}|w|^{p(x)}\,dx\bigg)\\
&\quad-\frac{\lambda t^{r_+}}{r_+}\int_{\mathbb{R}^N}a(x)|w|^{r(x)}\,dx.
\end{aligned}
$$

Since $r_+ < p^-$, it follows that $\mathcal{I}_\lambda(tw) < 0$ as $t \to 0^+$. The proof is completed. □

The following lemma is the variational principle of Ekeland type in [6, 37], initially developed by C.-K. Zhong [56].

Lemma 3.9 ([6, 37]) *Let E be a Banach space and x_0 be a fixed point of E. Suppose that $g: E \to \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous function, not identically $+\infty$, bounded from below. Then, for every $\varepsilon > 0$ and $y \in E$ such that*

$$g(y) < \inf_E g + \varepsilon$$

and every $\lambda > 0$, there exists some point $z \in E$ such that

$$g(z) \le g(y), \qquad \|z - x_0\|_E \le \bigl(1 + \|y\|_E\bigr)\bigl(e^{\lambda} - 1\bigr),$$

and

$$g(x) \ge g(z) - \frac{\varepsilon}{\lambda(1 + \|z\|_E)}\|x - z\|_E \quad \textit{for all } x \in E.$$

With the help of Lemmas 3.7, 3.8, and 3.9, we are in a position to derive our first main result.

Theorem 3.10 *Assume that* (H), (V), (A), *and* (F1)–(F6) *hold. Then there exists $\lambda^* > 0$ such that, for any $\lambda \in (0, \lambda^*)$, problem* (P) *admits at least two distinct nontrivial weak solutions.*

Proof By means of Lemmas 3.7 and 3.8, there is a positive real number λ^* such that $\mathcal{I}_\lambda$ ensures the mountain pass geometry and the (C)-condition for any $\lambda \in (0, \lambda^*)$. Thanks to the mountain pass theorem in [17], we deduce that $\mathcal{I}_\lambda$ has a critical point $z_0 \in \mathcal{X}$ with $\mathcal{I}_\lambda(z_0) = \overline{c} > 0 = \mathcal{I}_\lambda(0)$. Thus problem (P) possesses a nontrivial weak solution z_0.

Next we show the existence of the second weak solution of (P). Owing to Lemma 3.8, for fixed $\lambda \in (0, \lambda^*)$, there are positive constants R and $r \in (0, 1)$ such that $\mathcal{I}_\lambda(z) \ge R > 0$ for all $u \in \mathcal{X}$ with $\|z\|_{\mathcal{X}} = r$. Let us denote $c := \inf_{z \in \overline{B}_r} \mathcal{I}_\lambda(z)$ where $B_r := \{z \in \mathcal{X} : \|z\|_{\mathcal{X}} < r\}$ with a boundary ∂B_r. Then, by Lemma 3.8(3), we know $-\infty < c < 0$. If we put $0 < \varepsilon < \inf_{z \in \partial B_r} \mathcal{I}_\lambda(z) - c$, from Lemma 3.9 it follows that we can look for $z_\varepsilon \in \overline{B}_r$ such that

$$\begin{cases} \mathcal{I}_\lambda(z_\varepsilon) < c + \varepsilon, \\ \mathcal{I}_\lambda(z_\varepsilon) \le \mathcal{I}_\lambda(z) + \frac{\varepsilon}{1 + \|z_\varepsilon\|_{\mathcal{X}}}\|z - z_\varepsilon\|_{\mathcal{X}} \quad \text{for all } z \in \overline{B}_r \text{ with } z \ne z_\varepsilon. \end{cases} \tag{3.9}$$

This fact together with the estimate $\mathcal{I}_\lambda(z_\varepsilon) < c + \varepsilon < \inf_{z \in \partial B_r} \mathcal{I}_\lambda(z)$ gives that $z_\varepsilon \in B_r$. Hence it follows that z_ε is a local minimum of $\widetilde{I}_\lambda(z) := \mathcal{I}_\lambda(z) + \frac{\varepsilon}{1 + \|z_\varepsilon\|_{\mathcal{X}}}\|z - z_\varepsilon\|_{\mathcal{X}}$. Now, by taking $z = z_\varepsilon + t\omega$ for $\omega \in B_1$ and $t > 0$ small enough, we deduce from (3.9) that

$$0 \le \frac{\widetilde{I}_\lambda(z_\varepsilon + t\omega) - \widetilde{I}_\lambda(z_\varepsilon)}{t} = \frac{\mathcal{I}_\lambda(z_\varepsilon + t\omega) - \mathcal{I}_\lambda(z_\varepsilon)}{t} + \frac{\varepsilon}{1 + \|z_\varepsilon\|_{\mathcal{X}}}\|\omega\|_{\mathcal{X}}.$$

