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Convergence analysis of a novel iteration process with application to a fractional differential equation

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Abstract

The objective of this article is to study a three-step iteration process in the framework of Banach spaces and to obtain convergence results for Suzuki generalized nonexpansive mappings. We also provide numerical examples that support our main results and illustrate the convergence behavior of the proposed process. Further, we present a data-dependence result that is also supported by a nontrivial numerical example. Finally, we discuss the solution of a nonlinear fractional differential equation by utilizing our results.

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1 Introduction

Fixed-point theory has been gaining much attention among researchers as it provides useful tools to solve many nonlinear problems that have applications in different fields, like engineering, economics, chemistry, game theory, etc. Iteration processes play a crucial role in finding fixed points of a nonlinear mapping. Due to its simplicity and significance, the class of nonexpansive mappings is one of the most utilized class of nonlinear mappings. Let K be a nonempty closed convex subset of a Banach space E. A mapping $S : K \to K$ is said to be nonexpansive if $\|Sg - Sf\| \leq \|g - f\|$ for all $g, f \in K$. S is called quasinonexpansive if $F(S) \neq \emptyset$ and $\|Sg - q\| \leq \|g - q\|$ for all $g \in K$, $q \in F(S)$, where $F(S)$ is the set of fixed points of S, i.e., $F(S) = \{g \in S : Sg = g\}$. It is well known that every nonexpansive mapping with a fixed point is a quasinonexpansive mapping. One can observe that the famous Banach Contraction Principle is no longer true for nonexpansive mappings, i.e., a nonexpansive mapping need not admit a fixed point on a complete metric space. Also, Picard iteration need not be convergent for a nonexpansive map in a complete metric space. This led to the beginning of a new era of fixed-point theory for nonexpansive mappings by using geometric properties. In 1965, Browder [1], Göhde [2] and Kirk [3] gave three basic existence results in respect of nonexpansive mappings. With a view to locating fixed points of nonexpansive mappings, Mann [4], Ishikawa [5] and Halpern [6] introduced three basic iteration processes. In the same area we direct the reader to the recent works [7, 8].

Following this, several authors constructed numerous iteration processes to approximate the fixed points of different classes of nonlinear mappings, mainly Noor iteration [9], Agarwal et al. iteration [10], SP iteration [11], Normal-S iteration [12], Abbas and Nazir iteration [13], Thakur et al. iterations [14, 15], Karakaya et al. iteration [16] and many others.

In 2008, Suzuki [17] introduced a new class of mappings that is larger than the class of nonexpansive mappings and called the defining condition Condition (C), which is also referred to as generalized nonexpansive mappings. A mapping $S : K \to K$ defined on a nonempty subset K of a Banach space E is said to satisfy the Condition (C) if

$$\frac{1}{2}\|g - Sg\| \leq \|g - f\| \quad \Rightarrow \quad \|Sg - Sf\| \leq \|g - f\|$$

for all g and $f \in K$.

Suzuki obtained few results regarding the existence of fixed points for such mappings. In 2011, Phuengrattana [18] used Ishikawa iteration to obtain some convergence results for mappings satisfying Condition (C) in uniformly convex Banach spaces. In the last few years, many authors have studied this particular class of mappings in various domains and have obtained many convergence results (e.g., [14, 19–25]).

Motivated and inspired by such research, we introduce a new iteration process for approximating fixed points of Suzuki generalized nonexpansive mapping as follows:

$$\begin{cases} g_1 \in K, \\ e_n = S((1-\alpha_n)g_n + \alpha_n Sg_n), \\ f_n = Se_n, \\ g_{n+1} = Sf_n, \quad n \in \mathbb{N}, \end{cases} \tag{1.1}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$.

The aim of this paper is to prove some convergence results involving process (1.1) for Suzuki generalized nonexpansive mapping. Further, we provide a numerical example to show that our iteration (1.1) converges faster than a number of existing iteration processes, such as Thakur New, Vatan, M and M* iterations, etc. Further, we prove a data-dependence result along with an example to validate the analytical proof. In the last section, we use our results to find a solution to a nonlinear fractional differential equation.

2 Preliminaries

To make our paper self-contained, we collect some basic definitions and required results.

Definition 2.1 A Banach space E is said to be uniformly convex if for each $\epsilon \in (0, 2]$ there is a $\delta > 0$ such that for $g, f \in E$ with $\|g\| \leq 1$, $\|f\| \leq 1$ and $\|g - f\| > \epsilon$, we have

$$\left\| \frac{g + f}{2} \right\| < 1 - \delta.$$

Definition 2.2 A Banach space E is said to satisfy Opial's condition if for any sequence $\{g_n\}$ in E that converges weakly to $g \in E$, i.e., $g_n \rightharpoonup g$ implies that

$$\limsup_{n\to\infty} \|g_n - g\| < \limsup_{n\to\infty} \|g_n - f\|$$

for all $f \in E$ with $f \neq g$.

Examples of Banach spaces satisfying this condition are Hilbert spaces and all l^p spaces $(1 < p < \infty)$. On the other hand, $L^p[0, 2\pi]$ with $1 < p \neq 2$ fail to satisfy Opial's condition.

A mapping $S : K \to E$ is demiclosed at $f \in E$ if for each sequence $\{g_n\}$ in K and each $g \in E$, $g_n \rightharpoonup g$ and $Sg_n \to f$ imply that $g \in K$ and $Sg = f$.

Let K be a nonempty closed convex subset of a Banach E, and let $\{g_n\}$ be a bounded sequence in E. For $g \in E$, we denote:

$$r\bigl(g, \{g_n\}\bigr) = \limsup_{n\to\infty} \|g - g_n\|.$$

The asymptotic radius of $\{g_n\}$ relative to K is given by

$$r\bigl(K, \{g_n\}\bigr) = \inf\bigl\{r\bigl(g, \{g_n\}\bigr) : g \in K\bigr\}$$

and the asymptotic center $A(K, \{g_n\})$ of $\{g_n\}$ is defined as:

$$A\bigl(K, \{g_n\}\bigr) = \bigl\{g \in K : r\bigl(g, \{g_n\}\bigr) = r\bigl(K, \{g_n\}\bigr)\bigr\}.$$

Also, in a uniformly convex Banach space $A(K, \{g_n\})$ consists of exactly one point.

The following lemma due to Schu [26] will be very useful in our subsequent discussion.

