Wen and Xi *Advances in Continuous and Discrete Models* (2022) 2022:33
https://doi.org/10.1186/s13662-022-03706-8

Advances in Continuous and Discrete Models

RESEARCH Open Access

Complete controllability of nonlinear fractional neutral functional differential equations

Yanhua Wen[1*] and Xuan-Xuan Xi[2]

*Correspondence: wenyanhua0405@163.com
[1]School of Mathematical and Physics, Anhui Jianzhu University, Heifei, China
Full list of author information is available at the end of the article

Abstract

This paper is concerned with the complete controllability of a nonlinear fractional neutral functional differential equation. Some sufficient conditions are established for the complete controllability of the nonlinear fractional system. The conditions are established based on the fractional power of operators and the fixed-point theorem under the assumption that the associated linear system is completely controllable. Finally, an example is presented to illustrate our main result.

Keywords: Complete controllability; Fractional nonlinear neutral functional differential equation; Banach fixed-point theorem; Mild solution

1 Introduction

In this paper, we assume that X is a Hilbert space with the norm $|\cdot|$. Let $r > 0$ and C be the Banach space of all continuous functions from an interval $[-r, 0]$ into X with the norm $\|x\| = \sup_{t\in[-r,0]} |x(t)|$. The purpose of this paper is to study the complete controllability for the following nonlinear fractional neutral functional differential system

$$\begin{cases} {}^C D_t^q [x(t) - h(t, x_t)] = Ax(t) + Bu(t) + f(t, x_t, u(t)), & t \in I = [0, T], \\ x_0(\theta) = \phi(\theta), & -r \le \theta \le 0, \end{cases} \tag{1}$$

where ${}^C D_t^q$ is the Caputo fractional derivative of order $0 < q < 1$, the state variable $x(\cdot)$ takes values in the Hilbert space X, $h : I \times C \to X$ is a given function, $A : D(A) \subset X \to X$ is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\ge 0}$. $B : U \to X$ is a bounded linear operator, U is a Hilbert space, the control function $u(\cdot) \in L^2(I, U)$ and $f : I \times C \times U \to X$ is a given function satisfying some assumptions. If $x : [-r, T] \to X$ is a continuous function, then x_t is an element in C defined by $x_t(\theta) = x(t + \theta)$, $\theta \in [-r, 0]$ and $\phi \in C$.

In mathematical control theory, the controllability is one of the important concepts that has been studied by many authors (see [1–7] and the references therein). By means of semigroup theory and the fixed-point approach, various types of controllability problems have been investigated, for instance, approximate controllability [8–15] and complete controllability [16–25]. There are several papers devoted to the approximate or complete controllability for fractional differential systems, when the nonlinear term is independent of the control function. Wang and Zhou [22] studied the complete controllability of fractional evolution systems in infinite-dimensional spaces by using fractional calculus, the properties of characteristic solution operators and fixed-point theorems. Meanwhile, Sakthivel et al. [23] established a new set of sufficient conditions for the complete controllability of a fractional nonlinear neutral functional differential equation. Du and Jiang [24] investigated the approximate controllability of impulsive Hilfer fractional differential inclusions.

In [25, 26], the approximate controllability of first-order delay control systems has been proved when the nonlinear term is a function of both the state function and the control function by assuming that the corresponding linear system is approximately controllable. As far as we know, the fractional functional differential systems have been proved to be abstract formulations in many problems arising in engineering, physics, automatic control, etc. The delay differential equations had shown their efficiency in the study of the behavior of real populations. Thus, the study of controllability for such systems is important for many applications. In 2011, Sukavanam [27] investigated the approximate controllability of the following fractional-order semilinear delay system in which the nonlinear term is a function of both the state function and the control function.

$$\begin{cases} {}^C D_t^q x(t) = Ax(t) + Bu(t) + f(t, x_t, u(t)), & t \in I = [0, T], \\ x_0(\theta) = \phi(\theta), & -r \le \theta \le 0, \end{cases} \tag{2}$$

where $\frac{1}{2} < q < 1$, the state $x(\cdot)$ takes values in the Banach space X, the control function $u(\cdot)$ takes values in the Banach space Y, $A : D(A) \subset X \to X$ is a closed linear operator with dense domain $D(A)$ generating a C_0-semigroup $S(t)$; B is a bounded linear operator from $L^2([0, T]; Y)$ to $L^2([0, T]; X)$; the operator $f : [0, T] \times C([-r, 0]; X) \times Y \to X$ is nonlinear. If $x : [-r, T] \to X$ is a continuous function, then $x_t : [-r, 0] \to X$ is defined as $x_t(\theta) = x(t + \theta)$, $\theta \in [-r, 0]$ and $\phi \in C([-r, 0]; X)$.