Therefore, letting $t \to 0^+$, we get

$$\bigl\langle \mathcal{I}'_\lambda(z_\varepsilon), \omega \bigr\rangle + \frac{\varepsilon}{1 + \|z_\varepsilon\|_{\mathcal{X}}}\|\omega\|_{\mathcal{X}} \ge 0.$$

Changing ω into $-\omega$ in the argument above, one has

$$-\langle \mathcal{I}'_\lambda(z_\varepsilon), \omega\rangle + \frac{\varepsilon}{1 + \|z_\varepsilon\|_{\mathcal{X}}}\|\omega\|_{\mathcal{X}} \geq 0.$$

Thus, we have

$$(1 + \|z_\varepsilon\|_{\mathcal{X}})\left|\langle \mathcal{I}'_\lambda(z_\varepsilon), \omega\rangle\right| \leq \varepsilon\|\omega\|_{\mathcal{X}}$$

for any $\omega \in \overline{B}_1$. Hence we know

$$(1 + \|z_\varepsilon\|_{\mathcal{X}})\left\|\mathcal{I}'_\lambda(z_\varepsilon)\right\|_{\mathcal{X}^*} \leq \varepsilon. \tag{3.10}$$

Using (3.9) and (3.10), we can choose a sequence $\{z_n\} \subset B_r$ such that

$$\begin{cases} \mathcal{I}_\lambda(z_n) \to c \quad \text{as } n \to \infty, \\ (1 + \|z_\varepsilon\|_{\mathcal{X}})\|\mathcal{I}'_\lambda(z_\varepsilon)\|_{\mathcal{X}^*} \to 0 \quad \text{as } n \to \infty. \end{cases} \tag{3.11}$$

Thus, $\{z_n\}$ is a bounded (C)-sequence in the reflexive Banach space $\mathcal{X}$. By virtue of the fact that $\mathcal{I}'_\lambda$ is of type (S_+) as mentioned in the proof of Lemma 3.7, $\{z_n\}$ has a subsequence $\{z_{n_k}\}$ such that $z_{n_k} \to z_1$ in $\mathcal{X}$ as $k \to \infty$. This fact together with (3.11) leads to $\mathcal{I}_\lambda(z_1) = c < 0$ and $\mathcal{I}'_\lambda(z_1) = 0$. Hence there is a nontrivial solution z_1 which is different from z_0. Therefore we conclude that problem (P) possesses at least two distinct nontrivial weak solutions. □

Next, by applying the fountain theorem and the dual fountain theorem as essential tools which are originally provided by the papers [9] and [10], we establish two existence results of a sequence of infinitely many solutions for problem (P). Let E be a real reflexive and separable Banach space, then it is known (see [22, 57]) that there exist $\{e_n\} \subseteq W$ and $\{f_n^*\} \subseteq E^*$ such that

$$E = \overline{\operatorname{span}\{e_n : n = 1, 2, \dots\}}, \qquad E^* = \overline{\operatorname{span}\{f_n^* : n = 1, 2, \dots\}},$$

and

$$\langle f_i^*, e_j\rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Let us denote $E_n = \operatorname{span}\{e_n\}$, $Y_k = \bigoplus_{n=1}^k E_n$, and $Z_k = \overline{\bigoplus_{n=k}^\infty E_n}$.

Lemma 3.11 (Fountain theorem [9, 30, 52]) *Let E be a Banach space, the functional $\mathcal{I} \in C^1(E, \mathbb{R})$ satisfies the $(C)_c$-condition for any $c > 0$ and $\mathcal{I}$ is even. If for each sufficiently large $k \in \mathbb{N}$ there exist $\rho_k > \delta_k > 0$ such that the following properties hold:*

(1) $b_k := \inf\{\mathcal{I}(z) : z \in Z_k, \|z\|_E = \delta_k\} \to \infty$ *as* $k \to \infty$;

(2) $a_k := \max\{\mathcal{I}(z) : z \in Y_k, \|z\|_E = \rho_k\} \leq 0$,

then $\mathcal{I}$ possesses an unbounded sequence of critical values, i.e., there is a sequence $\{z_n\} \subset X$ such that $\mathcal{I}'(z_n) = 0$ and $\mathcal{I}(z_n) \to +\infty$ as $n \to +\infty$.

