Advances in Continuous and Discrete Models

RESEARCH Open Access

On the periodic boundary value problems for fractional nonautonomous differential equations with non-instantaneous impulses

Xue Wang[1*] and Bo Zhu[1]

[*]Correspondence:
xuew5056@163.com
[1]School of Mathematics and Quantitative Economics, Shandong University of Finance and Economics, Jinan, P.R. China

Abstract

In this paper, we investigate periodic boundary value problems for Caputo type fractional semilinear nonautonomous differential equations with non-instantaneous impulses. By using semigroup theory combined with the measure of noncompactness and some fixed point theorems, the existence of PC-mild solutions for the equations is established. At the end, an example is presented to illustrate the application of our main results.

MSC: 34A37; 34G20; 34K30; 35K90

Keywords: Fractional nonautonomous differential equations; Periodic boundary value problems; Resolvent operator; Non-instantaneous impulses

1 Introduction

Fractional differential equations have gained considerable significance during the past decades. Compared with integer order differential equations, fractional differential equations have memory in time and genetic properties, which are more suitable for describing many problems in anomalous diffusion, viscous fluid mechanics, porous media mechanics, electrical engineering and bioengineering, etc. In [1–6], the authors are committed to fractional differential equations with instantaneous impulsive effects, which can describe sudden changes at certain times such as earthquake, the closing of the switch in the circuit, and so on. Meanwhile, fractional differential equations with non-instantaneous impulses have currently been proven to be useful mathematical models to explain many phenomena occurring in biology, dynamics, control model, etc. For instance, the release and absorption of drugs in the bloodstream is a continuous and gradual process. As recent developments on fractional differential equations with non-instantaneous impulses, we mention the papers [7–16] and the references cited therein.

Cauchy problems for the abstract integer differential equations with non-instantaneous impulses were initially investigated by E. Hernandez and D. O'Regan [7], Pierri et al. [8] as follows:

$$\begin{cases} u'(t) = \mathrm{Au}(t) + f(t,u(t)), & t \in (s_i, t_{i+1}], i = 0,1,2,\ldots,m, \\ u(t) = g_i(t,u(t)), & t \in (t_i, s_i], i = 1,2,\ldots,m, \\ u(0) = u_0 \in E, \end{cases} \tag{1.1}$$

where $A : D(A) \subset E \to E$ is the generator of a C_0-semigroup $\{S(t), t \geq 0\}$ on a Banach space E, the prefixed numbers s_i, t_i satisfy $0 = s_0 < t_1 \leq s_1 < t_2 \leq s_2 < \cdots < t_m \leq s_m < t_{m+1} = T$, $f : [0,T] \times E \to E$ and $g_i : (t_i, s_i] \times E \to E$, $i = 1,2,\ldots,m$, are continuous functions, the existence of PC-mild solutions has been proved by a fixed point theorem.

Wang and Li [9] studied periodic boundary value problems for differential equations with non-instantaneous impulses via the fixed point theorem:

$$\begin{cases} u'(t) = f(t,u(t)), & t \in (s_i, t_{i+1}], i = 0,1,2,\ldots,m, \\ u(t) = g_i(t,u(t)), & t \in (t_i, s_i], i = 1,2,\ldots,m, \\ u(0) = u(T). \end{cases} \tag{1.2}$$

In [10–13], the authors studied the existence of solutions for non-instantaneous impulsive differential equations. Chen et al. [14] studied non-autonomous parabolic evolution equations with non-instantaneous impulses and obtained the existence results of mild solutions. Yu and Wang [15] investigated periodic boundary value problems for integer differential equations with non-instantaneous memory impulses; the existence of PC-mild solutions was established based on the theory of semigroup.

Inspired by these contributions, we consider the following periodic boundary value problems for fractional semilinear nonautonomous differential equations with non-instantaneous impulses:

$$\begin{cases} {}^cD_t^\beta x(t) = A(t)x(t) + f(t, x(t), \int_0^t g(t,s,x(s))\,ds), & t \in (s_i, t_{i+1}], i = 0,1,2,\ldots,m, \\ x(t) = h_i + U_\beta(t,t_i)\int_{t_i}^t g_i(s,x(s))\,ds, & t \in (t_i, s_i], i = 1,2,\ldots,m, \\ x(0) = x(T), \end{cases} \tag{1.3}$$

where ${}^cD_t^\beta$ is the Caputo's fractional derivative of order β, $\beta \in (0,1]$, $J = [0,T]$, $A(t)$ is a closed linear operator with domain $D(A)$ defined on a Banach space E, f, g, and U_β are to be specified later, the prefixed numbers s_i and t_i ($i = 1,2,\ldots,m$) satisfy $0 = s_0 < t_1 \leq s_1 < t_2 \leq \cdots < t_m \leq s_m < t_{m+1} = T$, $g_i : (t_i, s_i] \times E \to E$, $i = 1,2,\ldots,m$, are continuous and nonlinear functions, $h_i \in E$, $i = 1,2,\ldots,m$.

The rest of this paper is organized as follows. In Sect. 2, some basic definitions and auxiliary lemmas that will be needed in the remaining sections are collected. The existence of PC-mild solutions is shown in Sect. 3 based on the theory of resolvent operators, measure of noncompactness and various fixed point theorems. An example is presented to illustrate the main theorems in Sect. 4. Finally, Sect. 5 contains the summary of our results.

2 Auxiliary results

Let $(E, \|\cdot\|)$ be a Banach space, $J = [0,T]$ and $0 < T < +\infty$. $C(J,E)$ is the collection of all continuous functions from J into E equipped with the norm $\|x\|_C = \max\{\|x(t)\|, t \in J\}$.

Let $\mathrm{PC}(J,E) = \{x | x : J \to E : x \in C((t_k, t_{k+1}], E)$, and there exist $x(t_k^-)$ and $x(t_k^+)$ with $x(t_k) = x(t_k^-), k = 1, \dots, m\}$ endowed with the PC-norm $\|x\|_{\mathrm{PC}} = \sup\{\|x(t)\|, t \in J\}$.

Definition 2.1 ([17, 18]) The Caputo fractional derivative of order β of a function $f : (0, \infty) \to \mathbb{R}$ is defined as

$$ {}^cD_t^\beta f(t) = \frac{1}{\Gamma(n-\beta)} \int_0^t (t-s)^{n-\beta-1} f^{(n)}(s)\, ds, $$

where $n - 1 < \beta < n$, $n \in N$, $\Gamma(\cdot)$ denotes the gamma function. The Laplace transform of the Caputo fractional derivative of order β is given as

$$ \mathcal{L}\big({}^cD_t^\beta f(t)\big)(s) = s^\beta (\mathcal{L}f)(s) - \sum_{j=1}^{n-1} s^{\beta-j-1} x^{(j)}(0), \quad n-1 < \beta \le n, $$

where $(\mathcal{L}f)(s) = \int_0^\infty e^{-st} f(t)\, dt$ is the Laplace transform of the function $f(t)$.

Definition 2.2 ([19, 20]) Let $A(t)$ be a closed and linear operator with domain $D(A)$ defined on a Banach space E and $\beta > 0$. Let $\rho[A(t)]$ be the resolvent set of $A(t)$, $A(t)$ is called the generator of a β-resolvent family if there exist $\omega \ge 0$ and a strongly continuous function $U_\beta : \mathbb{R}_+^2 \to B(E)$ such that $\{\lambda^\beta : \mathrm{Re}\,\lambda > \omega\} \subset \rho(A)$ and

$$ \big(\lambda^\beta I - A(s)\big)^{-1} x = \int_0^\infty e^{-\lambda(t-s)} U_\beta(t,s) x\, dt, \quad \mathrm{Re}(\lambda) > \omega, x \in E. $$

In this case, $U_\beta(t,s)$ is called the β-resolvent family generated by $A(t)$, denote $M = \max_{0 \le s < t \le T} \|U_\beta(t,s)\|$.