Lemma 2.1 *Let E be a uniformly convex Banach space and $\{t_n\}$ be any sequence such that $0 < p \leq t_n \leq q < 1$ for some $p, r \in \mathbb{R}$ and for all $n \geq 1$. Let $\{g_n\}$ and $\{f_n\}$ be any two sequences of E such that $\limsup_{n\to\infty} \|g_n\| \leq r$, $\limsup_{n\to\infty} \|f_n\| \leq r$ and $\limsup_{n\to\infty} \|t_n g_n + (1 - t_n) f_n\| = r$ for some $r \geq 0$. Then, $\lim_{n\to\infty} \|g_n - f_n\| = 0$.*

Now, we list a few lemmas involving Suzuki generalized nonexpansive mapping.

Lemma 2.2 ([17]) *Let K be a nonempty subset of a Banach space E and $S : K \to K$ be any mapping. Then:*

(i) *If S is nonexpansive, then S is a Suzuki generalized nonexpansive mapping.*
(ii) *If S is a Suzuki generalized nonexpanisve mapping such that $F(S) \neq \emptyset$, then S is a quasinonexpansive mapping.*
(iii) *If S is a Suzuki generalized nonexpansive mapping, then $\|g - Sf\| \leq 3\|g - Sg\| + \|g - f\|$ for all g and $f \in K$.*

Lemma 2.3 ([27]) *Let S be a Suzuki generalized nonexpansive mapping defined on a subset K of a Banach space E with the Opial property. If a sequence $\{g_n\}$ converges weakly to e and $\lim_{n\to\infty} \|Sg_n - g_n\| = 0$, then $I - S$ is demiclosed at zero.*

Lemma 2.4 ([17]) *If S is a Suzuki generalized nonexpansive mapping defined on a compact convex subset K of a uniformly convex Banach space E, then S has a fixed point.*

In 1972, Zamfirescu [28] introduced Zamfirescu mappings that serve as an important generalization for the Banach contraction principle [29]. Later, in 2004, Berinde [30] gave a more general class of mappings known as quasicontractive mappings. Following this, Imoru and Olantiwo [31] gave the following definition:

Definition 2.3 A mapping $S : K \to K$ is known as a contractive-like mapping if there exists a strictly increasing and continuous function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$ and a constant $\delta \in [0,1)$ such that for all $g,f \in K$, we have

$$\|Sg - Sf\| \leq \delta\|g - f\| + \varphi\bigl(\|g - Sg\|\bigr).$$

Clearly, the class of contractive-like mappings is wider than the class of quasicontractive mappings. For more comparisons see [32].

Next, we recall the following definition and lemma that will be useful in proving our data-dependence result.

Definition 2.4 Let $S, \tilde{S} : K \to K$ be two operators, then $\tilde{S}$ is said to be an approximate operator of S if $\|Sg - \tilde{S}g\| \leq \epsilon$ for all $g \in K$ and $\epsilon > 0$ is a fixed number.

Lemma 2.5 ([33]) *If $\{a_n\}$ is a nonnegative real sequence and there exists an $m \in \mathbb{N}$ such that for all $n \geq m$ we have the following condition*:

$$a_{n+1} \leq (1 - u_n)a_n + u_n v_n$$

such that $u_n \in (0,1)$ for all $n \in \mathbb{N}$, $\sum_{n=0}^{\infty} u_n = \infty$ and $v_n \geq 0$ for all $n \in \mathbb{N}$, then the following inequality holds:

$$0 \leq \limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} v_n.$$

3 Convergence results

First, we prove a few lemmas that will be useful in obtaining convergence results.

Lemma 3.1 *Let S be a Suzuki generalized nonexpansive mapping defined on a nonempty closed convex subset K of a Banach space E with $F(S) \neq \emptyset$. Let $\{g_n\}$ be the sequence defined by the iteration process (1.1). Then, $\lim_{n\to\infty}\|g_n - q\|$ exists for all $q \in F(S)$.*

Proof Let $q \in F(S)$ and $e \in K$. Since S is a Suzuki generalized nonexpansive mapping, $\frac{1}{2}\|q - Sq\| = 0 \leq \|q - e\|$ implies that $\|Sq - Se\| \leq \|q - e\|$.
Consider,

$$\begin{aligned}\|e_n - q\| &= \bigl\|S\bigl((1-\alpha_n)g_n + \alpha_n Sg_n\bigr) - q\bigr\| \\ &\leq \bigl\|(1-\alpha_n)g_n + \alpha_n Sg_n - q\bigr\| \\ &\leq (1-\alpha_n)\|g_n - q\| + \alpha_n\|g_n - q\| \\ &= \|g_n - q\|\end{aligned} \tag{3.1}$$

and

$$\begin{aligned} \|f_n - q\| &= \|Se_n - q\| \\ &\le \|e_n - q\| \\ &\le \|g_n - q\|. \end{aligned} \tag{3.2}$$

Using (3.1) and (3.2), we obtain

$$\begin{aligned} \|g_{n+1} - q\| &= \|Sf_n - q\| \\ &\le \|f_n - q\| \\ &\le \|g_n - q\|. \end{aligned} \tag{3.3}$$

Thus, $\{\|g_n - q\|\}$ is a bounded and decreasing sequence of reals and hence $\lim_{n\to\infty} \|g_n - q\|$ exists. □

Lemma 3.2 *Let S be a Suzuki generalized nonexpansive mapping defined on a nonempty closed convex subset K of a Banach space E. Let $\{g_n\}$ be the sequence defined by the iteration process (1.1). Then, $F(S) \neq \emptyset$ if and only if $\{g_n\}$ is bounded and $\lim_{n\to\infty} \|Sg_n - g_n\| = 0$.*

Proof Suppose $F(S) \neq \emptyset$ and let $q \in F(S)$. Then, by Lemma 3.1, $\lim_{n\to\infty} \|g_n - q\|$ exists and $\{g_n\}$ is bounded. Let

$$\lim_{n\to\infty} \|g_n - q\| = r. \tag{3.4}$$

From (3.1) and (3.4), we have

$$\limsup_{n\to\infty} \|e_n - q\| \le r. \tag{3.5}$$

Also, from (3.2) and (3.4), we obtain

$$\limsup_{n\to\infty} \|f_n - q\| \le r. \tag{3.6}$$

Further,

$$\lim_{n\to\infty} \|g_{n+1} - q\| = r = \lim_{n\to\infty} \|Sf_n - q\|,$$

which gives

$$\lim_{n\to\infty} \|Sf_n - q\| = r.$$

Using Condition (C), we have

$$\|Sf_n - q\| \le \|f_n - q\|,$$

which yields

$$r \leq \liminf_{n\to\infty} \|f_n - q\|. \tag{3.7}$$

From (3.6) and (3.7), we obtain

$$\lim_{n\to\infty} \|f_n - q\| = r. \tag{3.8}$$

Now,

$$\lim_{n\to\infty} \|g_n - q\| = r = \lim_{n\to\infty} \|f_n - q\| = \lim_{n\to\infty} \|Se_n - q\|. \tag{3.9}$$