Motivated by the above work, we consider the functional differential equations in which the delay terms also occur in the derivative of the unknown solution. That is, we study the complete controllability of the nonlinear fractional neutral functional differential system (1) when the nonlinear term is a function of both the state function and the control function.

In the past decades, with the development of theories of fractional differential equations, there has been a great deal of interest in the study of solutions of fractional differential systems; see Byszewski [28], Podlubny [29], Kilbas et al. [30] and Lakshmikantham et al. [31]. Moreover, there are different types of mild solutions that have been investigated; see Byszewski and Lakshmikantham [32], Pazy [33], Zhou and Jiao [34, 35] and Wang and Zhou [36, 37]. In particular, Zhou and Jiao [34] obtained the existence and uniqueness of mild solutions for fractional neutral evolution equations by using the fractional power of operators and some fixed-point theorems. In addition, Wang and Zhou [36] introduced a new mild solution for semilinear fractional evolution equations and the existence and uniqueness of α-mild solutions are proved.

In order to derive the complete controllability of the nonlinear fractional neutral functional differential system (1), in this paper, we first give the concept of mild solutions of the system (1) in the light of [34, 36]. Then, we establish sufficient conditions for the complete controllability of the nonlinear fractional neutral functional differential system (1) in which the nonlinear term depends on the control function. To obtain the complete controllability of the system (1), we impose some necessary hypotheses on A, B, h, f and the assumption that the corresponding linear system of the system (1) is completely controllable.

The rest of our paper is organized as follows. Section 2 is devoted to some necessary preliminaries. In Sect. 3, the complete controllability for the system (1) is given. Finally, an example is presented to demonstrate our complete controllability result.

2 Preliminaries

In this section, we introduce some notations, definitions and lemmas that will be used throughout the paper.

Definition 2.1 ([29]) The fractional integral of order α with the lower limit 0 for a function f is defined as

$$I_0^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}}\, ds, \quad t > 0, \alpha > 0, \tag{3}$$

provided the right-hand side is pointwise defined on $[0, \infty)$, where Γ is the gamma function.

Definition 2.2 ([29]) The Caputo derivative of order α with the lower limit 0 for a function f can be written as

$${}^C D_0^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}}\, ds = I^{n-\alpha} f^{(n)}(t), \quad t > 0, 0 \le n-1 < \alpha < n. \tag{4}$$

Remark 2.1 If f is an abstract function with values in X, then integrals that appear in Definitions 2.1 and 2.2 are taken in Bochner's sense.

Throughout this paper, we assume that A is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\ge 0}$ of bounded operators on X. Let $0 \in \rho(A)$, where $\rho(A)$ is the resolvent set of A, then for $\eta \in (0,1]$, we define the fractional power A^η as a closed linear operator on its domain $D(A^\eta)$. Moreover, $T(t)$ and A^η have the following basic properties.

(i) There is a $M \ge 1$ such that

$$M = \sup_{t\in[0,T]} \| T(t) \| < \infty. \tag{5}$$

(ii) For any $\eta \in (0,1]$, there exists a constant $c_\eta > 0$ such that

$$\| A^\eta T(t) \| \le \frac{c_\eta}{t^\eta}, \quad 0 < t \le T. \tag{6}$$

For more details, see [33].

Lemma 2.1 ([33]) *There exists a constant* C *such that*

$$\| A^{-\alpha} \| \le C \quad \textit{for}\ 0 \le \alpha \le 1, \tag{7}$$

where $A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} T(t)\, dt$.

By comparison with the fractional differential equation given in [34], we give the following definition of the mild solution of the system (1).

Definition 2.3 A function $x \in C([-r,T];X)$ is called a mild solution of the system (1) if on $[-r,T]$ it satisfies

$$\begin{cases} x(t) = S_q(t)[\phi(0) - h(0,x_0)] + h(t,x_t) + \int_0^t (t-s)^{q-1} AT_q(t-s)h(s,x_s)\,ds \\ \qquad + \int_0^t (t-s)^{q-1} T_q(t-s)(Bu(s) + f(s,x_s,u(s)))\,ds, \quad t \in [0,T], \\ x_0(\theta) = \phi(\theta), \quad -r \le \theta \le 0, \end{cases} \tag{8}$$

where $S_q(t) = \int_0^\infty \phi_q(\theta) T(t^q\theta)\,d\theta$, $T_q(t) = q\int_0^\infty \theta\phi_q(\theta)T(t^q\theta)\,d\theta$ and for $\theta \in (0,\infty)$

$$\phi_q(\theta) = \frac{1}{q}\theta^{-1-\frac{1}{q}}\psi_q\bigl(\theta^{-\frac{1}{q}}\bigr), \qquad \psi_q(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-qn-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q).$$