Lemma 2.1 ([20, 21]) *$U_\beta(t,s)$ satisfies the following properties*:

(i) *$U_\beta(s,s) = I$, $U_\beta(t,s) = U_\beta(t,r)U_\beta(r,s)$ for $0 \le s \le r \le t \le a$;*

(ii) *$(t,s) \to U_\beta(t,s)$ is strongly continuous for $0 \le s \le t \le a$;*

(iii) *If $U_\beta(t,s)$ is compact for $t, s > 0$, then $U_\beta(t,s)$ is continuous in the uniform operator topology.*

Definition 2.3 A function $x \in \mathrm{PC}(J,E)$ is said to be a PC-mild solution of problem (1.3) if $x(t)$ satisfies the integral equation

$$ x(t) = \begin{cases} U_\beta(t,0)[U_\beta(T,s_m)h_m + U_\beta(T,t_m)\int_{t_m}^{s_m} g_m(s,x(s))\, ds \\ \qquad + \int_{s_m}^T U_\beta(T,s) f(s,x(s), \int_0^s g(s,\sigma,x(\sigma))\, d\sigma)\, ds] \\ \qquad + \int_0^t U_\beta(t,s) f(s,x(s), \int_0^s g(s,\sigma,x(\sigma))\, d\sigma)\, ds, \quad t \in [0,t_1], \\ h_i + U_\beta(t,t_i)\int_{t_i}^t g_i(s,x(s))\, ds, \quad t \in (t_i,s_i], i = 1,2,\dots,m, \\ U_\beta(t,s_i)h_i + U_\beta(t,t_i)\int_{t_i}^{s_i} g_i(s,x(s))\, ds \\ \qquad + \int_{s_i}^t U_\beta(t,s) f(s,x(s), \int_0^s g(s,\sigma,x(\sigma))\, d\sigma)\, ds, \\ \quad t \in (s_i,t_{i+1}], i = 1,2,\dots,m. \end{cases} \tag{2.1} $$

Lemma 2.2 ([22]) *Let $B \subset C(J,E)$ be equicontinuous and bounded, then $\overline{Co}B \subset C(J,E)$ is also equicontinuous and bounded.*

Lemma 2.3 ([22]) *Let E be a Banach space and $D \subset E$ be bounded, then there exists a countable set $D_0 \subset D$ such that $\alpha(D) \leq 2\alpha(D_0)$, where α denotes the measure of noncompactness.*

Lemma 2.4 ([23]) *Let $B \subset C(J, E)$ be equicontinuous and bounded, then $\alpha(B(t))$ is continuous on J and*

$$\alpha\left(\int_J B(s)\,ds\right) \leq \int_J \alpha\bigl(B(s)\bigr)\,ds, \quad \alpha(B) = \max_{t\in J} \alpha\bigl(B(t)\bigr).$$

3 Main results

First, we demonstrate the existence of PC-mild solutions for problem (1.3) based on the measure of noncompactness and fixed point theorem.

Theorem 3.1 *If the following assumptions (H_1)–(H_3) are satisfied.*

(H_1) *The function $g : D \times E \to E$ is continuous, $D = \{(t,s) | 0 \leq s \leq t \leq T\}$, there exists $h(t,\cdot) \in L^1(J, \mathbb{R}_+)$ with $h_0 = \max_{t\in[0,T]} \int_0^t h(t,s)\,ds$ for $(t,s) \in D$, $x \in E$ such that*

$$\bigl\|g(t,s,x)\bigr\| \leq h(t,s)\|x\|.$$

(H_2) *The function $f : J \times T_R \times T_R \to E$ is bounded and continuous for every $R > 0$ such that*

$$\lim_{R\to\infty} \sup \frac{M(R)}{R} < \frac{1}{\Delta},$$

where $M(R) = \max\{M_1(R), M_2(R)\}$, $M_1(R) = \sup\{\|f(t,x_1,x_2)\| : (t,x_1,x_2) \in J \times T_R \times T_R\}$, $M_2(R) = \sup\{\|g_i(t,x)\|, (t,x) \in J \times T_R, i = 1,2,\ldots,m\}$, $T_R = \{x \in E : \|x\| \leq R\}$, $\Delta = \max\{M^2 a_0(T - t_m) + Mt_1 a_0, Ma_0(t_{i+1} - t_i), i = 1,2,\ldots,m\}$, $a_0 = \max\{1, h_0\}$.

(H_3) *For all $R > 0$, there exist nonnegative Lebesgue integrable functions $L'_g, L'_{g_i}, L'_1, L'_2 \in L^1(J, \mathbb{R}_+)$ $(i = 1,2,\ldots,m)$ for all countable and equicontinuous sets $D, D_i \subset T_R$ $(i = 1,2)$ such that*

$$\alpha\bigl(g(t,s,D)\bigr) \leq L'_g(t)\alpha(D),$$
$$\alpha\bigl(g_i(t,D)\bigr) \leq L'_{g_i}(t)\alpha(D),$$

and

$$\alpha\bigl(f(t,D_1,D_2)\bigr) \leq L'_1(t)\alpha(D_1) + L'_2(t)\alpha(D_2).$$

Then problem (1.3) has at least one PC-mild solution on $\mathrm{PC}(J,E)$ provided that the resolvent operator $U_\beta(t,s)$ is compact for $t,s > 0$ and $\rho = \max\{2M^2 \int_{t_m}^{s_m} L'_{g_m}(s)\,ds + 2M^2 \int_{s_m}^T (L'_1(s) + L'_2(s) \int_0^T L'_g(\sigma)\,d\sigma)\,ds + 2M \int_0^{t_1} (L'_1(s) + L'_2(s) \int_0^T L'_g(\sigma)\,d\sigma)\,ds, 2M \int_{t_i}^{s_i} L'_{g_i}(s)\,ds + 2M \times \int_{s_i}^{t_{i+1}} (L'_1(s) + L'_2(s) \int_0^T L'_g(\sigma)\,d\sigma)\,ds, i = 1,2,\ldots,m\} < 1$.

Proof Consider an operator $\mathcal{F}: \mathrm{PC}(J,E) \to \mathrm{PC}(J,E)$ defined by

$$(\mathcal{F}x)(t) = \begin{cases} U_\beta(t,0)[U_\beta(T,s_m)h_m + U_\beta(T,t_m)\int_{t_m}^{s_m} g_m(s,x(s))\,ds \\ \qquad + \int_{s_m}^{T} U_\beta(T,s)f(s,x(s),\int_0^s g(s,\sigma,x(\sigma))\,d\sigma)\,ds] \\ \qquad + \int_0^t U_\beta(t,s)f(s,x(s),\int_0^s g(s,\sigma,x(\sigma))\,d\sigma)\,ds, \quad t \in [0,t_1], \\ h_i + U_\beta(t,t_i)\int_{t_i}^{t} g_i(s,x(s))\,ds, \quad t \in (t_i,s_i], i = 1,2,\dots,m, \\ U_\beta(t,s_i)h_i + U_\beta(t,t_i)\int_{t_i}^{s_i} g_i(s,x(s))\,ds \\ \qquad + \int_{s_i}^{t} U_\beta(t,s)f(s,x(s),\int_0^s g(s,\sigma,x(\sigma))\,d\sigma)\,ds, \\ \quad t \in (s_i,t_{i+1}], i = 1,2,\dots,m. \end{cases}$$

It is easy to see that the operator $\mathcal{F}$ is well defined in $\mathrm{PC}(J,E)$.