Since S is a Suzuki generalized nonexpansive mapping, we have

$$\|Se_n - q\| \leq \|e_n - q\|,$$

which gives

$$r \leq \liminf_{n\to\infty} \|e_n - q\|. \tag{3.10}$$

Using (3.5) and (3.10), we obtain

$$\lim_{n\to\infty} \|e_n - q\| = r. \tag{3.11}$$

Consider,

$$\begin{aligned} \|e_n - q\| &= \big\|S\big((1-\alpha_n)g_n + \alpha_n Sg_n\big) - q\big\| \\ &\leq \big\|(1-\alpha_n)g_n + \alpha_n Sg_n - q\big\| \\ &\leq \|g_n - q\|. \end{aligned}$$

From (3.4) and (3.11), we obtain

$$\lim_{n\to\infty} \big\|(1-\alpha_n)g_n + \alpha_n Sg_n - q\big\| = r. \tag{3.12}$$

On using Lemma 2.1 together with (3.4) and (3.12), we obtain

$$\lim_{n\to\infty} \|g_n - Sg_n\| = 0.$$

Conversely, suppose that $\{g_n\}$ is bounded and $\lim_{n\to\infty} \|g_n - Sg_n\| = 0$. Let $q \in A(K, \{g_n\})$, we have

$$\begin{aligned} r\big(Sq, \{g_n\}\big) &= \limsup_{n\to\infty} \|g_n - Sq\| \\ &\leq \limsup_{n\to\infty} \big(3\|Sg_n - g_n\| + \|g_n - q\|\big) \\ &= \limsup_{n\to\infty} \|g_n - q\| \\ &= r\big(q, \{g_n\}\big). \end{aligned}$$

This implies that $Sq \in A(K, \{g_n\})$. Since E is uniformly convex, $A(K, \{g_n\})$ is singleton, therefore we obtain $Sq = q$. □

Theorem 3.1 *Let S be a Suzuki generalized nonexpansive mapping defined on a nonempty closed convex subset K of a Banach space E that satisfies Opial's condition with $F(S) \neq \emptyset$. If $\{g_n\}$ is the sequence defined by the iteration process (1.1), then $\{g_n\}$ converges weakly to a fixed point of S.*

Proof Let $q \in F(S)$. Then, from Lemma 3.1 $\lim_{n\to\infty} \|g_n - q\|$ exists. In order to show the weak convergence of the iteration process (1.1) to a fixed point of S, we will prove that $\{g_n\}$ has a unique weak subsequential limit in $F(S)$. For this, let $\{g_{n_j}\}$ and $\{g_{n_k}\}$ be two subsequences of $\{g_n\}$ that converge weakly to u and v, respectively. By Lemma 3.2, we have $\lim_{n\to\infty} \|Sg_n - g_n\| = 0$ and using the Lemma 2.3, we have $I - S$ is demiclosed at zero. Hence, $u, v \in F(S)$.

Next, we show the uniqueness. Since $u, v \in F(S)$, $\lim_{n\to\infty} \|g_n - u\|$ and $\lim_{n\to\infty} \|g_n - v\|$ exists. Let $u \neq v$. Then, by Opial's condition, we obtain

$$\begin{aligned}
\lim_{n\to\infty} \|g_n - u\| &= \lim_{j\to\infty} \|g_{n_j} - u\| \\
&< \lim_{j\to\infty} \|g_{n_j} - v\| \\
&= \lim_{n\to\infty} \|g_n - v\| \\
&= \lim_{k\to\infty} \|g_{n_k} - v\| \\
&< \lim_{k\to\infty} \|g_{n_k} - u\| \\
&= \lim_{n\to\infty} \|g_n - u\|,
\end{aligned}$$

which is a contradiction, hence $u = v$. Thus, $\{g_n\}$ converges weakly to a fixed point of S. □

Next, we establish some strong convergence results for iteration process (1.1).

Theorem 3.2 *Let S be a Suzuki generalized nonexpansive mapping defined on a nonempty compact convex subset K of a uniformly convex Banach space E. If $\{g_n\}$ is the iterative sequence defined by the iteration process (1.1), then $\{g_n\}$ converges strongly to a fixed point of S.*

Proof Using Lemma 2.4, we obtain $F(S) \neq \emptyset$. Hence, by Lemma 3.2, we have $\lim_{n\to\infty} \|Sg_n - g_n\| = 0$. Since K is compact, there exists a subsequence $\{g_{n_k}\}$ of $\{g_n\}$ such that $\{g_{n_k}\}$ converges strongly to q for some $q \in K$. From Lemma 2.2(iii), we have

$$\|g_{n_k} - Sq\| \leq 3\|Sg_{n_k} - g_{n_k}\| + \|g_{n_k} - q\|$$

for all $n \geq 1$. Letting $k \to \infty$, we obtain that $\{g_{n_k}\}$ converges to Sq. This implies that $Sq = q$, i.e., $q \in F(S)$. Further, $\lim_{n\to\infty} \|g_n - q\|$ exists by Lemma 3.1. Hence, q is the strong limit of the sequence $\{g_n\}$.

A mapping $S : K \to K$ is said to satisfy the Condition (A) ([34]) if there exists a nondecreasing function $p : [0,\infty) \to [0,\infty)$ with $p(0) = 0$ and $p(r) > 0$ for all $r \in (0,\infty)$ such that $\|g - Sg\| \geq p(d(g, F(S)))$ for all $g \in K$, where $d(g, F(S)) = \inf\{\|g - q\| : q \in F(S)\}$. □

Theorem 3.3 *Let S be a Suzuki generalized nonexpansive mapping defined on a nonempty closed convex subset K of a uniformly convex Banach space E such that $F(S) \neq \emptyset$ and let $\{g_n\}$ be the sequence defined by (1.1). If S satisfies Condition (A), then $\{g_n\}$ converges strongly to a fixed point of S.*

Proof By Lemma 3.1, $\lim_{n\to\infty} \|g_n - q\|$ exists and $\|g_{n+1} - q\| \leq \|g_n - q\|$ for all $q \in F(S)$.

We obtain

$$\inf_{q\in F(S)} \|g_{n+1} - q\| \leq \inf_{q\in F(S)} \|g_n - q\|,$$

which yields

$$d\big(g_{n+1}, F(S)\big) \leq d\big(g_n, F(S)\big).$$

This shows that the sequence $\{d(g_n, F(S))\}$ is decreasing and bounded below, hence $\lim_{n\to\infty} d(g_n, F(S))$ exists.

Let $\lim_{n\to\infty} \|g_n - q\| = r$ for some $r \geq 0$. If $r = 0$, then the result follows. Assume $r > 0$. Also, by Lemma 3.2 we have $\lim_{n\to\infty} \|g_n - Sg_n\| = 0$.