In addition, $\phi_q(\theta)$ is the probability density function defined as

$$\phi_q(\theta) \ge 0, \quad \theta \in (0,\infty) \quad \text{and} \quad \int_0^\infty \phi_q(\theta)\,d\theta = 1.$$

Lemma 2.2 ([4, 34]) *The operators $S_q(t)$ and $T_q(t)$ have the following properties*:

(i) *For any $t \ge 0$, the operators $S_q(t)$ and $T_q(t)$ are linear and bounded operators, that is, for any $x \in X$,*

$$\|S_q(t)x\| \le M\|x\| \quad \textit{and} \quad \|T_q(t)x\| \le \frac{Mq}{\Gamma(1+q)}\|x\|.$$

(ii) *$\{S_q(t)\}_{t\ge0}$ and $\{T_q(t)\}_{t\ge0}$ are strongly continuous.*

Lemma 2.3 ([34]) *For any $x \in E$, E is a Banach space, $\beta \in (0,1)$ and $\eta \in (0,1]$, we have*

$$AT_q(t)x = A^{1-\beta}T_q(t)A^\beta x, \quad 0 \le t \le T \tag{9}$$

and

$$\|A^\eta T_q(t)\| \le \frac{qc_\eta}{t^{q\eta}}\frac{\Gamma(2-\eta)}{\Gamma(1+q(1-\eta))}, \quad 0 < t \le T. \tag{10}$$

Consider the following linear fractional differential system

$$\begin{cases} {}^CD_t^q x(t) = Ax(t) + Bu(t), \quad t \in [0,T], \\ x(0) = \phi(0) \end{cases} \tag{11}$$

and it is convenient to introduce the controllability operator associated with (11) as

$$\Gamma_0^T = \int_0^T (T-s)^{q-1}T_q(T-s)BB^*T_q^*(T-s)\,ds, \tag{12}$$

where B^*, $T_q^*(t)$ denote the adjoints of B and $T_q(t)$, respectively.

By [23, 38], the definition of complete controllability for the linear fractional differential system (11) is as follows.

Lemma 2.4 ([23, 38]) *The linear fractional control system (11) is completely controllable on I if and only if there exists a $\gamma > 0$ such that*

$$\langle \Gamma_0^T x, x\rangle \geq \gamma \|x\|^2 \quad \textit{in the Hilbert space } X, \textit{i.e.}, \left\|\left(\Gamma_0^T\right)^{-1}\right\| \leq \frac{1}{\gamma}.$$

3 Complete controllability

We present our main results of the paper in this section. We need the definition of complete controllability of the system (1).

Definition 3.1 The system (1) is said to be completely controllable on the interval I if $\mathcal{R}(T,\phi) = X$, where $\mathcal{R}(T,\phi) = \{x_T(\phi,u)(0) : u(\cdot) \in L^2(I,U)\}$.

To prove the main results, we impose the following hypotheses:

(P_1) The function $h : [0,T] \times C \to X$ is continuous and there exists a constant $\beta \in (0,1)$ and L, L_1, for any $x, y \in C$, $A^\beta h(\cdot,x)$ is strongly measurable and $A^\beta h(t,\cdot)$ satisfies the Lipchitz condition $\|A^\beta h(t,x) - A^\beta h(t,y)\| \leq L\|x - y\|$ and the inequality $\|A^\beta h(t,x)\| \leq L_1(\|x\| + 1)$.

(P_2) The nonlinear function $f : I \times C \times U \to X$ is continuous and there exists a constant $L_2 > 0$ such that

$$\left\|f(t,\varphi,u)\right\| \leq L_2\left(1 + \|\varphi\|_C + \|u\|\right), \quad (t,\varphi,u) \in I \times C \times U.$$

(P_3) The linear fractional control system (11) is completely controllable.

(P_4) The nonlinear function $f(t,x_t,u(t))$ satisfies the Lipschitz condition, that is, there exists a constant L_3 such that

$$\left\|f(t,\varphi_1,u_1) - f(t,\varphi_2,u_2)\right\| \leq L_3\left(\|\varphi_1 - \varphi_2\|_C + \|u_1 - u_2\|\right), \quad (\varphi_1,u_1),(\varphi_2,u_2) \in C \times U.$$

Define an operator Φ on $C(I,C) \times C(I,U)$ as

$$\Phi(x,u) = (z,v) \tag{13}$$

with the norm $\|(x,u)\| = \|x_t\|_C + \|u\|$, $(x,u) \in C(I,C) \times C(I,U)$, $t \in I$, where

$$v(t) = B^* T_q^*(T-t)\left(\Gamma_0^T\right)^{-1} p(x,u), \tag{14}$$

$$\begin{aligned} z(t) &= S_q(t)\left[\phi(0) - h(0,x_0)\right] + h(t,x_t) + \int_0^t (t-s)^{q-1} A T_q(t-s) h(s,x_s)\,ds \\ &\quad + \int_0^t (t-s)^{q-1} T_q(t-s)\left(Bv(s) + f\left(s,x_s,u(s)\right)\right) ds, \quad t \in [0,T], \end{aligned} \tag{15}$$

$$z_0(\theta) = \phi(\theta), \quad -r \leq \theta \leq 0,$$

$p(x,u) = x_T - S_q(T)[\phi(0) - h(0,x_0)] - h(T,x_T) - \int_0^T (T-s)^{q-1} A T_q(T-s) h(s,x_s)\,ds - \int_0^T (T-s)^{q-1} T_q(T-s) f(s,x_s,u(s))\,ds$.