According to condition (H_2), there exist $0 < r < \frac{1}{\Delta}$ and $R_0 > 0$ for every $R \geq a_0R_0$ such that

$$M(R) < rR.$$

Let $\eta = \max\{R_0, \frac{M^2\|h_m\|}{1-M^2ra_0(T-t_m)-Mra_0t_1}, \frac{\|h_i\|}{1-Mra_0(s_i-t_i)}, \frac{M\|h_i\|}{1-Mra_0(t_{i+1}-t_i)}, i = 1,2,\dots,m\}$. For all $x \in B_\eta = \{x \in \mathrm{PC}(J,E) : \|x\|_{\mathrm{PC}} \leq \eta\}$, $t \in (s_i,t_{i+1}]$, $i = 0,1,\dots,m$, then

$$\|x\|_{\mathrm{PC}} \leq \eta \leq a_0\eta,$$

which yields

$$\left\|\int_0^t g(t,s,x)\,ds\right\| \leq \int_0^t h(t,s)\|x\|_{\mathrm{PC}}\,ds \leq h_0\eta \leq a_0\eta.$$

First of all, we show that $\mathcal{F}x \in B_\eta$.

For $t \in [0,t_1]$,

$$\begin{aligned} \|(\mathcal{F}x)(t)\| &\leq M^2\|h_m\| + M^2\int_{t_m}^{s_m} \|g_m(s,x(s))\|\,ds \\ &\quad + M^2\int_{s_m}^{T} \left\|f\left(s,x(s),\int_0^s g(s,\sigma,x(\sigma))\,d\sigma\right)\right\|\,ds \\ &\quad + M\int_0^t \left\|f\left(s,x(s),\int_0^s g(s,\sigma,x(\sigma))\,d\sigma\right)\right\|\,ds \\ &\leq M^2\|h_m\| + M^2ra_0\eta(s_m - t_m) + M^2ra_0\eta(T - s_m) + Mra_0\eta t_1 \\ &\leq M^2\|h_m\| + \big(M^2ra_0(T-t_m) + Mra_0t_1\big)\eta \leq \eta. \end{aligned}$$

For $t \in (t_i,s_i]$, $i = 1,2,\dots,m$,

$$\|(\mathcal{F}x)(t)\| \leq \|h_i\| + M\int_{t_i}^{t} \|g_i(s,x(s))\|\,ds \leq \|h_i\| + Mra_0\eta(s_i - t_i) \leq \eta.$$

For $t \in (s_i, t_{i+1}]$, $i = 1, 2, \ldots, m$,

$$\begin{aligned}\big\|(\mathcal{F}x)(t)\big\| &\leq M\|h_i\| + M\int_{t_i}^{s_i}\big\|g_i\big(s, x(s)\big)\big\|\,ds \\ &\quad + M\int_{s_i}^{t}\bigg\|f\bigg(s, x(s), \int_0^s g\big(s, \sigma, x(\sigma)\big)\,d\sigma\bigg)\bigg\|\,ds \\ &\leq M\|h_i\| + Mra_0\eta(s_i - t_i) + Mra_0\eta(t_{i+1} - s_i) \\ &\leq M\|h_i\| + Mra_0\eta(t_{i+1} - t_i) \leq \eta.\end{aligned}$$

So $\mathcal{F} : B_\eta \to B_\eta$.

Furthermore, we prove that $\mathcal{F} : B_\eta \to B_\eta$ is continuous. Let $\{x_n\}_0^\infty$ with $x_n \to x$ in B_η.

For each $t \in [0, t_1]$, we obtain

$$\begin{aligned}&\big\|(\mathcal{F}x_n)(t) - (\mathcal{F}x)(t)\big\| \\ &\quad\leq M^2\int_{t_m}^{s_m}\big\|g_m\big(s, x_n(s)\big) - g_m\big(s, x(s)\big)\big\|\,ds \\ &\qquad + M^2\int_{s_m}^{T}\bigg\|f\bigg(s, x_n(s), \int_0^s g\big(s, \sigma, x_n(\sigma)\big)\,d\sigma\bigg) - f\bigg(s, x(s), \int_0^s g\big(s, \sigma, x(\sigma)\big)\,d\sigma\bigg)\bigg\|\,ds \\ &\qquad + M\int_0^{t_1}\bigg\|f\bigg(s, x_n(s), \int_0^s g\big(s, \sigma, x_n(\sigma)\big)\,d\sigma\bigg) - f\bigg(s, x(s), \int_0^s g\big(s, \sigma, x(\sigma)\big)\,d\sigma\bigg)\bigg\|\,ds \\ &\quad\leq M^2(s_m - t_m)\sup_{s\in J}\big\|g_m\big(s, x_n(s)\big) - g_m\big(s, x(s)\big)\big\| \\ &\qquad + M^2(T - s_m)\sup_{s\in J}\bigg\|f\bigg(s, x_n(s), \int_0^s g\big(s, \sigma, x_n(\sigma)\big)\,d\sigma\bigg) \\ &\qquad - f\bigg(s, x(s), \int_0^s g\big(s, \sigma, x(\sigma)\big)\,d\sigma\bigg)\bigg\| \\ &\qquad + Mt_1\sup_{s\in J}\bigg\|f\bigg(s, x_n(s), \int_0^s g\big(s, \sigma, x_n(\sigma)\big)\,d\sigma\bigg) - f\bigg(s, x(s), \int_0^s g\big(s, \sigma, x(\sigma)\big)\,d\sigma\bigg)\bigg\|.\end{aligned}$$

For each $t \in (t_i, s_i]$, $i = 1, 2, \ldots, m$, we obtain

$$\begin{aligned}\big\|(\mathcal{F}x_n)(t) - (\mathcal{F}x)(t)\big\| &\leq M\int_{t_i}^{s_i}\big\|g_i\big(s, x_n(s)\big) - g_i\big(s, x(s)\big)\big\|\,ds \\ &\leq M(s_i - t_i)\sup_{s\in J}\big\|g_i\big(s, x_n(s)\big) - g_i\big(s, x(s)\big)\big\|.\end{aligned}$$

For each $t \in (s_i, t_{i+1}]$, $i = 1, 2, \ldots, m$, we obtain

$$\begin{aligned}&\big\|(\mathcal{F}x_n)(t) - (\mathcal{F}x)(t)\big\| \\ &\quad\leq M\int_{t_i}^{s_i}\big\|g_i\big(s, x_n(s)\big) - g_i\big(s, x(s)\big)\big\|\,ds \\ &\qquad + M\int_{s_i}^{t_{i+1}}\bigg\|f\bigg(s, x_n(s), \int_0^s g\big(s, \sigma, x_n(\sigma)\big)\,d\sigma\bigg) - f\bigg(s, x(s), \int_0^s g\big(s, \sigma, x(\sigma)\big)\,d\sigma\bigg)\bigg\|\,ds \\ &\quad\leq M(s_i - t_i)\sup_{s\in J}\big\|g_i\big(s, x_n(s)\big) - g_i\big(s, x(s)\big)\big\|\end{aligned}$$

$$
\begin{aligned}
&+ M(t_{i+1} - s_i) \sup_{s\in J} \left\| f\left(s, x_n(s), \int_0^s g\big(s,\sigma,x_n(\sigma)\big)\,d\sigma\right) \right. \\
&\left. - f\left(s, x(s), \int_0^s g\big(s,\sigma,x(\sigma)\big)\,d\sigma\right) \right\|.
\end{aligned}
$$

Using the fact that the functions $f : J \times E \times E \to E$, $g : D \times E \to E$ and $g_i : (t_i, s_i] \times E \to E$ ($i = 1, 2, \ldots, m$) are continuous, we have

$$
\lim_{n\to\infty} \sup_{s\in J} \left\| f\left(s, x_n(s), \int_0^s g\big(s,\sigma,x_n(\sigma)\big)\,d\sigma\right) - f\left(s, x(s), \int_0^s g\big(s,\sigma,x(\sigma)\big)\,d\sigma\right) \right\| = 0
$$

and

$$
\lim_{n\to\infty} \sup_{t\in J} \left\| g_i\big(s, x_n(s)\big) - g_i\big(s, x(s)\big) \right\| = 0 \ (i = 1, 2, \ldots, m).
$$

From the above, we deduce that $\|\mathcal{F}x_n - \mathcal{F}x\|_{\mathrm{PC}} \to 0$ as $n \to \infty$. This shows that $\mathcal{F} : B_\eta \to B_\eta$ is continuous.