It follows from Condition (A) that

$$\lim_{n\to\infty} p\big(d\big(g_n, F(S)\big)\big) \leq \lim_{n\to\infty} \|g_n - Sg_n\| = 0,$$

hence $\lim_{n\to\infty} p(d(g_n, F(S))) = 0$.

Since p is a nondecreasing function satisfying $p(0) = 0$ and $p(r) > 0$ for all $r \in (0,\infty)$, $\lim_{n\to\infty} d(g_n, F(S)) = 0$. Hence, we have a subsequence $\{g_{n_k}\}$ of $\{g_n\}$ and a sequence $\{y_k\} \subset F(S)$ such that

$$\|g_{n_k} - y_k\| < \frac{1}{2^k}$$

for all $k \in \mathbb{N}$. Using (3.4), we obtain

$$\|g_{n_{k+1}} - y_k\| < \|g_{n_k} - y_k\| < \frac{1}{2^k}.$$

Therefore,

$$\begin{aligned}\|y_{k+1} - y_k\| &\leq \|y_{k+1} - g_{k+1}\| + \|g_{k+1} - y_k\| \\ &\leq \frac{1}{2^{k+1}} + \frac{1}{2^k} \\ &< \frac{1}{2^{k-1}} \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

This implies that $\{y_k\}$ is a Cauchy sequence in $F(S)$. Since $F(S)$ is closed, $\{y_k\}$ converges to a point $q \in F(S)$. Then, $\{g_{n_k}\}$ converges strongly to q. Since $\lim_{n\to\infty} \|g_n - q\|$ exists, we obtain $g_n \to q \in F(S)$. This completes the proof. □

Now, we will construct an example of a Suzuki generalized nonexpansive mapping that is not a nonexpansive mapping. Then, using that example, we will show that our iteration scheme (1.1) has a higher speed of convergence than a number of existing iteration schemes.

Example 1 Define a mapping $S : [0, 1] \to [0, 1]$ by

$$Sg = \begin{cases} 1 - g, & g \in [0, \frac{1}{14}), \\ \frac{g+13}{14}, & g \in [\frac{1}{14}, 1]. \end{cases}$$

First, we show that S is not a nonexpansive mapping. For this, take $g = \frac{7}{100}$ and $f = \frac{1}{14}$. Then,

$$\|Sg - Sf\| = \left\|(1 - g) - \left(\frac{f + 13}{14}\right)\right\| = \frac{72}{19{,}600}$$

and

$$\|g - f\| = |g - f| = \frac{2}{1400}.$$

Clearly, $\|Sg - Sf\| > \|g - f\|$, which proves that S is not a nonexpansive mapping.

Now, we show that S satisfies Condition (C). For this, consider the following cases:

Case-I: Let $g \in [0, \frac{1}{14})$, then $\frac{1}{2}\|g - Sg\| = \frac{1}{2}|2g - 1| = \frac{1}{2}(1 - 2g)$. For $\frac{1}{2}\|g - Sg\| \leq \|g - f\|$, we must have $\frac{1}{2}(1 - 2g) \leq \|g - f\|$, i.e., $\frac{1}{2}(1 - 2g) \leq |g - f|$. Here, note that the case $f < g$ is not possible. Hence, we are left with only one case when $f > g$, which gives $\frac{1}{2}(1 - 2g) \leq f - g$, which yields $f \geq \frac{1}{2}$. Hence, $f \in [\frac{1}{2}, 1]$. Now, we have $g \in [0, \frac{1}{14})$ and $f \in [\frac{1}{2}, 1]$. Hence,

$$\|Sg - Sf\| = \left\|(1 - g) - \frac{f + 13}{14}\right\| = \left|\frac{14g + f - 1}{14}\right| < \frac{1}{14}$$

and

$$\|g - f\| = |g - f| > \frac{6}{14}.$$

Hence,

$$\frac{1}{2}\|g - Sg\| \leq \|g - f\| \quad \Rightarrow \quad \|Sg - Sf\| \leq \|g - f\|.$$

Case-II: Let $g \in [\frac{1}{14}, 1]$, then $\frac{1}{2}\|g - Sg\| = \frac{1}{2}|g - \frac{g+13}{14}| = \frac{13-13g}{28}$. For $\frac{1}{2}\|g - Sg\| \leq \|g - f\|$, we must have $\frac{13-13g}{28} \leq \|g - f\|$, i.e., $\frac{13-13g}{28} \leq |g - f|$. Here, we have two possibilities.

A: When $g < f$, we obtain $\frac{13-13g}{28} \leq f - g$, i.e., $f \geq \frac{13+15g}{28}$. Hence, $f \in [\frac{197}{392}, 1] \subset [\frac{1}{14}, 1]$, which gives $\|Sg - Sf\| = \frac{1}{14}\|g - f\| \leq \|g - f\|$. Hence,

$$\frac{1}{2}\|g - Sg\| \leq \|g - f\| \quad \Rightarrow \quad \|Sg - Sf\| \leq \|g - f\|.$$

Table 1 Values of different iterations

Step	Thakur New Iteration	Vatan Iteration	M Iteration	M* Iteration	New Iteration
1	0.99211	0.99211	0.99211	0.99211	0.99211
10	0.0007541573	0.004154273	0.004575554	0.004605605	0.0003280728
20	0.001579318	0.003773194	0.004352545	0.004393863	0.0003119497
30	0.00238407	0.003501723	0.004186742	0.004235587	0.0002999722
40	0.003169161	0.003285412	0.004049977	0.004104486	0.0002900983
50	0.003935304	0.003103946	0.003931731	0.003990741	0.0002815657

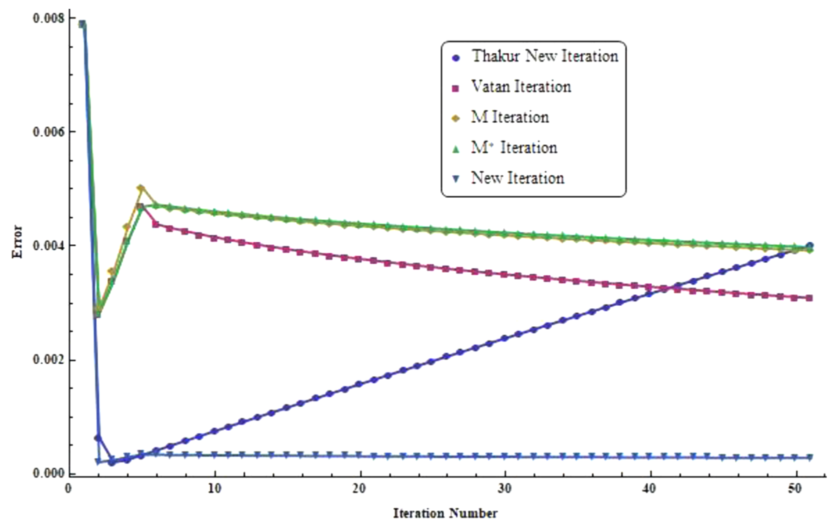

Figure 1 Graph corresponding to Table 1

B: When $g > f$, then $\frac{13-13g}{28} \leq g - f$, i.e., $f \leq \frac{41g-13}{28}$, which gives $f \in [0,1]$. Also, $\frac{28f+13}{41} \leq g$, which yields $g \in [\frac{13}{41}, 1]$. Here, for $g \in [\frac{13}{41}, 1]$ and $f \in [\frac{1}{14}, 1]$ Case IIA can be used. Hence, we only need to verify when $g \in [\frac{13}{41}, 1]$ and $f \in [0, \frac{1}{14})$. For this,

