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Partial-fraction decomposition of a rational function and its application

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Abstract

In this paper, by using the residue method of complex analysis, we obtain an explicit partial fraction decomposition for the *general* rational function $\frac{x^M}{(x+1)_n^\lambda}$ (*M* is any nonnegative integer, λ and *n* are any positive integers). As applications, we deduce the corresponding algebraic identities and combinatorial identities which are the corresponding extensions of Chu' results. We also give some explicit formulas of Apostol-type polynomials and harmonic Stirling numbers of the second kind.

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1 Introduction and the main results

The generalized harmonic numbers are defined by

$$H_0^{(r)} = 0 \quad \text{and} \quad H_n^{(r)} = \sum_{k=1}^{n} \frac{1}{k^r} \quad \text{for } n, r = 1, 2, \ldots,$$

when $r = 1$, they reduce to the classical harmonic numbers as $H_n = H_n^{(1)}$.

For $z \in \mathbb{C}$, the shifted factorial is defined by

$$(z)_0 = 1 \quad \text{and} \quad (z)_n = z(z+1)\cdots(z+n-1) \quad \text{for } n = 1, 2, \ldots$$

The complete Bell polynomials $\mathbf{B}_n(x_1, x_2, \ldots, x_n)$ are defined by [16, pp. 173–174]

$$\exp\left(\sum_{k=1}^{\infty} x_k \frac{z^k}{k!}\right) = \sum_{n=0}^{\infty} \mathbf{B}_n(x_1, x_2, \ldots, x_n) \frac{z^n}{n!}, \quad \mathbf{B}_0 := 1, \tag{1}$$

which exact expression is

$$\mathbf{B}_n(x_1, x_2, \ldots, x_n) = \sum_{\pi(n)} \frac{n!}{k_1! k_2! \cdots k_n!} \left(\frac{x_1}{1!}\right)^{k_1} \left(\frac{x_2}{2!}\right)^{k_2} \cdots \left(\frac{x_n}{n!}\right)^{k_n}, \tag{2}$$

where $\pi(n)$ denotes a partition of n, usually denoted $1^{k_1}2^{k_2}\cdots n^{k_n}$, with $k_1+2k_2+\cdots+nk_n=n$.

From (2) we easily obtain

$$\mathbf{B}_n\bigl(-x_1,x_2,\ldots,(-1)^n x_n\bigr)=(-1)^n\mathbf{B}_n(x_1,x_2,\ldots,x_n). \tag{3}$$

For convenience, we define the above sum as equal to zero for $n<0$, i.e., $\mathbf{B}_n(x_1,x_2,\ldots,x_n)=0$ when $n<0$.

The partial fraction decomposition plays an important role in the study of the combinatorial identities and related questions (for example, see [14, 15, 17–19, 22–24] and the references therein). Chu [5] established the partial fraction decompositions of two rational functions $\frac{1}{(x)_{n+1}^{\lambda}}$ and $\frac{x^{M-1}}{(x+1)_n^{\lambda}}$, thereby completely resolving the open problem of Driver et al. [7].

When $\frac{x^{M-1}}{(x+1)_n^{\lambda}}$ is a proper rational fraction, i.e., $M-1<\lambda n$, we can use the method of the partial fraction decomposition. But when $\frac{x^{M-1}}{(x+1)_n^{\lambda}}$ is an improper rational fraction, i.e., $M-1\geq\lambda n$, how do we decompose $\frac{x^{M-1}}{(x+1)_n^{\lambda}}$ into partial fractions?

In the present paper, by using the contour integral and Cauchy's residue theorem, we will answer the above question and give an explicit decomposition for the *general* rational function $\frac{x^{M}}{(x+1)_n^{\lambda}}$. As applications, we deduce the corresponding algebraic and combinatorial identities which are just some extensions of Chu's results. We give some explicit formulas of Apostol-type polynomials and harmonic Stirling numbers of the second kind.