With the aid of Lemma 3.11, we are in a position to derive the existence of multiple large energy solutions.

Theorem 3.12 *Assume that* (H), (V), (A), *and* (F1)–(F4) *hold. If* $f(x,-t) = -f(x,t)$ *holds for all* $(x,t) \in \mathbb{R}^N \times \mathbb{R}$, *then for any* $\lambda > 0$ *problem* (P) *possesses a sequence of nontrivial weak solutions* $\{z_n\}$ *in* $\mathcal{X}$ *such that* $\mathcal{I}_\lambda(z_n) \to \infty$ *as* $n \to \infty$.

Proof Clearly, $\mathcal{I}_\lambda$ is an even functional that ensures the $(C)_c$-condition. It is enough to prove that there are $\rho_k > \delta_k > 0$ such that

(1) $b_k := \inf\{\mathcal{I}_\lambda(z) : z \in Z_k, \|z\|_{\mathcal{X}} = \delta_k\} \to \infty$ as $k \to \infty$;

(2) $a_k := \max\{\mathcal{I}_\lambda(z) : z \in Y_k, \|z\|_{\mathcal{X}} = \rho_k\} \le 0$,

for sufficiently large k. Denote

$$\alpha_k := \sup_{z\in Z_k, \|z\|_{\mathcal{X}}=1} \|z\|_{L^{q(\cdot)}(\mathbb{R}^N)}.$$

Then we assert $\alpha_k \to 0$ as $k \to \infty$. Indeed, suppose to the contrary that we can choose $\varepsilon_0 > 0$, $k_0 \ge 0$, and the sequence $\{z_k\}$ in Z_k such that

$$\|z_k\|_{\mathcal{X}} = 1, \qquad \|z\|_{L^{q(\cdot)}(\mathbb{R}^N)} \ge \varepsilon_0$$

for all $k \ge k_0$. From the boundedness of the sequence $\{z_k\}$ in $\mathcal{X}$, we look for $z \in \mathcal{X}$ such that $z_k \rightharpoonup z$ in $\mathcal{X}$ as $n \to \infty$ and

$$\langle f_j^*, z\rangle = \lim_{k\to\infty} \langle f_j^*, z_k\rangle = 0$$

for $j = 1, 2, \dots$. Hence we get $z = 0$. However, we obtain

$$\varepsilon_0 \le \lim_{k\to\infty} \|z_k\|_{L^{q(\cdot)}(\mathbb{R}^N)} = \|z\|_{L^{q(\cdot)}(\mathbb{R}^N)} = 0,$$

that is a contradiction.

For any $z \in Z_k$, suppose that $\|z\|_{\mathcal{X}} > 1$. From (F2), Lemma 2.1, and (3.4), it follows that

$$\begin{aligned}
\mathcal{I}_\lambda(z) &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{p(x,y)} \bigl|z(x) - z(y)\bigr|^{p(x,y)} \mathcal{K}(x,y)\,dx\,dy + \int_{\mathbb{R}^N} \frac{V(x)}{p(x)} |z|^{p(x)}\,dx \\
&\quad - \lambda \int_{\mathbb{R}^N} \frac{a(x)}{r(x)} |z|^{r(x)}\,dx - \int_{\mathbb{R}^N} F(x,z)\,dx \\
&\ge \frac{1}{p^+}\|z\|_{\mathcal{X}}^{p^-} - \frac{\lambda}{r_-}\|a\|_{L^{\frac{p(\cdot)}{p(\cdot)-r(\cdot)}}(\mathbb{R}^N)} \max\bigl\{\|z\|_{L^{p(\cdot)}(\mathbb{R}^N)}^{r_+}, \|z\|_{L^{p(\cdot)}(\mathbb{R}^N)}^{r_-}\bigr\} \\
&\quad - \int_{\mathbb{R}^N} \frac{|b(x)|}{q(x)} |z|^{q(x)}\,dx \\
&\ge \frac{1}{p^+}\|z\|_{\mathcal{X}}^{p^-} - \frac{\lambda}{r_-}\|a\|_{L^{\frac{p(\cdot)}{p(\cdot)-r(\cdot)}}(\mathbb{R}^N)} \max\bigl\{\|z\|_{L^{p(\cdot)}(\mathbb{R}^N)}^{r_+}, \|z\|_{L^{p(\cdot)}(\mathbb{R}^N)}^{r_-}\bigr\} \\
&\quad - \frac{\|b\|_{L^\infty(\mathbb{R}^N)}}{q_-} \int_{\mathbb{R}^N} |z|^{q(x)}\,dx \\
&\ge \frac{1}{p^+}\|z\|_{\mathcal{X}}^{p^-} - \frac{\lambda}{r_-} C_4 \|z\|_{\mathcal{X}}^{r_+} - \frac{1}{q_-}\alpha_k^{q_-} C_5 \|z\|_{\mathcal{X}}^{q_+},
\end{aligned}$$