It will be shown that the system (1) is completely controllable on I if the operator Φ has a fixed point in $C(I,C) \times C(I,U)$.

Theorem 3.1 *Assume that the hypotheses* (P_1)–(P_4) *are satisfied. Then, the problem* (1) *has a unique mild solution in* $C([-r,T];X)$ *provided that*

$$|A^{-\beta}|L + \left(1 + \frac{MM_BT^q}{\Gamma(1+q)}\right)\left(\frac{d\Gamma(1+\beta)c_{1-\beta}LT^{q\beta}}{\beta\Gamma(1+q\beta)} + \frac{dMT^qL_3}{\Gamma(1+q)}\right) + \frac{MT^qL_3}{\Gamma(1+q)} < 1, \tag{16}$$

where $\beta \in (0,1)$, $M_B = |B|$, $d = \frac{M_BMq}{\gamma\Gamma(1+q)}$.

Proof Obviously, $x \in C([-r,T];X)$ is a mild solution of the system (1) if and only if the operator Φ has a fixed point in $C(I,C) \times C(I,U)$. Therefore, it is sufficient to prove that Φ has a fixed point in $C(I,C) \times C(I,U)$. We first show that Φ maps $C(I,C) \times C(I,U)$ into itself. Based on Lemma 2.3 and the condition (P_1), we have

$$\begin{aligned}
&\left\| \int_0^t (t-s)^{q-1} AT_q(t-s)h(s,x_s)\,ds \right\| \\
&\quad\le \int_0^t (t-s)^{q-1} A^{1-\beta}T_q(t-s)A^{\beta}h(s,x_s)\,ds \\
&\quad\le \int_0^t (t-s)^{q-1} \frac{q\Gamma(1+\beta)c_{1-\beta}}{(t-s)^{q(1-\beta)}\Gamma(1+q\beta)} L_1\big(\|x_s\|_C + 1\big)\,ds \\
&\quad\le \frac{\Gamma(1+\beta)c_{1-\beta}}{\beta\Gamma(1+q\beta)} L_1\big(\|x_t\|_C + 1\big)T^{q\beta}, \quad \beta \in (0,1).
\end{aligned} \tag{17}$$

According to Lemma 2.2(i) and the hypothesis (P_2), we have

$$\left\| \int_0^T (T-s)^{q-1} T_q(T-s)f\big(s,x_s,u(s)\big)\,ds \right\| \le \frac{MT^q}{\Gamma(1+q)} L_2\big(1 + \|x_t\|_C + \|u\|\big). \tag{18}$$

By using (17) and (18), Lemmas 2.1 and 2.4, and hypothesis (P_3), it can be shown that there exist two constants $C_1, C_2 > 0$ such that

$$\begin{aligned}
\|v(t)\| &= \bigg\| B^*T_q^*(T-t)\big(\Gamma_0^T\big)^{-1}\bigg(x_T - S_q(T)\big[\phi(0) - h(0,x_0)\big] - h(T,x_T) \\
&\qquad - \int_0^T (T-s)^{q-1} AT_q(T-s)h(s,x_s)\,ds \\
&\qquad - \int_0^T (T-s)^{q-1} T_q(T-s)f\big(s,x_s,u(s)\big)\,ds \bigg) \bigg\| \\
&\le \frac{M_BMq}{\gamma\Gamma(1+q)} \bigg[|x_T| + M\|\phi\| + M|A^{-\beta}|L_1\big(\|\phi\| + 1\big) + |A^{-\beta}|L_1\big(\|x_T\| + 1\big) \\
&\qquad + \frac{\Gamma(1+\beta)c_{1-\beta}}{\beta\Gamma(1+q\beta)} L_1\big(\|x_t\|_C + 1\big)T^{q\beta} + \frac{MT^q}{\Gamma(1+q)} L_2\big(1 + \|x_t\|_C + \|u\|\big) \bigg] \\
&\le C_1\big(1 + \|x_t\|_C + \|u\|\big)
\end{aligned} \tag{19}$$