$$\|Sg - Sf\| = \left|\frac{g+13}{14} - (1-f)\right| = \frac{1}{14}|14f + g - 1| \leq \frac{1}{14}$$

and

$$\|g - f\| = |g - f| > \frac{141}{574}.$$

Hence, $\|Sg - Sf\| \leq \|g - f\|$. Thus, mapping S satisfies the Condition (C) for all the possible cases.

Now, using the above example, we will show that the iteration algorithm (1.1) converges faster than Thakur New, Vatan, M and M* iterations. Let $\alpha_n = \tau_n = \sqrt{\frac{n}{n+100}}$ for all $n \in \mathbb{N}$ and $g_1 = 0.00789$, then we obtain Table 1 and Fig. 1 showing the errors.

It is evident from Table 1 and Fig. 1 that our iteration process (1.1) converges at a higher speed than the above-mentioned schemes.

4 Data dependence

In this section, we prove a data-dependence result for the iteration scheme (1.1) and we verify our theoretical result with the help of a numerical example.

Theorem 4.1 *Let S be a contractive-like mapping defined on a nonempty closed convex subset K of a Banach space E with $F(S) \neq \emptyset$. If $\{g_n\}$ is a sequence defined by (1.1), then $\{g_n\}$ converges to the fixed point of S.*

Proof From (1.1), for any $q \in F(S)$, by using the fact that $(1-(1-\delta)\alpha_n) < 1$ for all $n \in \mathbb{N}$ we have

$$\begin{aligned}
\|e_n - q\| &= \big\|S\big((1-\alpha_n)g_n + \alpha_n Sg_n\big) - q\big\| \\
&\leq \delta\big\|(1-\alpha_n)g_n + \alpha_n Sg_n - q\big\| \\
&\leq \delta\big((1-\alpha_n)\|g_n - q\| + \alpha_n\delta\|g_n - q\|\big) \\
&= \delta\big(1-(1-\delta)\alpha_n\big)\|g_n - q\| \\
&\leq \delta\|g_n - q\|
\end{aligned}$$

and

$$\begin{aligned}
\|f_n - q\| &= \|Se_n - q\| \\
&\leq \delta\|e_n - q\| \\
&\leq \delta^2\|g_n - q\|.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|g_{n+1} - q\| &= \|Sf_n - q\| \\
&\leq \delta\|f_n - q\| \\
&\leq \delta^3\|g_n - q\| \\
&\ \ \vdots \\
&\leq \delta^{3n}\|g_1 - q\|.
\end{aligned}$$

Since, $0 \leq \delta < 1$, we obtain

$$\lim_{n\to\infty} \delta^{3n} = 0.$$

Hence,

$$\lim_{n\to\infty} \|g_{n+1} - q\| = 0. \qquad \square$$

Theorem 4.2 *Let S be a contractive-like mapping defined on a nonempty closed convex subset K of a Banach space E with $F(S) \neq \emptyset$ and let $\tilde{S}$ be an approximate operator of S. Let $\{g_n\}$ be a sequence defined by (1.1) for S, we define a sequence $\{\tilde{g_n}\}$ involving $\tilde{S}$ as follows:*

$$\begin{cases}
\tilde{g_1} \in K, \\
\tilde{e_n} = \tilde{S}((1-\alpha_n)\tilde{g_n} + \alpha_n\tilde{S}\tilde{g_n}), \\
\tilde{f_n} = \tilde{S}\tilde{e_n}, \\
\tilde{g_{n+1}} = \tilde{S}\tilde{f_n}, \quad n \in \mathbb{N},
\end{cases} \tag{4.1}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ such that $\frac{1}{2} \leq \alpha_n$ for all $n \in \mathbb{N}$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. If q and $\tilde{q}$ are fixed points of S and $\tilde{S}$, respectively, such that $\{\tilde{g_n}\} \to \tilde{q}$ as $n \to \infty$, then we have

$$\|q - \tilde{q}\| \leq \frac{8\epsilon}{1-\delta}.$$

Proof From (1.1) and (4.1), we have

$$\begin{aligned}
\|e_n - \tilde{e_n}\| &= \big\|S\big((1-\alpha_n)g_n + \alpha_n Sg_n\big) - \tilde{S}\big((1-\alpha_n)\tilde{g_n} + \alpha_n \tilde{S}\tilde{g_n}\big)\big\| \\
&\leq \big\|S\big((1-\alpha_n)g_n + \alpha_n Sg_n\big) - S\big((1-\alpha_n)\tilde{g_n} + \alpha_n \tilde{S}\tilde{g_n}\big)\big\| \\
&\quad + \big\|S\big((1-\alpha_n)\tilde{g_n} + \alpha_n \tilde{S}\tilde{g_n}\big) - \tilde{S}\big((1-\alpha_n)\tilde{g_n} + \alpha_n \tilde{S}\tilde{g_n}\big)\big\| \\
&\leq \delta\big(\big\|\big((1-\alpha_n)g_n + \alpha_n Sg_n\big) - \big((1-\alpha_n)\tilde{g_n} + \alpha_n \tilde{S}\tilde{g_n}\big)\big\|\big) \\
&\quad + \varphi\big(\big\|\big((1-\alpha_n)g_n + \alpha_n Sg_n\big) - S\big((1-\alpha_n)g_n + \alpha_n Sg_n\big)\big\|\big) + \epsilon \\
&\leq \delta\big((1-\alpha_n)\|g_n - \tilde{g_n}\| + \alpha_n\|Sg_n - \tilde{S}\tilde{g_n}\|\big) \\
&\quad + \varphi\big(\big\|\big((1-\alpha_n)g_n + \alpha_n Sg_n\big) - S\big((1-\alpha_n)g_n + \alpha_n Sg_n\big)\big\|\big) + \epsilon \\
&\leq \delta(1-\alpha_n)\|g_n - \tilde{g_n}\| + \alpha_n\delta\|Sg_n - S\tilde{g_n}\| + \alpha_n\delta\|S\tilde{g_n} - \tilde{S}\tilde{g_n}\| \\
&\quad + \varphi\big(\big\|\big((1-\alpha_n)g_n + \alpha_n Sg_n\big) - S\big((1-\alpha_n)g_n + \alpha_n Sg_n\big)\big\|\big) + \epsilon \\
&\leq \delta\big(1-\alpha_n(1-\delta)\big)\|g_n - \tilde{g_n}\| + \alpha_n\delta\varphi\big(\|g_n - Sg_n\|\big) + (1+\delta\alpha_n)\epsilon \\
&\quad + \varphi\big(\big\|\big((1-\alpha_n)g_n + \alpha_n Sg_n\big) - S\big((1-\alpha_n)g_n + \alpha_n Sg_n\big)\big\|\big)
\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}
\|f_n - \tilde{f_n}\| &= \|Se_n - \tilde{S}\tilde{e_n}\| \\
&\leq \|Se_n - S\tilde{e_n}\| + \|S\tilde{e_n} - \tilde{S}\tilde{e_n}\| \\
&\leq \delta\|e_n - \tilde{e_n}\| + \varphi\big(\|e_n - Se_n\|\big) + \epsilon.
\end{aligned} \tag{4.3}$$