where C_4 and C_5 are positive constants. Now, let us choose $\delta_k = (q_+ C_5 \alpha_k^{q_-}/q_-)^{1/(p^- - q_+)}$. Since $p^- < q_+$ and $\alpha_k \to 0$ as $k \to \infty$, we assert $\delta_k \to \infty$ as $k \to \infty$. Hence, if $z \in Z_k$ and $\|z\|_{\mathcal{X}} = \delta_k$, then we arrive at

$$\mathcal{I}_\lambda(z) \geq \left(\frac{1}{p^+} - \frac{1}{q_+}\right)\delta_k^{p^-} - \frac{\lambda}{r_-} C_4 \delta_k^{r_+} \to \infty \quad \text{as } k \to \infty,$$

which implies condition (1).

Now we prove condition (2) arguing by contradiction. Then, let us assume that condition (2) is not satisfied for some k. Then we can find a sequence $\{z_n\}$ in Y_k such that

$$\|z_n\|_{\mathcal{X}} \to \infty \quad \text{as } n \to \infty \quad \text{and} \quad I_\lambda(z_n) \geq 0. \tag{3.12}$$

Let $w_n = z_n/\|z_n\|_{\mathcal{X}}$. Then, clearly, we have $\|w_n\|_{\mathcal{X}} = 1$. Since $\dim Y_k < \infty$, there is an element w in $Y_k \setminus \{0\}$ such that, up to a subsequence still denoted by $\{w_n\}$,

$$\|w_n - w\|_{\mathcal{X}} \to 0 \quad \text{and} \quad w_n(x) \to w(x)$$

for almost all $x \in \mathbb{R}^N$ as $n \to \infty$. We claim that $w(x) = 0$ for almost all $x \in \mathbb{R}^N$. If $w(x) \neq 0$, then $|z_n(x)| \to \infty$ for all $x \in \mathbb{R}^N$ as $n \to \infty$. Hence, by means of assumption (F3) it follows that

$$\lim_{n\to\infty} \frac{F(x, z_n(x))}{\|z_n\|_{\mathcal{X}}^{p^+}} = \lim_{n\to\infty} \frac{F(x, z_n(x))}{|z_n(x)|^{p^+}} \left|w_n(x)\right|^{p^+} = \infty \tag{3.13}$$

for all $x \in \Omega_2 := \{x \in \mathbb{R}^N : w(x) \neq 0\}$. Proceeding as in the proof of Lemma 3.7, it can be shown that there is $\mathcal{M}_2 \in \mathbb{R}$ such that $F(x,t) \geq \mathcal{M}_2$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$, and so

$$\frac{F(x, z_n) - \mathcal{M}_2}{\|z_n\|_{\mathcal{X}}^{p^+}} \geq 0$$

for all $x \in \mathbb{R}^N$ and $n \in \mathbb{N}$. Using (3.13) and the Fatou lemma, one has

$$\begin{aligned}
\liminf_{n\to\infty} \int_{\mathbb{R}^N} \frac{F(x, z_n)}{\|z_n\|_{\mathcal{X}}^{p^+}}\,dx &\geq \liminf_{n\to\infty} \int_{\Omega_2} \frac{F(x, z_n)}{\|z_n\|_{\mathcal{X}}^{p^+}}\,dx - \limsup_{n\to\infty} \int_{\Omega_2} \frac{\mathcal{M}_2}{\|z_n\|_{\mathcal{X}}^{p^+}}\,dx \\
&= \liminf_{n\to\infty} \int_{\Omega_2} \frac{F(x, z_n) - \mathcal{M}_2}{\|z_n\|_{\mathcal{X}}^{p^+}}\,dx \\
&\geq \int_{\Omega_2} \liminf_{n\to\infty} \frac{F(x, z_n) - \mathcal{M}_2}{\|z_n\|_{\mathcal{X}}^{p^+}}\,dx \\
&= \int_{\Omega_2} \liminf_{n\to\infty} \frac{F(x, z_n)}{\|z_n\|_{\mathcal{X}}^{p^+}}\,dx - \int_{\Omega_2} \limsup_{n\to\infty} \frac{\mathcal{M}_2}{\|z_n\|_{\mathcal{X}}^{p^+}}\,dx.
\end{aligned}$$