and

$$\begin{aligned}\|z(t)\| &= \left\| S_q(t)\big[\phi(0)-h(0,x_0)\big] + h(t,x_t) + \int_0^t (t-s)^{q-1} A T_q(t-s) h(s,x_s)\,ds \right.\\ &\quad \left. + \int_0^t (t-s)^{q-1} T_q(t-s)\big(Bv(s) + f\big(s,x_s,u(s)\big)\big)\,ds \right\| \\ &\le M\|\phi\| + M\big|A^{-\beta}\big|L_1\big(\|\phi\|+1\big) + \big|A^{-\beta}\big|L_1\big(\|x_t\|_C+1\big) \\ &\quad + \frac{\Gamma(1+\beta)c_{1-\beta}}{\beta\Gamma(1+q\beta)} L_1\big(\|x_t\|_C+1\big)T^{q\beta} \\ &\quad + \frac{MM_BT^q}{\Gamma(1+q)} C_1\big(1+\|x_t\|_C+\|u\|\big) + \frac{MT^q}{\Gamma(1+q)} L_2\big(1+\|x_t\|_C+\|u\|\big) \\ &\le C_2\big(1+\|x_t\|_C+\|u\|\big),\end{aligned} \tag{20}$$

where $\beta \in (0,1)$. It follows from (15), (19) and (20) that there exists a constant C_3 such that

$$\big\|\Phi(x,u)\big\| = \|z\|_{C([-r,T];X)} + \|v\| \le C_3\big(1+\|x_t\|_C+\|u\|\big), \tag{21}$$

which means that Φ maps $C(I,C)\times C(I,U)$ into itself.

We next prove that the operator Φ is a contraction mapping on $C(I,C)\times C(I,U)$. For any $(x,u),(y,w)\in C(I,C)\times C(I,U)$, it holds that

$$\begin{aligned}&\big\|\Phi(x,u)-\Phi(y,w)\big\| \\ &\quad = \|v_1-v_2\| + \|z_1-z_2\|_{C([-r,T];X)} \\ &\quad \le \|v_1-v_2\| + \big\|h(t,x_t)-h(t,y_t)\big\| + \left\| \int_0^t (t-s)^{q-1} T_q(t-s) B\big(v_1(s)-v_2(s)\big)\,ds \right\| \\ &\qquad + \left\| \int_0^t (t-s)^{q-1} T_q(t-s)\big(f\big(s,x_s,u(s)-f\big(s,y_s,w(s)\big)\big)\,ds \right\| \\ &\quad = I_1+I_2+I_3+I_4.\end{aligned} \tag{22}$$

By hypotheses (P_1)–(P_4), Lemma 2.2(i), and (17) and (18), we have

$$\begin{aligned}I_1 &= \|v_1-v_2\| \\ &= \left\| B^* T_q^*(T-t)\big(\Gamma_0^T\big)^{-1}\left(\int_0^T (T-s)^{q-1} A T_q(T-s)\big(h(s,x_s)-h(s,y_s)\big)\,ds \right.\right.\\ &\quad \left.\left. + \int_0^T (T-s)^{q-1} T_q(T-s)\big(f\big(s,x_s,u(s)-f\big(s,y_s,w(s)\big)\big)\,ds\right)\right\| \\ &\le \frac{M_BMq}{\gamma\Gamma(1+q)}\left(\frac{\Gamma(1+\beta)c_{1-\beta}LT^{q\beta}}{\beta\Gamma(1+q\beta)}\|x_t-y_t\|_C + \frac{MT^qL_3}{\Gamma(1+q)}\big(\|x_t-y_t\|_C+\|u-w\|\big)\right) \\ &\le \left(\frac{d\Gamma(1+\beta)c_{1-\beta}LT^{q\beta}}{\beta\Gamma(1+q\beta)} + \frac{dMT^qL_3}{\Gamma(1+q)}\right)\big(\|x-y\|_{C([-r,T];X)}+\|u-w\|\big),\end{aligned} \tag{23}$$

where $d=\frac{M_BMq}{\gamma\Gamma(1+q)}$. The condition (P_1) implies

$$I_2 = \big\|h(t,x_t)-h(t,y_t)\big\| \le \big|A^{-\beta}\big|L\|x-y\|_{C([-r,T];X)}. \tag{24}$$

Based on Lemma 2.2(i) and (23), one can obtain

$$\begin{aligned} I_3 &= \left\| \int_0^t (t-s)^{q-1} T_q(t-s) B\big(v_1(s) - v_2(s)\big)\, ds \right\| \\ &\le \frac{MM_B T^q}{\Gamma(1+q)} \|v_1 - v_2\| \\ &\le \frac{MM_B T^q}{\Gamma(1+q)} \left(\frac{d\Gamma(1+\beta) c_{1-\beta} L T^{q\beta}}{\beta \Gamma(1+q\beta)} + \frac{dMT^q L_3}{\Gamma(1+q)} \right) \big(\|x-y\|_{C([-r,T];X)} + \|u-w\| \big). \end{aligned} \tag{25}$$