Theorem 1 *Suppose M is any nonnegative integer, λ and n are any positive integers such that $N=\lambda n$, and x is a complex number such that $x\in\mathbb{C}\setminus\{-1,-2,\ldots,-n\}$. Then the following partial fraction decomposition holds*:

$$\begin{aligned}\frac{x^M}{(x+1)_n^{\lambda}}&=\sum_{j=0}^{M-N}\frac{\mathbf{B}_j(x_1,\ldots,x_j)}{j!}x^{M-N-j}\\&\quad+\sum_{k=1}^{n}\frac{(-1)^{\lambda k}}{(n!)^{\lambda}}\binom{n}{k}^{\lambda}(-k)^{\lambda+M}\sum_{j=0}^{\lambda-1}\frac{\mathbf{B}_j(y_1,y_2,\ldots,y_j)}{j!(x+k)^{\lambda-j}},\end{aligned} \tag{4}$$

where

$$x_i=\lambda(-1)^i(i-1)!\sum_{j=1}^{n}j^i,\quad i=1,2,\ldots,M-N,$$

$$y_i=(i-1)!\left[\lambda\bigl(H_k^{(i)}+(-1)^iH_{n-k}^{(i)}\bigr)-\frac{\lambda+M}{k^i}\right],\quad i=1,2,\ldots,\lambda-1.$$

• When $M-N\geq0$ in Theorem 1, i.e., the degree of the numerator polynomial, M, is not smaller than the degree of the denominator polynomial, $N=\lambda n$, we say that Theorem 1 is a more general and new decomposition of the rational function $\frac{x^M}{(x+1)_n^{\lambda}}$. For example, below we give the first four special cases.

When $M-N=0$, we have

$$\frac{x^{\lambda n}}{(x+1)_n^{\lambda}}=1+\sum_{k=1}^{n}\frac{(-1)^{\lambda k}}{(n!)^{\lambda}}\binom{n}{k}^{\lambda}(-k)^{\lambda(n+1)}\sum_{j=0}^{\lambda-1}\frac{\mathbf{B}_j(y_1,y_2,\ldots,y_j)}{j!(x+k)^{\lambda-j}},$$

where

$$y_i = \lambda(i-1)!\left[\left(H_k^{(i)} + (-1)^i H_{n-k}^{(i)}\right) - \frac{n+1}{k^i}\right], \quad i = 1,2,\ldots,\lambda-1.$$

When $M-N=1$, we have

$$\frac{x^{\lambda n+1}}{(x+1)_n^\lambda} = x - \frac{\lambda n(n+1)}{2} + \sum_{k=1}^{n} \frac{(-1)^{\lambda k}}{(n!)^\lambda}\binom{n}{k}^\lambda (-k)^{\lambda(n+1)+1} \sum_{j=0}^{\lambda-1} \frac{\mathbf{B}_j(y_1,y_2,\ldots,y_j)}{j!(x+k)^{\lambda-j}},$$

where $y_i = (i-1)![\lambda(H_k^{(i)} + (-1)^i H_{n-k}^{(i)}) - \frac{\lambda(n+1)+1}{k^i}]$, $i = 1,2,\ldots,\lambda-1$.

When $M-N=2$,

$$\begin{aligned}\frac{x^{\lambda n+2}}{(x+1)_n^\lambda} &= x^2 - \frac{\lambda n(n+1)}{2}x + \frac{\lambda n(n+1)[3n(n+1)\lambda+4n+2]}{24} \\ &\quad + \sum_{k=1}^{n} \frac{(-1)^{\lambda k}}{(n!)^\lambda}\binom{n}{k}^\lambda (-k)^{\lambda(n+1)+2} \sum_{j=0}^{\lambda-1} \frac{\mathbf{B}_j(y_1,y_2,\ldots,y_j)}{j!(x+k)^{\lambda-j}},\end{aligned}$$

where $y_i = (i-1)![\lambda(H_k^{(i)} + (-1)^i H_{n-k}^{(i)}) - \frac{\lambda(n+1)+2}{k^i}]$, $i = 1,2,\ldots,\lambda-1$.