Thus we infer

$$\int_{\mathbb{R}^N} \frac{F(x, z_n(x))}{\|z_n\|_{\mathcal{X}}^{p^+}}\,dx \to \infty \quad \text{as } n \to \infty.$$

We may assume that $\|z_n\|_{\mathcal{X}} > 1$. Therefore, we have

$$\begin{aligned}
\mathcal{I}_\lambda(z_n) &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{p(x,y)}\big|z_n(x)-z_n(y)\big|^{p(x,y)}\mathcal{K}(x,y)\,dx\,dy + \int_{\mathbb{R}^N}\frac{V(x)}{p(x)}|z_n|^{p(x)}\,dx \\
&\quad - \lambda\int_{\mathbb{R}^N}\frac{a(x)}{r(x)}|z_n|^{r(x)}\,dx - \int_{\mathbb{R}^N} F(x,z_n)\,dx \\
&\le \frac{1}{p^-}\|z_n\|_{\mathcal{X}}^{p^+} - \int_{\Omega_2} F(x,z_n)\,dx \\
&\le \|z_n\|_{\mathcal{X}}^{p^+}\left(\frac{1}{p^-} - \int_{\Omega_2}\frac{F(x,z_n(x))}{\|z_n\|_{\mathcal{X}}^{p^+}}\,dx\right) \to -\infty \quad \text{as } n\to\infty,
\end{aligned}$$

which is a contradiction to (3.12). This completes the proof. □

Definition 3.13 Let E be a real separable and reflexive Banach space. We say that $\mathcal{I}$ satisfies the $(C)_c^*$-condition (with respect to Y_n) if any sequence $\{z_n\}_{n\in\mathbb{N}} \subset E$ for which $z_n \in Y_n$, for any $n \in \mathbb{N}$,

$$\mathcal{I}(z_n) \to c \quad \text{and} \quad \big\|(\mathcal{I}|_{Y_n})'(z_n)\big\|_{E^*}\big(1+\|z_n\|_E\big) \to 0 \quad \text{as } n\to\infty,$$

has a subsequence converging to a critical point of $\mathcal{I}$.

Lemma 3.14 (Dual fountain theorem [10, 30]) *Assume that E is a Banach space, $\mathcal{I} \in C^1(E,\mathbb{R})$ is an even functional. If there is $k_0 > 0$ so that, for each $k \ge k_0$, there exist $\rho_k > \delta_k > 0$ such that the following properties hold*:

(H1) $\inf\{\mathcal{I}(\omega) : \omega \in Z_k, \|\omega\|_E = \rho_k\} \ge 0$;
(H2) $b_k := \max\{\mathcal{I}(\omega) : \omega \in Y_k, \|\omega\|_E = \delta_k\} < 0$;
(H3) $d_k := \inf\{\mathcal{I}(\omega) : \omega \in Z_k, \|\omega\|_E \le \rho_k\} \to 0$ *as* $k \to \infty$;
(H4) *$\mathcal{I}$ satisfies the $(C)_c^*$-condition for every $c \in [d_{k_0}, 0)$*,

then $\mathcal{I}$ has a sequence of negative critical values $c_n < 0$ satisfying $c_n \to 0$ as $n \to \infty$.

Lemma 3.15 *Suppose that* (H), (V), (A), *and* (F1)–(F5) *hold. Then the functional $\mathcal{I}_\lambda$ satisfies the $(C)_c^*$-condition for any $\lambda > 0$.*

Proof Since $\mathcal{X}$ is a reflexive Banach space, and $\mathcal{A}'$ and Ψ_λ' are of type (S_+), the proof is almost identical to that of Lemma 3.12 in [30]. □

With the help of Lemmas 3.14 and 3.15 we are ready to establish our final consequence.

Theorem 3.16 *Assume that* (H), (V), (A), *and* (F1)–(F5) *hold. Then problem* (P) *admits a sequence of nontrivial weak solutions $\{\omega_n\}$ in $\mathcal{X}$ such that $\mathcal{I}_\lambda(\omega_n) \to 0$ as $n \to \infty$ for any* $\lambda > 0$.

Proof By means of (F4) and Lemma 3.15, we infer that the functional $\mathcal{I}_\lambda$ is even and ensures the $(C)_c^*$-condition for all $c \in \mathbb{R}$. Now we will prove that properties (H1), (H2), and (H3) of the dual fountain theorem hold.