Similar to the discussion of I_1, we obtain

$$\begin{aligned} I_4 &= \left\| \int_0^t (t-s)^{q-1} T_q(t-s) \big(f\big(s, x_s, u(s) - f\big(s, y_s, w(s)\big)\big)\big)\, ds \right\| \\ &\le \frac{MT^q L_3}{\Gamma(1+q)} \big(\|x-y\|_{C([-r,T];X)} + \|u-w\| \big). \end{aligned} \tag{26}$$

Then, (22)–(26) imply

$$\begin{aligned} &\big\| \Phi(x,u) - \Phi(y,w) \big\|_{C([-r,T];X)} \\ &\quad \le \left[\big|A^{-\beta}\big| L + \left(1 + \frac{MM_B T^q}{\Gamma(1+q)}\right) \left(\frac{d\Gamma(1+\beta) c_{1-\beta} L T^{q\beta}}{\beta\Gamma(1+q\beta)} \right) \right. \\ &\qquad \left. + \frac{MT^q L_3}{\Gamma(1+q)} \right] \big(\|x-y\|_{C([-r,T];X)} + \|u-w\| \big). \end{aligned} \tag{27}$$

In view of (16), we obtain that Φ is a contraction. Consequently, Φ has a fixed point in $C(I,C) \times C(I,U)$ by the Banach fixed-point theorem, which is a mild solution of the system (1). This completes the proof. □

Theorem 3.2 *If all the assumptions of Theorem 3.1 hold, then the system (1) is completely controllable on I.*

Proof Let $(\bar{x}(\cdot), \bar{u})$ be a fixed point of the operator Φ in (13), that is

$$\Phi\big(\bar{x}(\cdot), \bar{u}\big) = \big(\bar{x}(\cdot), \bar{u}\big), \tag{28}$$

where

$$\begin{aligned} \bar{x}(t) &= S_q(t)\big[\phi(0) - h(0, x_0)\big] + h(t, \bar{x}_t) + \int_0^t (t-s)^{q-1} A T_q(t-s) h(s, \bar{x}_s)\, ds \\ &\quad + \int_0^t (t-s)^{q-1} T_q(t-s) \big(B\bar{u}(s) + f\big(s, \bar{x}_s, \bar{u}(s)\big)\big)\, ds, \quad t \in [0,T], \\ \bar{x}_0(\theta) &= \phi(\theta), \quad -r \le \theta \le 0 \end{aligned} \tag{29}$$

and the control function

$$\bar{u}(t) = B^* T_q^*(T-t) \big(\Gamma_0^T\big)^{-1} p(\bar{x}, \bar{u}), \tag{30}$$

here $p(\bar{x},\bar{u}) = x_T - S_q(T)[\phi(0) - h(0,x_0)] + h(T,x_T) + \int_0^T (T-s)^{q-1} A T_q(T-s) h(s,\bar{x}_s)\,ds - \int_0^T (T-s)^{q-1} T_q(T-s) f(s,\bar{x}_s,\bar{u}(s))\,ds$.

According to Theorem 3.1, any fixed point of Φ is a mild solution of the system (1). Then, by (12), (29) and (30), we have

$$\begin{aligned} \bar{x}(T) &= x_T - p(\bar{x},\bar{u}) + \int_0^T (T-s)^{q-1} T_q(T-s) B\bar{u}(s)\,ds \\ &= x_T - p(\bar{x},\bar{u}) + \int_0^T (T-s)^{q-1} T_q(T-s) BB^* T_q^*(T-s) \left(\Gamma_0^T\right)^{-1} p(\bar{x},\bar{u})\,ds \\ &= x_T - p(\bar{x},\bar{u}) + p(\bar{x},\bar{u}) \\ &= x_T. \end{aligned} \tag{31}$$

Thus, the system (1) is approximately controllable on I by Definition 3.1. The proof is completed. □

4 Example

As an application of our complete controllability result, we consider the following fractional partial differential equation

$$\begin{cases} \partial_t^q [x(t,z) - \int_0^\pi g(z,y) x_t(\theta,y)\,dy] = \partial_z^2 x(t,z) + \mu(t,z) + f(t, x_t(\theta,z), u(t)), \\ t \in I := [0,1], \quad z \in [0,\pi], \\ x(t,0) = x(t,\pi) = 0, \quad 0 < t \leq 1, \\ x(\theta,z) = \phi(\theta,z), \quad -r \leq \theta \leq 0, \end{cases} \tag{32}$$

where ∂_t^q is the Caputo fractional partial derivative of order $0 < q < 1$, g, f are given continuous functions, $\mu : I \times [0,\pi] \to [0,\pi]$ is continuous in t, $x_t(\theta,z) = x(t+\theta,z)$ and $\phi(\theta,z)$ is continuous.