Now we prove that $\mathcal{F}(B_\eta)$ is equicontinuous.

For the interval $[0, t_1]$, $0 \le e_1 < e_2 \le t_1$, $x \in B_\eta$, we get

$$
\begin{aligned}
&\left\| (\mathcal{F}x)(e_2) - (\mathcal{F}x)(e_1) \right\| \\
&\quad\le \left\| U_\beta(e_2, 0) - U_\beta(e_1, 0) \right\| \left(M\|h_m\| + M \left\| \int_{t_m}^{s_m} g_m\big(s, x(s)\big)\,ds \right\| \right. \\
&\qquad \left. + M \left\| \int_{s_m}^T f\left(s, x(s), \int_0^s g\big(s,\sigma,x(\sigma)\big)\,d\sigma\right) ds \right\| \right) \\
&\qquad + \left\| \int_{e_1}^{e_2} U_\beta(e_2, s) f\left(s, x(s), \int_0^s g\big(s,\sigma,x(\sigma)\big)\,d\sigma\right) ds \right\| \\
&\qquad + \left\| \int_0^{e_1} \big(U_\beta(e_2, s) - U_\beta(e_1, s)\big) f\left(s, x(s), \int_0^s g\big(s,\sigma,x(\sigma)\big)\,d\sigma\right) ds \right\| \\
&\quad\le \left\| U_\beta(e_2, 0) - U_\beta(e_1, 0) \right\| M \left(\|h_m\| + \left\| \int_{t_m}^{s_m} g_m\big(s, x(s)\big)\,ds \right\| \right. \\
&\qquad \left. + \left\| \int_{s_m}^T f\left(s, x(s), \int_0^s g\big(s,\sigma,x(\sigma)\big)\,d\sigma\right) ds \right\| \right) \\
&\qquad + M r a_0 \eta (e_2 - e_1) + \sup_{s\in[0,t_1]} \left\| U_\beta(e_2, s) - U_\beta(e_1, s) \right\| r a_0 \eta t_1.
\end{aligned}
$$

For the interval $(t_i, s_i]$, $i = 1, 2, \ldots, m$, $t_i < e_1 < e_2 \le s_i$, $x \in B_\eta$, we get

$$
\begin{aligned}
\left\| (\mathcal{F}x)(e_2) - (\mathcal{F}x)(e_1) \right\| &\le \left\| U_\beta(e_2, t_i) \int_{e_1}^{e_2} g_i\big(s, x(s)\big)\,ds \right\| \\
&\quad + \left\| \big(U_\beta(e_2, t_i) - U_\beta(e_1, t_i)\big) \int_{t_i}^{e_1} g_i\big(s, x(s)\big)\,ds \right\| \\
&\le M r a_0 \eta (e_2 - e_1) + \left\| U_\beta(e_2, t_i) - U_\beta(e_1, t_i) \right\| r a_0 \eta (s_i - t_i).
\end{aligned}
$$

For interval $(s_i, t_{i+1}]$, $i = 1, 2, \dots, m$, $s_i < e_1 < e_2 \le t_{i+1}$, $x \in B_\eta$, we get

$$\begin{aligned}
&\|(\mathcal{F}x)(e_2) - (\mathcal{F}x)(e_1)\| \\
&\le \|U_\beta(e_2, s_i) - U_\beta(e_1, s_i)\| \|h_i\| + \|U_\beta(e_2, t_i) - U_\beta(e_1, t_i)\| \left\| \int_{t_i}^{s_i} g_i\big(s, x(s)\big)\, ds \right\| \\
&\quad + \left\| \int_{e_1}^{e_2} U_\beta(e_2, s) f\left(s, x(s), \int_0^s g\big(s, \sigma, x(\sigma)\big)\, d\sigma\right) ds \right\| \\
&\quad + \left\| \int_{s_i}^{e_1} \big(U_\beta(e_2, s) - U_\beta(e_1, s)\big) f\left(s, x(s), \int_0^s g\big(s, \sigma, x(\sigma)\big)\, d\sigma\right) ds \right\| \\
&\le \|U_\beta(e_2, s_i) - U_\beta(e_1, s_i)\| \|h_i\| + \|U_\beta(e_2, t_i) - U_\beta(e_1, t_i)\| \left\| \int_{t_i}^{s_i} g_i\big(s, x(s)\big)\, ds \right\| \\
&\quad + Mra_0\eta(e_2 - e_1) + \sup_{s \in (s_i, t_{i+1}]} \|U_\beta(e_2, s) - U_\beta(e_1, s)\| ra_0\eta(t_{i+1} - s_i).
\end{aligned}$$

We deduce that $\|(\mathcal{F}x)(e_2) - (\mathcal{F}x)(e_1)\| \to 0$ independently of $x \in B_\eta$ as $e_2 \to e_1$, since the compactness of $U_\beta(t, s)$ $(t, s > 0)$ implies the continuity in the uniform operator topology. This shows that $\mathcal{F}(B_\eta)$ is equicontinuous. In view of Lemma 2.2, $\overline{Co}\mathcal{F}(B_\eta) \subset B_\eta$ is equicontinuous and bounded.

It remains to prove that $F : \overline{Co}\mathcal{F}(B_\eta) \to \overline{Co}\mathcal{F}(B_\eta)$ is a condensing operator. For any $D \subset \overline{Co}\mathcal{F}(B_\eta)$, by Lemma 2.3, there exists a countable set $D_0 = \{x_n\} \subset D$ such that

$$\alpha\big(\mathcal{F}(D)\big) \le 2\alpha\big(\mathcal{F}(D_0)\big).$$

Using the fact that $\overline{Co}\mathcal{F}(B_\eta)$ is equicontinuous, $D_0 \subset \overline{Co}\mathcal{F}(B_\eta)$ is equicontinuous. By (H_3), for $s \in (s_i, t_{i+1}]$, $i = 0, 1, \dots, m$, then

$$\begin{aligned}
\alpha\left(f\left(s, D_0(s), \int_0^s g\big(s, \sigma, D_0(\sigma)\big)\, d\sigma\right)\right) &\le L_1'(s)\alpha\big(D_0(s)\big) + L_2'(s)\int_0^s L_g'(\sigma)\alpha\big(D_0(\sigma)\big)\, d\sigma \\
&\le \left(L_1'(s) + L_2'(s)\int_0^T L_g'(\sigma)\, d\sigma\right)\alpha(D).
\end{aligned}$$

For each $t \in [0, t_1]$,

$$\begin{aligned}
&\alpha\big(\mathcal{F}(D_0)(t)\big) \\
&\le M^2\alpha\left(\int_{t_m}^{s_m} g_m\big(s, D_0(s)\big)\, ds\right) + M^2\alpha\left(\int_{s_m}^T f\left(s, D_0(s), \int_0^s g\big(s, \sigma, D_0(\sigma)\big)\, d\sigma\right) ds\right) \\
&\quad + M\alpha\left(\int_0^t f\left(s, D_0(s), \int_0^s g\big(s, \sigma, D_0(\sigma)\big)\, d\sigma\right) ds\right) \\
&\le M^2\int_{t_m}^{s_m} L_{g_m}'(s)\alpha\big(D_0(s)\big)\, ds + M^2\int_{s_m}^T \alpha\left(f\left(s, D_0(s), \int_0^s g\big(s, \sigma, D_0(\sigma)\big)\, d\sigma\right)\right) ds \\
&\quad + M\int_0^t \alpha\left(f\left(s, D_0(s), \int_0^s g\big(s, \sigma, D_0(\sigma)\big)\, d\sigma\right)\right) ds
\end{aligned}$$

$$\leq \left(M^2 \int_{t_m}^{s_m} L'_{g_m}(s)\,ds + M^2 \int_{s_m}^{T} \left(L'_1(s) + L'_2(s) \int_0^T L'_g(\sigma)\,d\sigma \right) ds + M \int_0^{t_1} \left(L'_1(s) + L'_2(s) \int_0^T L'_g(\sigma)\,d\sigma \right) ds \right) \alpha(D).$$