(H1): In accordance with (F1), we have

$$\big|F(x,\tau)\big| \le \frac{b(x)}{q(x)}|\tau|^{q(x)}, \quad (x,\tau) \in \mathbb{R}^N \times \mathbb{R}.$$

For convenience, we denote

$$\theta_{1,k} = \sup_{\|\omega\|_{\mathcal{X}}=1, \omega\in Z_k} \|\omega\|_{L^{p(\cdot)}(\mathbb{R}^N)}, \qquad \theta_{2,k} = \sup_{\|\omega\|_{\mathcal{X}}=1, \omega\in Z_k} \|\omega\|_{L^{q(\cdot)}(\mathbb{R}^N)}.$$

Then it is easy to verify that $\theta_{1,k} \to 0$ and $\theta_{2,k} \to 0$ as $k \to \infty$ (see [30]). Set $\vartheta_k = \max\{\theta_{1,k}, \theta_{2,k}\}$. Then it follows that

$$\begin{aligned}
\mathcal{I}_\lambda(\omega) &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{p(x,y)} \left|\omega(x)-\omega(y)\right|^{p(x,y)} \mathcal{K}(x,y)\,dx\,dy + \int_{\mathbb{R}^N} \frac{V(x)}{p(x)} |\omega|^{p(x)}\,dx \\
&\quad - \lambda \int_{\mathbb{R}^N} \frac{a(x)}{r(x)} |\omega|^{r(x)}\,dx - \int_{\mathbb{R}^N} F(x,\omega)\,dx \\
&\ge \frac{1}{p^+}\|\omega\|_{\mathcal{X}}^{p^-} - \frac{\lambda}{r_-}\|a\|_{L^{\frac{p(\cdot)}{p(\cdot)-r(\cdot)}}(\mathbb{R}^N)} \max\left\{\|\omega\|_{L^{p(\cdot)}(\mathbb{R}^N)}^{r_+}, \|\omega\|_{L^{p(\cdot)}(\mathbb{R}^N)}^{r_-}\right\} \\
&\quad - \frac{\|b\|_{L^\infty(\mathbb{R}^N)}}{q_-} \max\left\{\|\omega\|_{L^{q(\cdot)}(\mathbb{R}^N)}^{q_-}, \|\omega\|_{L^{q(\cdot)}(\mathbb{R}^N)}^{q_+}\right\} \\
&\ge \frac{1}{p^+}\|\omega\|_{\mathcal{X}}^{p^-} - \frac{\lambda}{r_-}\|a\|_{L^{\frac{p(\cdot)}{p(\cdot)-r(\cdot)}}(\mathbb{R}^N)} \vartheta_{1,k}^{r_-}\|\omega\|_{\mathcal{X}}^{r_+} - \frac{\|b\|_{L^\infty(\mathbb{R}^N)}}{q_-}\vartheta_{2,k}^{q_-}\|\omega\|_{\mathcal{X}}^{q_+} \\
&\ge \frac{1}{p^+}\|\omega\|_{\mathcal{X}}^{p^-} - \left(\frac{2\lambda}{r_-}\|a\|_{L^{\frac{p(\cdot)}{p(\cdot)-r(\cdot)}}(\mathbb{R}^N)} + \frac{\|b\|_{L^\infty(\mathbb{R}^N)}}{q_-}\right)\vartheta_k^{r_-}\|\omega\|_{\mathcal{X}}^{q_+}
\end{aligned}$$

for k large enough and $\|\omega\|_{\mathcal{X}} \ge 1$. Choose

$$\rho_k = \left[\left(\frac{4\lambda}{r_-}\|a\|_{L^{\frac{p(\cdot)}{p(\cdot)-r(\cdot)}}(\mathbb{R}^N)} + \frac{2\|b\|_{L^\infty(\mathbb{R}^N)}}{q_-}\right)p^+\vartheta_k^{r_-}\right]^{\frac{1}{p^- - 2q_+}}.$$