Now, (4.2) and (4.3) give

$$\begin{aligned}
&\|g_{n+1} - \tilde{g_{n+1}}\| \\
&\quad = \|Sf_n - \tilde{S}\tilde{f_n}\| \\
&\quad \leq \delta\|f_n - \tilde{f_n}\| + \varphi\big(\|f_n - Sf_n\|\big) + \epsilon \\
&\quad \leq \delta\big(\delta\|e_n - \tilde{e_n}\| + \varphi\big(\|e_n - Se_n\|\big) + \epsilon\big) + \varphi\big(\|f_n - Sf_n\|\big) + \epsilon \\
&\quad = \delta^2\|e_n - \tilde{e_n}\| + \delta\varphi\big(\|e_n - Se_n\|\big) + \varphi\big(\|f_n - Sf_n\|\big) + (1+\delta)\epsilon \\
&\quad \leq \delta^2\big(\delta\big(1-\alpha_n(1-\delta)\big)\|g_n - \tilde{g_n}\| + \alpha_n\delta\varphi\big(\|g_n - Sg_n\|\big) \\
&\qquad + (1+\delta\alpha_n)\epsilon + \varphi\big(\big\|\big((1-\alpha_n)g_n + \alpha_n Sg_n\big) - S\big((1-\alpha_n)g_n + \alpha_n Sg_n\big)\big\|\big)\big) \\
&\qquad + \delta\varphi\big(\|e_n - Se_n\|\big) + \varphi\big(\|f_n - Sf_n\|\big) + (1+\delta)\epsilon \\
&\quad = \delta^3\big(1-\alpha_n(1-\delta)\big)\|g_n - \tilde{g_n}\| \\
&\qquad + \delta^2\varphi\big(\big\|\big((1-\alpha_n)g_n + \alpha_n Sg_n\big) - S\big((1-\alpha_n)g_n + \alpha_n Sg_n\big)\big\|\big)
\end{aligned} \tag{4.4}$$

$$+\alpha_n\delta^3\varphi\big(\|g_n - Sg_n\|\big) + \delta\varphi\big(\|e_n - Se_n\|\big) + \varphi\big(\|f_n - Sf_n\|\big) + \big(1 + \delta + \delta^2 + \delta^3\alpha_n\big)\epsilon.$$

Since S is a contractive-like operator with $q \in F(S)$, from Theorem 4.1 it follows that $\lim_{n\to\infty} \|g_n - q\| = 0$. Hence,

$$\begin{aligned} 0 &\le \|g_n - Sg_n\| \\ &\le \|g_n - q\| + \|Sq - Sg_n\| \\ &\le (1+\delta)\|g_n - q\| \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{4.5}$$

Also,

$$\begin{aligned} 0 &\le \|e_n - Se_n\| \\ &\le \|e_n - q\| + \|Sq - Se_n\| \\ &\le (1+\delta)\|e_n - q\| \\ &= (1+\delta)\big\|S\big((1-\alpha_n)g_n + Sg_n\big) - q\big\| \\ &\le (1+\delta)\delta\|g_n - q\| \to 0 \quad \text{as } n \to \infty \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} 0 &\le \|f_n - Sf_n\| \\ &\le (1+\delta)\|f_n - q\| \\ &= (1+\delta)\|Se_n - q\| \\ &\le (1+\delta)\delta^2\|g_n - q\| \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{4.7}$$

As φ is a continuous function (4.5), (4.6) and (4.7) yield

$$\lim_{n\to\infty}\varphi\big(\|g_n - Sg_n\|\big) = \lim_{n\to\infty}\varphi\big(\|e_n - Se_n\|\big) = \lim_{n\to\infty}\varphi\big(\|f_n - Sf_n\|\big) = 0. \tag{4.8}$$

On using (4.5) and (4.8), we obtain

$$\begin{aligned} 0 &\le \big\|\big((1-\alpha_n)g_n + \alpha_n Sg_n\big) - S\big((1-\alpha_n)g_n + \alpha_n Sg_n\big)\big\| \\ &\le (1-\alpha_n)\big\|g_n - S\big((1-\alpha_n)g_n + \alpha_n Sg_n\big)\big\| + {\alpha_n}^2\delta\|g_n - Sg_n\| + \alpha_n\varphi\big(\|g_n - Sg_n\|\big) \\ &\le (1-\alpha_n)\|g_n - q\| + \big\|Sq - S\big((1-\alpha_n)g_n + \alpha_n Sg_n\big)\big\| \\ &\quad + {\alpha_n}^2\delta\|g_n - Sg_n\| + \alpha_n\varphi\big(\|g_n - Sg_n\|\big) \\ &\le (1-\alpha_n)\|g_n - q\| + \delta\big\|q - \big((1-\alpha_n)g_n + \alpha_n Sg_n\big)\big\| + {\alpha_n}^2\delta\|g_n - Sg_n\| \\ &\quad + \alpha_n\varphi\big(\|g_n - Sg_n\|\big) \\ &\le (1-\alpha_n)\big(1 + \delta - \alpha_n\delta + \delta^2\alpha_n\big)\|g_n - q\| + {\alpha_n}^2\delta\|g_n - Sg_n\| + \alpha_n\varphi\big(\|g_n - Sg_n\|\big) \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Again, using the fact that φ is a continuous function, we have

$$\lim_{n\to\infty}\varphi\big(\big\|\big((1-\alpha_n)g_n + \alpha_n Sg_n\big) - S\big((1-\alpha_n)g_n + \alpha_n Sg_n\big)\big\|\big) = 0. \tag{4.9}$$