For each $t \in (t_i, s_i]$, $i = 1, \dots, m$,

$$\alpha\big(\mathcal{F}(D_0)(t)\big) \leq M\alpha\left(\int_{t_i}^{t} g_i\big(s, D_0(s)\big)\,ds \right) \leq M \int_{t_i}^{s_i} L'_{g_i}(s)\,ds \alpha(D).$$

For each $t \in (s_i, t_{i+1}]$, $i = 1, \dots, m$,

$$\begin{aligned}\alpha\big(\mathcal{F}(D_0)(t)\big) &\leq M\alpha\left(\int_{t_i}^{s_i} g_i\big(s, D_0(s)\big)\,ds \right) \\ &\quad + M\alpha\left(\int_{s_i}^{t} f\left(s, D_0(s), \int_0^s g\big(s,\sigma, D_0(\sigma)\big)\,d\sigma \right) ds \right) \\ &\leq \left(M \int_{t_i}^{s_i} L'_{g_i}(s)\,ds + M \int_{s_i}^{t_{i+1}} \left(L'_1(s) + L'_2(s) \int_0^T L'_g(\sigma)\,d\sigma \right) ds \right) \alpha(D).\end{aligned}$$

By Lemma 2.4,

$$\alpha\big(\mathcal{F}(D_0)\big) = \max_{t\in J} \alpha\big(\mathcal{F}(D_0)(t)\big).$$

Hence

$$\alpha\big(\mathcal{F}(D)\big) \leq \rho\alpha(D) < \alpha(D).$$

These arguments enable us to infer that $\mathcal{F} : \overline{Co}\mathcal{F}(B_\eta) \to \overline{Co}\mathcal{F}(B_\eta)$ is a condensing operator and by the fixed point theorem of Sadovskii, there exists one fixed point $x^\star \in \overline{Co}\mathcal{F}(B_\eta) \subset \mathrm{PC}(J, E)$ for $\mathcal{F}$. In conclusion, problem (1.3) has at least one PC-mild solution. This completes the proof. □

Now we establish the existence results of PC-mild solutions for problem (1.3) via Krasnoselskii's fixed point theorem.

Theorem 3.2 *Assume that* (G_1)*–*(G_4) *hold and the resolvent operator* $U_\beta(t,s)$ *is compact for* $t, s > 0$.

(G_1) *The function* $f : J \times E \times E \to E$ *is continuous, there exist nonnegative Lebesgue integrable functions* $a, L_1, L_2 \in L^1(J, \mathbb{R}_+)$ *for* $t \in (s_i, t_{i+1}]$ $(i = 0, 1, \dots, m)$ *and* $x_1, x_2 \in E$ *such that*

$$\big\| f(t, x_1, x_2) \big\| \leq a(t) + L_1(t)\|x_1\| + L_2(t)\|x_2\|.$$

(G_2) *The function* $g : D \times E \to E$ *is continuous,* $D = \{(t,s) | 0 \leq s \leq t \leq T\}$, *there exist nonnegative Lebesgue integrable functions* $b, L_3 \in L^1(J, \mathbb{R}_+)$ *for* $(t,s) \in D$, $x \in E$ *such that*

$$\big\| g(t,s,x) \big\| \leq b(t) + L_3(t)\|x\|.$$

(G_3) *There exists a function $\omega_i(t)$ with $\varpi_i = \sup_{t\in[t_i,s_i]} \omega_i(t) < +\infty$ for $t \in (t_i, s_i]$ $(i = 1,2,\ldots,m)$ and $x \in E$ such that*

$$\|g_i(t,x)\| \le \omega_i(t).$$

(G_4) *There exist nonnegative constants $L_{g_i} > 0$ for $t \in (t_i, s_i]$ $(i = 1,2,\ldots,m)$ and $x, x' \in E$ such that*

$$\|g_i(t,x) - g_i(t,x')\| \le L_{g_i}\|x - x'\|.$$

Then problem (1.3) has at least one PC-mild solution on $\mathrm{PC}(J,E)$ provided that $\vartheta = \max\{M^2\int_{s_m}^T b_1(s)\,ds + M\int_0^{t_1} b_1(s)\,ds, M\int_{s_i}^{t_{i+1}} b_1(s)\,ds, M^2L_{g_m}(s_m - t_m), ML_{g_i}(s_i - t_i), i = 1,\ldots,m\} < 1$, where $b_1(s) = L_1(s) + L_2(s)\int_0^T L_3(\sigma)\,d\sigma$.

Proof We decompose $\mathcal{F}$ as $\mathcal{F} = \mathcal{G} + \mathcal{H}$, where

$$(\mathcal{G}x)(t) = \begin{cases} U_\beta(t,0)[U_\beta(T,s_m)h_m + U_\beta(T,t_m)\int_{t_m}^{s_m} g_m(s,x(s))\,ds], & t \in [0,t_1], \\ h_i + U_\beta(t,t_i)\int_{t_i}^t g_i(s,x(s))\,ds, & t \in (t_i,s_i], i = 1,2,\ldots,m, \\ U_\beta(t,s_i)h_i + U_\beta(t,t_i)\int_{t_i}^{s_i} g_i(s,x(s))\,ds, & t \in (s_i,t_{i+1}], i = 1,2,\ldots,m, \end{cases}$$

and

$$(\mathcal{H}x)(t) = \begin{cases} U_\beta(t,0)\int_{s_m}^T U_\beta(T,s)f(s,x(s),\int_0^s g(s,\sigma,x(\sigma))\,d\sigma)\,ds \\ \quad + \int_0^t U_\beta(t,s)f(s,x(s),\int_0^s g(s,\sigma,x(\sigma))\,d\sigma)\,ds, & t \in [0,t_1], \\ 0, \quad t \in (t_i,s_i], i = 1,2,\ldots,m, \\ \int_{s_i}^t U_\beta(t,s)f(s,x(s),\int_0^s g(s,\sigma,x(\sigma))\,d\sigma)\,ds, & t \in (s_i,t_{i+1}], i = 1,2,\ldots,m. \end{cases}$$

Let us fix $R^\star > 0$ such that

$$R^\star \ge \max\left\{\frac{M^2\|h_m\| + M^2\varpi_m(s_m - t_m) + M^2\int_{s_m}^T a_1(s)\,ds + M\int_0^{t_1} a_1(s)\,ds}{1-\vartheta},\right.$$
$$\left.\|h_i\| + M\varpi_i(s_i - t_i), \frac{M\|h_i\| + M\varpi_i(s_i - t_i) + M\int_{s_i}^{t_{i+1}} a_1(s)\,ds}{1-\vartheta}, i = 1,2,\ldots,m\right\},$$

where $a_1(s) = a(s) + L_2(s)\int_0^T b(\sigma)\,d\sigma$.

We consider the set $B_{R^\star} = \{x \in \mathrm{PC}(J,E) : \|x\|_{\mathrm{PC}} \le R^\star\}$ for any $x \in B_{R^\star}$. From conditions (G_1) and (G_2), for all $s \in (s_i, t_{i+1}]$, $i = 0,1,\ldots,m$, one can find that

$$\begin{aligned} \left\|f\left(s,x(s),\int_0^s g\big(s,\sigma,x(\sigma)\big)\,d\sigma\right)\right\| &\le a(s) + L_1(s)R^\star + L_2(s)\int_0^s \big(b(\sigma) + L_3(\sigma)R^\star\big)\,d\sigma \\ &\le a(s) + L_2(s)\int_0^T b(\sigma)\,d\sigma \\ &\quad + \left(L_1(s) + L_2(s)\int_0^T L_3(\sigma)\,d\sigma\right)R^\star \\ &= a_1(s) + b_1(s)R^\star. \end{aligned}$$

Obviously, $a_1(s)$ and $b_1(s)$ are nonnegative Lebesgue integrable functions.