Let $\omega \in Z_k$ with $\|\omega\|_{\mathcal{X}} = \rho_k > 1$ for sufficiently large k. Then there is $k_0 \in \mathbb{N}$ such that

$$\begin{aligned}
\mathcal{I}_\lambda(\omega) &\ge \frac{1}{p^+}\|\omega\|_{\mathcal{X}}^{p^-} - \left(\frac{2\lambda}{r_-}\|a\|_{L^{\frac{p(\cdot)}{p(\cdot)-r(\cdot)}}(\mathbb{R}^N)} + \frac{\|b\|_{L^\infty(\mathbb{R}^N)}}{q_-}\right)\vartheta_k^{r_-}\|\omega\|_{\mathcal{X}}^{2q_+} \\
&\ge \frac{1}{2p^+}\rho_k^{p^-} \ge 0
\end{aligned}$$

for all $k \in \mathbb{N}$ with $k \ge k_0$, being

$$\lim_{k\to\infty} \frac{1}{2p^+}\rho_k^{p^-} = \infty.$$

Therefore,

$$\inf\left\{\mathcal{I}_\lambda(\omega) : \omega \in Z_k, \|\omega\|_{\mathcal{X}} = \rho_k\right\} \ge 0.$$

(H2): Observe that $\|\cdot\|_{L^{p(\cdot)}(\mathbb{R}^N)}$, $\|\cdot\|_{L^{p^+}(\mathbb{R}^N)}$ and $\|\cdot\|_{\mathcal{X}}$ are equivalent on Y_k. Then we can choose some constants $\varsigma_{1,k} > 0$ and $\varsigma_{2,k} > 0$ such that

$$\|\omega\|_{L^{p(\cdot)}(\mathbb{R}^N)} \le \varsigma_{1,k}\|\omega\|_{\mathcal{X}} \quad \text{and} \quad \|\omega\|_{\mathcal{X}} \le \varsigma_{2,k}\|\omega\|_{L^{p^+}(\mathbb{R}^N)} \tag{3.14}$$

for any $\omega \in Y_k$. From (F2)–(F3) and (F5), for any $\mathcal{M} > 0$, there are some constants $C_6 > 0$ and $C_7(\mathcal{M}) > 0$ such that

$$F(x,\tau) \geq \mathcal{M}\varsigma_{2,k}^{p^+}|\tau|^{p^+} - C_6|\tau|^{p(x)} - C_7(\mathcal{M})b(x) \tag{3.15}$$

for almost all $(x,\tau) \in \mathbb{R}^N \times \mathbb{R}$ where b comes from (H2). Then, from (3.14) and (3.15), it follows that

$$\begin{aligned}
\mathcal{I}_\lambda(\omega) &\leq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{p(x,y)}\left|\omega(x)-\omega(y)\right|^{p(x,y)}\mathcal{K}(x,y)\,dx\,dy \\
&\quad + \int_{\mathbb{R}^N} \frac{V(x)}{p(x)}|\omega|^{p(x)}\,dx - \int_{\mathbb{R}^N} F(x,\omega)\,dx \\
&\leq \frac{1}{p^-}\|\omega\|_{\mathcal{X}}^{p^+} - \mathcal{M}\varsigma_{2,k}^{p^+}\int_{\mathbb{R}^N}|\omega|^{p^+}\,dx + C_6\int_{\mathbb{R}^N}|\omega|^{p(x)} + C_7(\mathcal{M})\int_{\mathbb{R}^N} b(x)\,dx \\
&\leq \frac{1}{p^-}\|\omega\|_{\mathcal{X}}^{p^+} - \mathcal{M}\|\omega\|_{\mathcal{X}}^{p^+} + C_6\left(\varsigma_{1,k}^{p^+} + \varsigma_{1,k}^{p^-}\right)\|\omega\|_{\mathcal{X}}^{p^+} + C_8
\end{aligned}$$

for any $\omega \in Y_k$ with $\|\omega\|_{\mathcal{X}} \geq 1$ and positive constant C_8. Let $f(t) = \frac{1}{p^-}t^{p^+} - \mathcal{M}t^{p^+} + C_6(\varsigma_{1,k}^{p^+} + \varsigma_{1,k}^{p^-})t^{p^+} + C_8$. If $\mathcal{M}$ is large enough, then $\lim_{t\to\infty} f(t) = -\infty$, and so there is $t_0 \in (1,\infty)$ such that $f(t) < 0$ for all $t \in [t_0,\infty)$. Hence $\mathcal{I}_\lambda(\omega) < 0$ for all $\omega \in Y_k$ with $\|\omega\|_{\mathcal{X}} = t_0$. Choosing $\delta_k = t_0$ for all $k \in \mathbb{N}$, one has

$$b_k := \max\left\{\mathcal{J}_\lambda(\omega) : \omega \in Y_k, \|\omega\|_{\mathcal{X}} = \delta_k\right\} < 0.$$

If necessary, we can change k_0 to a large value, so that $\rho_k > \delta_k > 0$ for all $k \geq k_0$.