Now, using the fact that $\delta \in (0,1)$ and $\alpha_n \in (0,1)$ with $\alpha_n \geq \frac{1}{2}$, (4.4) transforms into

$$\begin{aligned} &\|g_{n+1} - \tilde{g_{n+1}}\| \\ &\quad \leq \bigl(1-\alpha_n(1-\delta)\bigr)\|g_n - \tilde{g}_n\| + \varphi\bigl(\bigl\|\bigl((1-\alpha_n)g_n + \alpha_n Sg_n\bigr) - S\bigl((1-\alpha_n)g_n + \alpha_n Sg_n\bigr)\bigr\|\bigr) \\ &\qquad + \alpha_n\varphi\bigl(\|g_n - Sg_n\|\bigr) + \varphi\bigl(\|e_n - Se_n\|\bigr) + \varphi\bigl(\|f_n - Sf_n\|\bigr) + \bigl(1+\delta+\delta^2+\delta^3\alpha_n\bigr)\epsilon \qquad (4.10) \\ &\quad \leq \bigl(1-\alpha_n(1-\delta)\bigr)\|g_n - \tilde{g}_n\| + 2\alpha_n\varphi\bigl(\bigl\|\bigl((1-\alpha_n)g_n + \alpha_n Sg_n\bigr) - S\bigl((1-\alpha_n)g_n + \alpha_n Sg_n\bigr)\bigr\|\bigr) \\ &\qquad + \alpha_n\varphi\bigl(\|g_n - Sg_n\|\bigr) + 2\alpha_n\varphi\bigl(\|e_n - Se_n\|\bigr) + 2\alpha_n\varphi\bigl(\|f_n - Sf_n\|\bigr) + 8\alpha_n\epsilon. \end{aligned}$$

Let $a_n = \|g_n - \tilde{g}_n\|$, $u_n = \alpha_n(1-\delta) \in (0,1)$ and

$$\begin{aligned} v_n = {}& \bigl(2\varphi\bigl(\bigl\|\bigl((1-\alpha_n)g_n + \alpha_n Sg_n\bigr) - S\bigl((1-\alpha_n)g_n + Sg_n\bigr)\bigr\|\bigr) + \varphi\bigl(\|g_n - Sg_n\|\bigr) \\ & + 2\varphi\bigl(\|e_n - Se_n\|\bigr) + 2\varphi\bigl(\|f_n - Sf_n\|\bigr) + 8\epsilon\bigr) \\ & /(1-\delta). \end{aligned}$$

On using (4.8) and (4.9) along with Lemma 2.5, we obtain

$$0 \leq \limsup_{n\to\infty} \|g_n - \tilde{g}_n\| \leq \limsup_{n\to\infty} v_n = \frac{8\epsilon}{1-\delta}$$

that along with Theorem 4.1 yields

$$\|q - \tilde{q}\| \leq \frac{8\epsilon}{1-\delta}.$$

Now, we present an example to validate Theorem 4.2 numerically. □

Example 2 Let $E = \mathbb{R}$ and $K = [0,6]$. Let $S : K \to K$ be a mapping defined as

$$S(g) = \begin{cases} \frac{g}{4}, & g \in [0,3), \\ \frac{g}{8}, & g \in [3,6]. \end{cases}$$

Proof Clearly $g = 0$ is the fixed point of S. First, we prove that S is a contractive-like mapping but not a contraction. Since S is not continuous at $g = 3 \in [0,6]$, S is not a contraction. We show that S is a contractive-like mapping. For this, define $\varphi : [0,\infty) \to [0,\infty)$ as $\varphi(g) = \frac{g}{6}$. Then, φ is a strictly increasing and continuous function. Also, $\varphi(0) = 0$.

We need to show that

$$\|Sg - Sf\| \leq \delta\|g - f\| + \varphi\bigl(\|g - Sg\|\bigr) \qquad (A)$$

for all $g,f \in [0,6]$ and δ is a constant in $[0,1)$.

Before going ahead, let us note the following. When $g \in [0,3)$,

$$\|g - Sg\| = \left\|g - \frac{g}{4}\right\| = \frac{3g}{4}$$

and

$$\varphi\left(\frac{3g}{4}\right) = \frac{3g}{24} = \frac{g}{8}. \tag{4.11}$$

Similarly, when $g \in [3, 6]$, then

$$\|g - Sg\| = \left\|g - \frac{g}{8}\right\| = \frac{7g}{8}$$

and

$$\varphi\left(\frac{7g}{8}\right) = \frac{7g}{48}. \tag{4.12}$$

Now consider the following cases:

Case A: Let $g, f \in [0, 3)$. Using (4.11) we obtain

$$\begin{aligned}
\|Sg - Sf\| &= \left\|\frac{g}{4} - \frac{f}{4}\right\| \\
&\leq \frac{1}{4}\|g - f\| \\
&\leq \frac{1}{4}\|g - f\| + \frac{g}{8} \\
&= \frac{1}{4}\|g - f\| + \varphi\left(\frac{3g}{4}\right) \\
&= \frac{1}{4}\|g - f\| + \varphi\big(\|g - Sg\|\big).
\end{aligned}$$

Hence, (A) is satisfied with $\delta = \frac{1}{4}$.

Case B: Let $g \in [0, 3)$ and $f \in [3, 6]$. Using (4.11), we obtain

$$\begin{aligned}
\|Sg - Sf\| &= \left\|\frac{g}{4} - \frac{f}{8}\right\| \\
&= \left\|\frac{g}{8} + \frac{g}{8} - \frac{f}{8}\right\| \\
&\leq \frac{1}{8}\|g - f\| + \left\|\frac{g}{8}\right\| \\
&\leq \frac{1}{4}\|g - f\| + \varphi\left(\frac{3g}{4}\right) \\
&= \frac{1}{4}\|g - f\| + \varphi\big(\|g - Sg\|\big).
\end{aligned}$$

Hence, (A) is satisfied with $\delta = \frac{1}{4}$.