According to condition (G_3) and the above inequities, for any $t \in [0, t_1]$, we obtain

$$\begin{aligned}
\|(\mathcal{F}x)(t)\| &\leq \|U_\beta(t,0)\| \biggl(\|U_\beta(T,s_m)h_m\| + \|U_\beta(T,t_m)\| \biggl\| \int_{t_m}^{s_m} g_m(s,x(s))\,ds \biggr\| \\
&\quad + \biggl\| \int_{s_m}^{T} U_\beta(T,s) f\biggl(s, x(s), \int_0^s g(s,\sigma,x(\sigma))\,d\sigma \biggr)\,ds \biggr\| \biggr) \\
&\quad + \biggl\| \int_0^t U_\beta(t,s) f\biggl(s, x(s), \int_0^s g(s,\sigma,x(\sigma))\,d\sigma \biggr)\,ds \biggr\| \\
&\leq M^2\|h_m\| + M^2\varpi_m(s_m - t_m) + M^2 \int_{s_m}^{T} (a_1(s) + b_1(s)R^\star)\,ds \\
&\quad + M \int_0^t (a_1(s) + b_1(s)R^\star)\,ds \\
&\leq M^2\|h_m\| + M^2\varpi_m(s_m - t_m) + M^2 \int_{s_m}^{T} a_1(s)\,ds + M \int_0^{t_1} a_1(s)\,ds + \vartheta R^\star \\
&\leq R^\star.
\end{aligned}$$

For any $t \in (t_i, s_i]$, $i = 1, 2, \ldots, m$, we have

$$\begin{aligned}
\|(\mathcal{F}x)(t)\| &\leq \|h_i\| + \biggl\| U_\beta(t,t_i) \int_{t_i}^{t} g_i(s,x(s))\,ds \biggr\| \\
&\leq \|h_i\| + M \int_{t_i}^{t} \|g_i(s,x(s))\|\,ds \\
&\leq \|h_i\| + M\varpi_i(s_i - t_i) \leq R^\star.
\end{aligned}$$

For any $t \in (s_i, t_{i+1}]$, $i = 1, 2, \ldots, m$, we have

$$\begin{aligned}
\|(\mathcal{F}x)(t)\| &\leq \|U_\beta(t,s_i)h_i\| + \biggl\| U_\beta(t,t_i) \int_{t_i}^{s_i} g_i(s,x(s))\,ds \biggr\| \\
&\quad + \biggl\| \int_{s_i}^{t} U_\beta(t,s) f\biggl(s, x(s), \int_0^s g(s,\sigma,x(\sigma))\,d\sigma \biggr)\,ds \biggr\| \\
&\leq M\|h_i\| + M\varpi_i(s_i - t_i) + M \int_{s_i}^{t} (a_1(s) + b_1(s)R^\star)\,ds \\
&\leq M\|h_i\| + M\varpi_i(s_i - t_i) + M \int_{s_i}^{t_{i+1}} a_1(s)\,ds + \vartheta R^\star \leq R^\star.
\end{aligned}$$

From the above inequities, we conclude $\mathcal{F}x = \mathcal{G}x + \mathcal{H}x \in B_{R^\star}$.

Next we prove that the operator $\mathcal{G}$ is a contraction on $B_{R^\star}$. By (G_4), for $x, x' \in B_{R^\star}$, for any $t \in [0, t_1]$, we get

$$\begin{aligned}
\|(\mathcal{G}x)(t) - (\mathcal{G}x')(t)\| &\leq M^2 \int_{t_m}^{s_m} \|g_m(s,x(s)) - g_m(s,x'(s))\|\,ds \\
&\leq M^2 L_{g_m} \|x - x'\|_{\mathrm{PC}}(s_m - t_m).
\end{aligned}$$

For any $t \in (t_i, s_i]$, $i = 1, 2, \dots, m$, we get

$$\begin{aligned}\left\|(\mathcal{G}x)(t) - (\mathcal{G}x')(t)\right\| &\le M \int_{t_i}^{t} \left\|g_i(s, x(s)) - g_i(s, x'(s))\right\| ds \\ &\le ML_{g_i} \left\|x - x'\right\|_{\mathrm{PC}} (s_i - t_i).\end{aligned}$$

For any $t \in (s_i, t_{i+1}]$, $i = 1, 2, \dots, m$, we get

$$\begin{aligned}\left\|(\mathcal{G}x)(t) - (\mathcal{G}x')(t)\right\| &\le M \int_{t_i}^{s_i} \left\|g_i(s, x(s)) - g_i(s, x'(s))\right\| ds \\ &\le ML_{g_i} \left\|x - x'\right\|_{\mathrm{PC}} (s_i - t_i).\end{aligned}$$

From the above inequities with $\vartheta < 1$, we have $\|\mathcal{G}x - \mathcal{G}x'\|_{\mathrm{PC}} < \|x - x'\|_{\mathrm{PC}}$. This implies that $\mathcal{G}$ is a contraction.

To prove that $\mathcal{H}$ is completely continuous on $B_{R^\star}$, first we claim that $\mathcal{H}$ is continuous applying the arguments employed in the proof of Theorem 3.1. Moreover, $\mathcal{H}$ is uniformly bounded on $B_{R^\star}$ since $\|\mathcal{H}x\|_{\mathrm{PC}} \le R^\star$. Next we show that $\mathcal{H}(B_{R^\star})$ is equicontinuous. To do this, for $x \in B_{R^\star}$, $e_1, e_2 \in [0, t_1]$ with $e_1 < e_2$, we have

$$\begin{aligned}&\left\|(\mathcal{H}x)(e_2) - (\mathcal{H}x)(e_1)\right\| \\ &\quad\le \left\|U_\beta(e_2, 0) - U_\beta(e_1, 0)\right\| M \int_{s_m}^{T} \left\|f\left(s, x(s), \int_0^s g(s, \sigma, x(\sigma))\, d\sigma\right)\right\| ds \\ &\qquad+ \left\|\int_{e_1}^{e_2} U_\beta(e_2, s) f\left(s, x(s), \int_0^s g(s, \sigma, x(\sigma))\, d\sigma\right) ds\right\| \\ &\qquad+ \left\|\int_0^{e_1} \bigl(U_\beta(e_2, s) - U_\beta(e_1, s)\bigr) f\left(s, x(s), \int_0^s g(s, \sigma, x(\sigma))\, d\sigma\right) ds\right\| \\ &\quad\le \left\|U_\beta(e_2, 0) - U_\beta(e_1, 0)\right\| M \int_{s_m}^{T} \bigl(a_1(s) + b_1(s) R^\star\bigr)\, ds \\ &\qquad+ M \int_{e_1}^{e_2} \bigl(a_1(s) + b_1(s) R^\star\bigr)\, ds \\ &\qquad+ \sup_{s \in [0, t_1]} \left\|U_\beta(e_2, s) - U_\beta(e_1, s)\right\| \int_0^{e_1} \bigl(a_1(s) + b_1(s) R^\star\bigr)\, ds.\end{aligned}$$