(H3): Because $Y_k \cap Z_k \neq \emptyset$ and $0 < \delta_k < \rho_k$, we have $d_k \leq b_k < 0$ for all $k \geq k_0$. For any $\omega \in Z_k$ with $\|\omega\|_{\mathcal{X}} = 1$ and $0 < t < \rho_k$, one has

$$\begin{aligned}
\mathcal{I}_\lambda(t\omega) &\geq \frac{1}{p^+}\|t\omega\|_{\mathcal{X}}^{p^-} - \frac{2\lambda}{r_-}\|a\|_{L^{\frac{p(\cdot)}{p(\cdot)-r(\cdot)}}(\mathbb{R}^N)} \max\left\{\|t\omega\|_{L^{p(\cdot)}(\mathbb{R}^N)}^{r_-}, \|t\omega\|_{L^{p(\cdot)}(\mathbb{R}^N)}^{r_+}\right\} \\
&\quad - \frac{\|b\|_{L^\infty(\mathbb{R}^N)}}{q_-}\max\left\{\|t\omega\|_{L^{q(\cdot)}(\mathbb{R}^N)}^{q_-}, \|t\omega\|_{L^{q(\cdot)}(\mathbb{R}^N)}^{q_+}\right\} \\
&\geq -\frac{2\lambda}{r_-}\|a\|_{L^{\frac{p(\cdot)}{p(\cdot)-r(\cdot)}}(\mathbb{R}^N)}\rho_k^{r_+}\vartheta_k^{r_-} - \frac{\|b\|_{L^\infty(\mathbb{R}^N)}}{q_-}\rho_k^{q_+}\vartheta_k^{q_-}
\end{aligned}$$

for large enough k. Hence, from the definition of ρ_k, it follows that

$$\begin{aligned}
d_k &\geq -\frac{2\lambda}{r_-}\|a\|_{L^{\frac{p(\cdot)}{p(\cdot)-r(\cdot)}}(\mathbb{R}^N)}\rho_k^{r_+}\vartheta_k^{r_-} - \frac{\|b\|_{L^\infty(\mathbb{R}^N)}}{q_-}\rho_k^{q_+}\vartheta_k^{q_-} \\
&= -\frac{2\lambda}{r_-}\|a\|_{L^{\frac{p(\cdot)}{p(\cdot)-r(\cdot)}}(\mathbb{R}^N)}\left[p^+\left(\frac{4\lambda}{r_-}\|a\|_{L^{\frac{p(\cdot)}{p(\cdot)-r(\cdot)}}(\mathbb{R}^N)} + \frac{2\|b\|_{L^\infty(\mathbb{R}^N)}}{q_-}\right)\right]^{\frac{r_+}{p^--2q_+}}\vartheta_k^{\frac{r_-r_++(p^--2q_+)r_-}{p^--2q_+}} \\
&\quad - \frac{\|b\|_{L^\infty(\mathbb{R}^N)}}{q_-}\left[p^+\left(\frac{4\lambda}{r_-}\|a\|_{L^{\frac{p(\cdot)}{p(\cdot)-r(\cdot)}}(\mathbb{R}^N)} + \frac{2\|b\|_{L^\infty(\mathbb{R}^N)}}{q_-}\right)\right]^{\frac{q_+}{p^--2q_+}}\vartheta_k^{\frac{r_-q_++(p^--2q_+)q_-}{p^--2q_+}}.
\end{aligned}$$

Since $p^- < q_+$, $r_+ + p^- < 2q_+$, $r_-q_+ + q_-p^- < 2q_-q_+$, and $\vartheta_k \to 0$ as $k \to \infty$, we arrive at $\lim_{k\to\infty} d_k = 0$.

Then, all the properties of Lemma 3.14 are satisfied. Consequently we conclude that problem (P) admits a sequence of nontrivial weak solutions $\{\omega_n\}$ in $\mathcal{X}$ such that $\mathcal{I}_\lambda(\omega_n) \to 0$ as $n \to \infty$ for any $\lambda > 0$. □

Remark 3.17 In order to obtain a result similar to Theorem 3.16, the authors in [10, 41, 51, 52] have applied the dual fountain theorem when ρ_k—defined in Lemma 3.14—converges to 0 as $k \to \infty$. For this reason, the proof of Theorem 3.16 is different from that of the papers [10, 41, 51, 52] because we get this result when $\rho_k \to \infty$ as $k \to \infty$.

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