Taking $X = U = L^2[0,\pi]$. Let $A : D(A) \subset X \to X$ be an operator defined by $Aw = w''$ with the domain

$$\begin{aligned} D(A) = \{ & w \in X \mid w(\cdot) \in L^2[0,\pi], w, w' \text{ are absolutely continuous}, \\ & w'' \in X, w(0) = w(\pi) = 0 \}. \end{aligned}$$

Then,

$$Aw = -\sum_{n=1}^\infty n^2 (w,e_n) e_n, \quad z \in D(A),$$

where $e_n(z) = \sqrt{\frac{2}{\pi}} \sin(nz)$, $0 \leq z \leq \pi$, $n = 1,2,\ldots$. It is known that $\{e_n\}$, $n = 1,2,\ldots$ is an orthonormal base for U and A generates a compact semigroup $T(t)$, $t > 0$ in X that is given by

$$T(t)w = \sum_{n=1}^\infty e^{-n^2 t} (w,e_n) e_n, \quad w \in X.$$

For more details, please refer to [33].

Put $x_t = x_t(\theta, \cdot)$, i.e., $(x(t+\theta))(z) = x(t+\theta, z)$, $\theta \in [-r, 0]$. Define the functions $h : [0,1] \times C \to X$ by $(h(t,x_t))(z) = \int_0^\pi g(z,y)x_t(\theta,y)\,dy$ and $f : [0,1] \times C \times U \to X$ as $f(t,x_t,u)(z) = f(t,x_t(\theta,z),u(t))$. Moreover, define $B : U \to X$ by $(Bu(t))(z) = \mu(t,z)$, $z \in [0,\pi]$. Let us take $f(t,x_t,u) = \|x_t\|_C e_3 + \|u\| e_4$, then the conditions (P_2) and (P_4) are satisfied.

Consequently, the system (32) can be written in the abstract form (1) with the appropriate choices of A, B, h and f, and its associated linear system

$$\begin{cases} {}^C D_t^q x(t) = Ax(t) + Bu(t), & t \in [0,T], \\ x(0) = \phi(0) \end{cases} \tag{33}$$

is completely controllable according to Lemma 2.4, which means that the condition (P_4) is satisfied. Meanwhile, the inequality (3.4) is also satisfied with the appropriate choices of A, B, h and f. Therefore, all the conditions of Theorem 3.2 are satisfied. Hence, the system (32) is completely controllable on $[0,1]$.

Acknowledgements
The authors would like to thank the reviewers very much for their valuable comments for improvement of the paper. We would also like to acknowledge the valuable comments and suggestions from the editors, which contributed to an improvement in the presentation of this paper.

Funding
This work is supported by the National Natural Science Foundation of China (11471015 and 11601003); the Natural Science Foundation of Anhui Province (1708085MA15 and 2008085QA12); the Natural Science Fund of Colleges and Universities in Anhui Province (KJ2021A0967) and the Scientific Research Project of Anhui Jianzhu University (2019QDZ37).

Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this article. They read and approved the final manuscript.

Author details
[1]School of Mathematical and Physics, Anhui Jianzhu University, Heifei, China. [2]Faculty of Mathematics and Computational Science, Xiangtan University, Hunan, China.

Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 9 October 2021 Accepted: 27 March 2022 Published online: 18 April 2022

References
1. Curtain, R., Zwart, H.J.: An Introduction to Infinite Dimensional Linear Systems Theory. Springer, New York (1995)
2. Ahmed, N.: Dynamic Systems and Control with Applications. World Scientific, Hackensack (2006)
3. Klamka, J.: Controllability of non-densely defined functional differential systems with delays. Nonlinear Dyn. **56**, 169–177 (2009)
4. Wang, J., Zhou, Y.: Existence and controllability results for fractional semilinear differential inclusions. Nonlinear Anal., Real World Appl. **12**, 3642–3653 (2011)
5. Jiang, W.: The controllability of fractional control systems with control delay. Comput. Math. Appl. **64**, 3153–3159 (2012)
6. Zhou, X.-F., Jiang, W., Hu, L.-G.: Controllability of a fractional linear time-invariant neutral dynamical system. Appl. Math. Lett. **26**, 418–424 (2013)
7. Liang, J., Yang, H.: Controllability of fractional integro-differential evolution equations with nonlocal conditions. Appl. Math. Comput. **254**, 20–29 (2015)
8. Mahmudov, N.I.: Approximate controllability of evolution systems with nonlocal conditions. Nonlinear Anal. **68**, 536–546 (2008)