For $e_1, e_2 \in (t_i, s_i]$ with $e_1 < e_2$, $i = 1, 2, \dots, m$, we have

$$\left\|(\mathcal{H}x)(e_2) - (\mathcal{H}x)(e_1)\right\| = 0.$$

For $e_1, e_2 \in (s_i, t_{i+1}]$ with $e_1 < e_2$, $i = 1, 2, \dots, m$, we have

$$\begin{aligned}&\left\|(\mathcal{H}x)(e_2) - (\mathcal{H}x)(e_1)\right\| \\ &\quad\le \int_{e_1}^{e_2} \left\|U_\beta(e_2, s)\right\| \left\|f\left(s, x(s), \int_0^s g(s, \sigma, x(\sigma))\, d\sigma\right)\right\| ds \\ &\qquad+ \int_{s_i}^{e_1} \left\|U_\beta(e_2, s) - U_\beta(e_1, s)\right\| \left\|f\left(s, x(s), \int_0^s g(s, \sigma, x(\sigma))\, d\sigma\right)\right\| ds\end{aligned}$$

$$\begin{aligned}&\leq M\int_{e_1}^{e_2}\bigl(a_1(s)+b_1(s)R^\star\bigr)\,ds\\&\quad+\sup_{s\in(s_i,t_{i+1}]}\bigl\|U_\beta(e_2,s)-U_\beta(e_1,s)\bigr\|\int_{s_i}^{e_1}\bigl(a_1(s)+b_1(s)R^\star\bigr)\,ds.\end{aligned}$$

By Lemma 2.1, the compactness of the resolvent operator $U_\beta(t,s)$ implies the continuity in the uniform operator topology and together with $a_1(s)$, $b_1(s)\in L^1(J,\mathbb{R}_+)$, we infer that $\|(\mathcal{H}x)(e_2)-(\mathcal{H}x)(e_1)\|\to 0$ as $e_2\to e_1$. Consequently, $\mathcal{H}(B_{R^\star})$ is equicontinuous.

Third, we prove that $\mathcal{H}(B_{R^\star})$ is precompact.

For $t\in[0,t_1]$, $0<\epsilon<t$, $x\in B_{R^\star}$, define

$$\begin{aligned}(\mathcal{H}_\epsilon x)(t)&=U_\beta(t,0)\int_{s_m}^{T}U_\beta(T,s)f\biggl(s,x(s),\int_0^s g\bigl(s,\sigma,x(\sigma)\bigr)\,d\sigma\biggr)\,ds\\&\quad+\int_0^{t-\epsilon}U_\beta(t,s)f\biggl(s,x(s),\int_0^s g\bigl(s,\sigma,x(\sigma)\bigr)\,d\sigma\biggr)\,ds.\end{aligned}$$

Hence

$$\begin{aligned}\bigl\|(\mathcal{H}x)(t)-(\mathcal{H}_\epsilon x)(t)\bigr\|&\leq\biggl\|\int_{t-\epsilon}^{t}U_\beta(t,s)f\biggl(s,x(s),\int_0^s g\bigl(s,\sigma,x(\sigma)\bigr)\,d\sigma\biggr)\,ds\biggr\|\\&\leq M\int_{t-\epsilon}^{t}\bigl(a_1(s)+b_1(s)R^\star\bigr)\,ds.\end{aligned}$$

For $t\in(t_i,s_i]$, $0<\epsilon<t$, $x\in B_{R^\star}$, $i=1,2,\dots,m$, define $(\mathcal{H}_\epsilon x)(t)=0$.

Obviously, $\|(\mathcal{H}x)(t)-(\mathcal{H}_\epsilon x)(t)\|=0$.

For $t\in(s_i,t_{i+1}]$, $0<\epsilon<t$, $x\in B_{R^\star}$, $i=1,2,\dots,m$, define

$$(\mathcal{H}_\epsilon x)(t)=\int_{s_i}^{t-\epsilon}U_\beta(t,s)f\biggl(s,x(s),\int_0^s g\bigl(s,\sigma,x(\sigma)\bigr)\,d\sigma\biggr)\,ds.$$

Thus

$$\begin{aligned}\bigl\|(\mathcal{H}x)(t)-(\mathcal{H}_\epsilon x)(t)\bigr\|&\leq\biggl\|\int_{t-\epsilon}^{t}U_\beta(t,s)f\biggl(s,x(s),\int_0^s g\bigl(s,\sigma,x(\sigma)\bigr)\,d\sigma\biggr)\,ds\biggr\|\\&\leq M\int_{t-\epsilon}^{t}\bigl(a_1(s)+b_1(s)R^\star\bigr)\,ds.\end{aligned}$$

Since $U_\beta(t,s)$ is a compact resolvent operator, then the set $Y_\epsilon(t)=\{(\mathcal{H}_\epsilon x)(t):x\in B_{R^\star}\}$ is relatively compact in E for every $0<\epsilon<t$. Thus $Y(t)=\{(\mathcal{H}x)(t):x\in B_{R^\star}\}$ is totally bounded. Hence, $Y(t)$ is relatively compact in E, and so, with the help of the Arzelá–Ascoli theorem, $\mathcal{H}$ is completely continuous on $B_{R^\star}$. Therefore, by Krasnoselskii's fixed point theorem, there exists a fixed point for $\mathcal{F}=\mathcal{G}+\mathcal{H}$, which corresponds to a PC-mild solution of problem (1.3) on $\mathrm{PC}(J,E)$. This completes the proof. □

4 An application

In order to show the application of the main results, we consider the following problem:

$$\begin{cases} {}^cD_t^\beta x(z,t) = t\frac{\partial^2}{\partial z^2}x(z,t) + \frac{t}{4M^2(1+t^2)}x(z,t) + \int_0^t \frac{e^{t-s}|x(z,s)|}{8M^2e^4}\,ds, \\ \quad t \in [0,1)\cup(2,3], z\in(0,1), \\ \frac{\partial}{\partial z}x(0,t) = \frac{\partial}{\partial z}x(1,t) = 0, \quad t\in[0,1)\cup(2,3], \\ x(z,t) = y_1 z + U_\beta(t,1)\int_1^t \frac{|x(z,s)|}{8M^2(1+t)}\,ds, \quad t\in(1,2], z\in(0,1), \\ x(0,t) = x(3,t), \quad t\in(0,1), \end{cases} \tag{4.1}$$

where $E = L^2[0,3]$, $0 = t_0 = s_0$, $t_1 = 1$, $s_1 = 2$, ${}^cD_t^\beta$ is the Caputo's fractional derivative of order β, $0 < \beta < 1$. The operator $A : D(A) \subset E \to E$ is defined as $A(t)(z) = t\frac{\partial^2 x}{\partial z^2}$, where $D(A) = \{x \in E : x'' \in E,\ x(0) = x(1) = 0\}$. It is well known that the operator $A(t)$ generates a β-resolvent family $U_\beta(t,s)$ and $\max_{0\le s<t\le T}\|U_\beta(t,s)\| \le M$, $(M > 1)$.

By setting

$$x(t)(z) = x(z,t), \quad h_1 z = y_1 z, \qquad g_1\big(t,x(t)\big)(z) = \frac{|x(z,s)|}{8M^2(1+t)},$$

$$f\left(t,x(t),\int_0^t g\big(t,s,x(s)\big)\,ds\right)(z) = \frac{t}{4M^2(1+t^2)}x(z,t) + \int_0^t \frac{e^{t-s}|x(z,s)|}{8M^2e^4}\,ds,$$

problem (4.1) can be rewritten as the following abstract form:

$$\begin{cases} {}^cD_t^\beta x(t) = A(t)x(t) + f(t,x(t),\int_0^t g(t,s,x(s))\,ds), & t\in[0,1)\cup(2,3], \\ x(t) = h_1 + U_\beta(t,1)\int_1^t g_1(s,x(s))\,ds, & t\in(1,2], \\ x(0) = x(3). \end{cases} \tag{4.2}$$

The function $f : J \times T_R \times T_R \to E$ is bounded and continuous, for every $R > 0$, such that

$$\lim_{R\to\infty}\sup\frac{M(R)}{R} < \frac{1}{Ma_0(M+1)}, \tag{4.3}$$

where $M(R) = \max\{M_1(R), M_2(R)\}$, $M_1(R) = \sup\{\|f(t,x_1,x_2)\| : (t,x_1,x_2) \in J \times T_R \times T_R\}$, $M_2(R) = \sup\{\|g_1(t,x)\|, (t,x) \in J \times T_R,\}$, $T_R = \{x \in E : \|x\| \le R\}$, $a_0 = \max\{1, h_0\}$.