Case C: Let $g \in [3, 6]$ and $f \in [0, 3)$. Using (4.12), we obtain

$$\begin{aligned}
\|Sg - Sf\| &= \left\|\frac{g}{8} - \frac{f}{4}\right\| \\
&= \left\|\frac{g}{4} - \frac{g}{8} - \frac{f}{4}\right\|
\end{aligned}$$

Table 2 Iteration values involving S and $\tilde{S}$

Step	Mapping S	Mapping $\tilde{S}$	Difference
1	5	5	0
2	0.0234375	0 0.02357	0.0001325
3	0.0001373291	0.00027065755	0.00013332845
4	7.6634543×10^{-7}	0.00013409965	0.00013333331
5	4.1161132×10^{-9}	0.00013333745	0.00013333333
6	$2.1438089 \times 10^{-11}$	0.00013333335	0.00013333333
7	1.088653×10^{-13}	0.00013333333	0.00013333333
8	$5.4123372 \times 10^{-16}$	0.00013333333	0.00013333333

$$\begin{aligned}
&\leq \frac{1}{4}\|g-f\| + \left\|\frac{g}{8}\right\| \\
&\leq \frac{1}{4}\|g-f\| + \left\|\frac{7g}{48}\right\| \\
&= \frac{1}{4}\|g-f\| + \varphi\big(\|g-Sg\|\big).
\end{aligned}$$

Hence, (A) is satisfied with $\delta = \frac{1}{4}$.

Case D: Let $g,f \in [3,6]$. Using (4.12), we obtain

$$\begin{aligned}
\|Sg-Sf\| &= \left\|\frac{g}{8} - \frac{f}{8}\right\| \\
&\leq \frac{1}{8}\|g-f\| + \left\|\frac{7g}{48}\right\| \\
&\leq \frac{1}{4}\|g-f\| + \left\|\frac{7g}{48}\right\| \\
&= \frac{1}{4}\|g-f\| + \varphi\big(\|g-Sg\|\big).
\end{aligned}$$

Hence, (A) is satisfied with $\delta = \frac{1}{4}$.

Consequently, (A) is satisfied for $\delta = \frac{1}{4}$ and $\varphi(g) = \frac{g}{6}$ in all the possible cases. Thus, S is a contractive-like mapping.

Next, we define another operator $\tilde{S} : K \to K$ as

$$\tilde{S}(g) = \begin{cases} \frac{g}{4} + \frac{1}{10{,}000}, & g \in [0,3), \\ \frac{g}{8} + \frac{1}{10{,}000}, & g \in [3,6], \end{cases}$$

for all $g \in K$. Then, it is easy to see that $\tilde{S}$ is an approximate operator for S with $\epsilon = 0.0001$ as $\|Sg - \tilde{S}g\| \leq 0.0001$ for all $g \in K$. Also, $q = 0$ is the fixed point of S and $\tilde{q} = 0.00013333333$ is the fixed point of $\tilde{g}$. We obtain Table 2 of iteration values with an initial approximation of 5 and $\alpha_n = \frac{n+3}{n+4}$ for all $n \in \mathbb{N}$. Also, we have $\|q - \tilde{q}\| = \|0 - 0.00013333333\| = 0.00013333333 \leq 0.0010666667 = \frac{8\epsilon}{1-\delta}$. Hence, we can say that in a situation when it is difficult to calculate the fixed point of a mapping S we can choose a mapping $\tilde{S}$ closer to S and the distance between the two fixed points will reduce too. □

5 Application

Fractional differential equations have applications in many fields of engineering and science including diffusive transport, electrical networks, fluid flow, probability and electromagnetic theory. Numerous fixed-point results are available for finding solutions to differential/integral equations involving fractional operators (see [35–43]). This section is devoted to the study of a solution of a nonlinear fractional differential equation with the help of iteration process (1.1).

Here, we consider the following fractional differential equation:

$$\begin{cases} D^{\lambda}g(u) + f(u,g(u)) = 0 \quad (0 \leq u \leq 1, 1 < \lambda < 2), \\ g(0) = g(1) = 0, \end{cases} \tag{5.1}$$

where D^{λ} is the Caputo fractional derivative of order λ and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Let $E = C[0,1]$ be a Banach space of a continuous function endowed with the maximum norm and the Green's function associated to (5.1) is defined as:

$$G(u,v) = \begin{cases} \frac{1}{\Gamma(\lambda)}(u(1-v)^{\lambda-1} - (u-v)^{\lambda-1}), & 0 \leq v \leq u \leq 1, \\ \frac{u(1-v)^{\lambda-1}}{\Gamma(\lambda)}, & 0 \leq u \leq v \leq 1. \end{cases}$$

Assume that

$$\left|f(u,a) - f(u,b)\right| \leq |a-b|, \tag{5.2}$$

for all $u \in [0,1]$ and $a, b \in \mathbb{R}$.

Theorem 5.1 *Let $E = C[0,1]$ and the operator $S : E \to E$ be defined as*

$$S\bigl(g(u)\bigr) = \int_0^1 G(u,v) f\bigl(v, g(v)\bigr)\, dv,$$

for all $u \in [0,1]$ and $g \in E$. If the condition (5.2) is satisfied then the iteration process (1.1) converges to the solution of (5.1).

Proof It is easy to see that $g \in E$ is a solution of (5.1) if and only if g is a solution of the following integral equation

$$g(u) = \int_0^1 G(u,v) f\bigl(v, g(v)\bigr)\, dv.$$

Let $g, h \in E$ and $u \in [0,1]$. Then,

$$\begin{aligned} \left|S\bigl(g(u)\bigr) - S\bigl(h(u)\bigr)\right| &\leq \int_0^1 G(u,v) \left|f\bigl(v,g(v)\bigr) - f\bigl(v,h(v)\bigr)\right| dv \\ &\leq \int_0^1 G(u,v) \left|g(v) - h(v)\right| dv \\ &\leq \|g-h\| \left(\int_0^1 G(u,v)\, dv \right) \\ &\leq \|g-h\|, \end{aligned}$$

which yields

$$\|Sg - Sh\| \leq \|g - h\|,$$

for all $g, h \in E$ and for all $u \in [0,1]$. With the help of Lemma 2.2(i) and Theorem 3.2, we can say that process (1.1) converges to the solution of (5.1). □

6 Conclusion

A new iteration process (1.1) is obtained for approximating fixed points of Suzuki generalized nonexpansive mappings. It is proved that it has a higher rate of convergence than the M^*-iteration process for contractive-like mappings. Strong and weak convergence to the fixed point of Suzuki generalized nonexpansive mappings in the setting of uniformly convex Banach spaces are proved. The results have been supported by a newly introduced Example 1. We then presented a data-dependence result that is again followed by a numerical example. We ended by providing an application to nonlinear fractional differential equations in the framework of Caputo with a power-law singular kernel. The Riemann–Liouville integral operators used in obtaining the solution representation have semigroup properties that may make them more appropriate when we apply iterative techniques. Also, there are no restrictions on the right-hand side of the considered initial-value problem, as in the case of fractional operators with nonsingular kernels [44–46]. Our new iteration process can be used by engineers, computer scientists, physicists as well as mathematicians to solve different problems more efficiently and effectively.

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