9. Sakthivel, R., Ren, Y., Mahmudov, N.I.: On the approximate controllability of semilinear fractional differetial systems. Comput. Math. Appl. **62**, 1451–1459 (2011)
10. Kumar, S., Sukavanam, N.: Approximate controllability of fractional order semilinear systems with bounded delay. J. Differ. Equ. **252**, 6163–6174 (2012)
11. Sakthivel, R., Ganesh, R., Anthoni, S.M.: Approximate controllability of fractional nonlinear differential inclusions. Appl. Math. Comput. **225**, 708–717 (2013)
12. Mahmudov, N.I., Zorlu, S.: On the approximate controllability of fractional evolution equations with compact analytic semigroup. J. Comput. Appl. Math. **259**, 194–204 (2014)
13. Yang, M., Wang, Q.: Approximate controllability of Riemann–Liouville fractional differential inclusions. Appl. Math. Comput. **274**, 267–281 (2016)
14. Debbouche, A., Antonov, V.: Approximate controllability of semilinear Hilfer fractional differential inclusions with impulsive control inclusion conditions in Banach spaces. Chaos Solitons Fractals **102**, 140–148 (2017)
15. Mahmudov, N.I.: Partial-approximate controllability of nonlocal fractional evolution equations via approximating method. Appl. Math. Comput. **334**, 227–238 (2018)
16. Balachandran, K., Sakthivel, R.: Controllability of functional semilinear integrodifferential systems in Banach spaces. J. Math. Anal. Appl. **225**, 447–457 (2001)
17. Balachandran, K., Sakthivel, R.: Controllability of integrodifferential systems in Banach spaces. Appl. Math. Comput. **118**, 63–71 (2001)
18. Chukwu, E.N., Lenhart, S.M.: Controllability questions for nonlinear systems in abstract spaces. J. Optim. Theory Appl. **68**, 437–462 (1991)
19. Fu, X.: Controllability of neutral functional differential systems in abstract space. Appl. Math. Comput. **141**, 281–296 (2003)
20. Fu, X.: Controllability of abstract neutral functional differential systems with unbounded delay. Appl. Math. Comput. **151**, 299–314 (2004)
21. Chang, Y.K., Nieto, J.J., Li, W.S.: Controllability of semilinear differential systems with nonlocal initial conditions in Banach spaces. J. Optim. Theory Appl. **142**, 267–273 (2008)
22. Wang, J., Zhou, Y.: Complete controllability of fractional evolution systems. Commun. Nonlinear Sci. Numer. Simul. **17**, 4346–4355 (2012)
23. Sakthivel, R., Mahmudov, N.I., Nieto, J.J.: Controllability for a class of fractional-order neutral evolution control systems. Appl. Math. Comput. **218**, 10334–10340 (2012)
24. Du, J., Jiang, W., Ullah Khan Niazi, A.: Approximate controllability of impulsive Hilfer fractional differential inclusions. J. Nonlinear Sci. Appl. **10**, 595–611 (2017)
25. Dauer, J.P., Mahmudov, N.I.: Approximate controllability of semilinear functional equations in Hilbert spaces. J. Math. Anal. Appl. **273**, 310–327 (2002)
26. Sukavanam, N., Tomar, N.K.: Approximate controllability of semilinear delay control systems. Nonlinear Funct. Anal. Appl. **12**, 53–59 (2007)
27. Sukavanam, N., Kumar, S.: Approximate controllability of fractional order semilinear delay systems. J. Optim. Theory Appl. **151**, 373–384 (2011)
28. Byszewski, L.: Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem. J. Math. Anal. Appl. **162**, 494–505 (1991)
29. Podlubny, I.: Fractional Differential Equations. Academic Press, New York (1993)
30. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)
31. Lakshmikantham, V., Leela, S., Vasundhara, J.: Theory of Fractional Dynamic Systems. Cambridge Academic Publ., Cambridge (2009)
32. Byszewski, L., Lakshmikantham, V.: Theorem about the existence and uniqueness of a solution of a nonlocal abstract Cauchy problem in a Banach space. Appl. Anal. **40**, 11–19 (1991)
33. Pazy, A.: Semigroup of Linear Operators and Applications to Partial Differential Equations. Springer, New York (1993)
34. Zhou, Y., Jiao, F.: Existence of mild solutions for fractional neutral evolution equations. Comput. Math. Appl. **59**, 1063–1077 (2010)
35. Zhou, Y., Jiao, F.: Nonlocal Cauchy problem for fractional evolution equations. Nonlinear Anal. **11**, 4465–4475 (2010)
36. Wang, J., Zhou, Y.: A class of fractional evolution equations and optimal controls. Nonlinear Anal., Real World Appl. **12**, 262–272 (2011)
37. Wang, J., Zhou, Y.: Analysis of nonlinear fractional control systems in Banach spaces. Nonlinear Anal. **74**, 5929–5942 (2011)
38. Mahmudov, N.I., Denker, A.: On controllability of linear stochastic systems. Int. J. Control **73**, 144–151 (2000)