Let

$$\big\|g(t,s,x)\big\| \le \frac{e^{t-s}}{8M^2e^4}\|x\|,$$

$$\alpha\big(g(t,s,D)\big) \le e^t\alpha(D),$$

$$\alpha\big(g_1(t,s,D)\big) \le \frac{1}{8M^2(1+t)}\alpha(D),$$

$$\alpha\big(f(t,D_1,D_2)\big) \le \frac{t}{4M^2(1+t^2)}\alpha(D_1) + e^t\alpha(D_2).$$

Then

$$\begin{aligned}
&2M^2\int_1^2 \frac{1}{8M^2(1+s)}\,ds + 2M^2\int_2^3\left(\frac{s}{4M^2(1+s^2)} + e^s\int_0^3 \frac{e^{s-\sigma}}{8M^2e^4}\,d\sigma\right)ds \\
&\quad + 2M\int_0^1\left(\frac{s}{4M^2(1+s^2)} + e^s\int_0^3 \frac{e^{s-\sigma}}{8M^2e^4}\,d\sigma\right)ds \\
&\quad < \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1, \\
&2M\int_1^2 \frac{1}{8M^2(1+s)}\,ds + 2M\int_2^3\left(\frac{s}{4M^2(1+s^2)} + e^s\int_0^3 \frac{e^{s-\sigma}}{8M^2e^4}\,d\sigma\right)ds \\
&\quad < \frac{1}{4M} + \frac{1}{4M} + \frac{1}{4M} < 1.
\end{aligned}$$

We have

$$\begin{aligned}
\rho = \max\Bigg\{&2M^2\int_1^2 \frac{1}{8M^2(1+s)}\,ds + 2M^2\int_2^3\left(\frac{s}{4M^2(1+s^2)} + e^s\int_0^3 \frac{e^{s-\sigma}}{8M^2e^4}\,d\sigma\right)ds \\
&+ 2M\int_0^1\left(\frac{s}{4M^2(1+s^2)} + e^s\int_0^3 \frac{e^{s-\sigma}}{8M^2e^4}\,d\sigma\right)ds, \\
&2M\int_1^2 \frac{1}{8M^2(1+s)}\,ds + 2M\int_2^3\left(\frac{s}{4M^2(1+s^2)} + e^s\int_0^3 \frac{e^{s-\sigma}}{8M^2e^4}\,d\sigma\right)ds\Bigg\} < 1.
\end{aligned}$$

Therefore, problem (4.2) satisfies the conditions of Theorem 3.1, then problem (4.2) has a PC-mild solution, which means that problem (4.1) has a mild solution.

5 Conclusion

In this paper, we demonstrate sufficient conditions on the existence of PC-mild solutions for periodic boundary value problems for fractional semilinear nonautonomous differential equations with non-instantaneous impulses. For the proofs of the main theorems, we use the measure of noncompactness together with Sadovskii's fixed point theorem and Krasnoselskii's fixed point theorem. Finally, an example is given to illustrate the application of our main results.

Acknowledgements
Not applicable.

Funding
This paper was supported by the National Natural Science Foundation of P.R. China (No.62073190, 11871302).

Availability of data and materials
Not applicable.

Declarations

Ethics approval and consent to participate
Not applicable.

Consent for publication
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors have contributed equally and significantly to the contents of the paper. All authors read and approved the final manuscript.

Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 4 August 2021 Accepted: 12 April 2022 Published online: 27 April 2022

References
1. Li, H., Kao, Y.: Mittag-Leffler stability for a new coupled system of fractional-order differential equations with impulses. Appl. Math. Comput. **361**, 22–31 (2019)
2. Yan, Z., Lu, F.: Approximate controllability of a multi-valued fractional impulsive stochastic partial integro-differential equation with infinite delay. Appl. Math. Comput. **292**, 425–447 (2017)
3. Ge, F., Zhou, H., Kou, C.: Approximate controllability of semilinear evolution equations of fractional order with nonlocal and impulsive conditions via an approximating technique. Appl. Math. Comput. **275**, 107–120 (2016)
4. Gou, H., Li, B.: Local and global existence of mild solution to impulsive fractional semilinear integro-differential equation with noncompact semigroup. Commun. Nonlinear Sci. Numer. Simul. **42**, 204–214 (2017)
5. Liu, Y.: Piecewise continuous solutions of initial value problems of singular fractional differential equations with impulse effects. Acta Math. Sci. **36B**, 1492–1508 (2016)
6. Yang, X., Li, C., Huang, T., Song, Q.: Mittag-Leffler stability analysis of nonlinear fractional-order systems with impulses. Appl. Math. Comput. **293**, 416–422 (2017)
7. Hernandez, E., O'Regan, D.: On a new class of abstract impulsive differential equations. Proc. Am. Math. Soc. **141**, 1641–1649 (2013)
8. Pierri, M., O'Regan, D., Rolnik, V.: Existence of solutions for semilinear abstract differential equations with not instantaneous impulses. Appl. Math. Comput. **219**, 6743–6749 (2013)
9. Wang, J., Li, X.: Periodic BVP for integer/fractional order nonlinear differential equations with noninstantaneous impulses. Appl. Math. Comput. **46**, 321–334 (2014)
10. Tian, Y., Wang, J., Zhou, Y.: Almost periodic solutions for a class of non-instantaneous impulsive differential equations. Quaest. Math. **42**(7), 885–905 (2019)
11. Malik, M., Kumar, V.: Existence, uniqueness and UHR stability of solutions to nonlinear ordinary differential equations with noninstantaneous impulses. IMA J. Math. Control Inf. **37**(1), 276–299 (2020)
12. Pierri, M., Hernan, R., Prokopczyk, A.: Global solutions for abstract differential equations with non-instantaneous impulses. Mediterr. J. Math. **13**(4), 1685–1708 (2016)
13. Abbas, S., Benchohra, M.: Uniqueness and Ulam stabilities results for partial fractional differential equations with not instantaneous impulses. Appl. Math. Comput. **257**, 190–198 (2015)
14. Chen, P., Zhang, X., Li, Y.: Non-autonomous parabolic evolution equations with noninstantaneous impulses governed by noncompact evolution families. J. Fixed Point Theory Appl. **21**(3), 84–101 (2019)
15. Yu, X., Wang, J.: Periodic boundary value problems for nonlinear impulsive evolution equations on Banach spaces. Commun. Nonlinear Sci. Numer. Simul. **22**, 980–989 (2015)
16. Wang, J., Feckan, M., Debbouche, A.: Time optimal control of a system governed by non-instantaneous impulsive differential equations. J. Optim. Theory Appl. **182**(2), 573–587 (2019)
17. Samko, S., Kilbas, A., Marichev, O.: Fractional Integrals and Derivatives: Theory and Applications. Gordon and Breach, Amsterdam (1993)
18. Caputo, M.: Linear models of dissipation whose q is almost frequency independent. J. R. Astron. Soc. **13**(2), 529–539 (1967)
19. Araya, D., Lizama, C.: Almost automorphic mild solutions to fractional differential equations. Nonlinear Anal. **69**, 3692–3705 (2008)
20. Debbouche, A., Baleanu, D.: Controllability of fractional evolution nonlocal impulsive quasilinear delay integro-differential systems. Comput. Math. Appl. **62**, 1442–1450 (2011)
21. Lizama, C., Pereira, A., Ponec, R.: On the compactness of fractional resolvent operator functions. Semigroup Forum **93**, 363–374 (2016)
22. Liu, L., Guo, F., Wu, Y.: Existence theorems of global solutions for nonlinear Volterra type integral equations in Banach spaces. J. Math. Anal. Appl. **309**, 638–649 (2005)
23. Liu, L.: Iterative method for solutions and coupled quasi-solutions of nonlinear integro-differential equations of mixed type in Banach spaces. Nonlinear Anal. **42**, 583–598 (2000)