When $M-N=3$,

$$\begin{aligned}\frac{x^{\lambda n+3}}{(x+1)_n^\lambda} &= x^3 - \frac{\lambda n(n+1)}{2}x^2 + \frac{\lambda n(n+1)[3n(n+1)\lambda+4n+2]}{24}x - \frac{\lambda n^2(n+1)^2}{8} \\ &\quad \times \left[n(n+1)\lambda^2 + 2(2n+1)\lambda + 4\right] \\ &\quad + \sum_{k=1}^{n} \frac{(-1)^{\lambda k}}{(n!)^\lambda}\binom{n}{k}^\lambda (-k)^{\lambda(n+1)+3} \sum_{j=0}^{\lambda-1} \frac{\mathbf{B}_j(y_1,y_2,\ldots,y_j)}{j!(x+k)^{\lambda-j}},\end{aligned}$$

where $y_i = (i-1)![\lambda(H_k^{(i)} + (-1)^i H_{n-k}^{(i)}) - \frac{\lambda(n+1)+3}{k^i}]$, $i = 1,2,\ldots,\lambda-1$.

Remark 2 When $M-N \geq 0$ in Theorem 1, if put $a_j = \frac{\mathbf{B}_j(x_1,\ldots,x_j)}{j!}$ and

$$a_{k,j} = \frac{(-1)^{\lambda k}}{j!(n!)^\lambda}\binom{n}{k}^\lambda (-k)^{\lambda+M}\mathbf{B}_j(y_1,y_2,\ldots,y_j),$$

then Theorem 1 becomes the following explicit form:

$$\frac{x^M}{(x+1)_n^\lambda} = \sum_{j=0}^{M-N} a_{M-N-j}x^j + \sum_{k=1}^{n}\sum_{j=0}^{\lambda-1} \frac{a_{k,j}(x+k)^j}{(x+k)^\lambda},$$

which is an explicit result when the polynomial x^M is divided by polynomial $(x+1)_n^\lambda$, i.e., the improper rational fraction $\frac{x^M}{(x+1)_n^\lambda}$ is decomposed into a polynomial of order $M-N$ plus a proper rational fraction. Therefore we say that Theorem 1 implies a new and interesting method for division of two polynomials.

- When $M-N<0$, i.e., the degree of the numerator polynomial, M, is smaller than the degree of the denominator polynomial, $N=\lambda n$, we obtain the following Chu's result:

Corollary 3 ([5, Theorem 5]) *Suppose M is any nonnegative integer, λ and n are any positive integers such that $M < N = \lambda n$, and x is a complex number such that $x \in \mathbb{C} \setminus \{-1,-2,\dots,-n\}$. Then the following partial fraction decomposition holds*:

$$\frac{x^M}{(x+1)_n^\lambda} = \sum_{k=1}^{n} \frac{(-1)^{\lambda k}}{(n!)^\lambda} \binom{n}{k}^\lambda (-k)^{\lambda+M} \sum_{j=0}^{\lambda-1} \frac{\mathbf{B}_j(y_1,y_2,\dots,y_j)}{j!(x+k)^{\lambda-j}},$$

where

$$y_i = (i-1)!\left[\lambda\left(H_k^{(i)} + (-1)^i H_{n-k}^{(i)}\right) - \frac{\lambda+M}{k^i}\right], \quad i = 1,2,\dots,\lambda-1.$$

Remark 4 If put $M = 0$ and then let $x \longmapsto x-1$ and $n \longmapsto n+1$ in Theorem 1, noting that the empty sum is zero, we observe that Theorem 1 reduces to Theorem 2 of Chu [5, p. 44, (1.5)]:

$$\frac{1}{(x)_{n+1}^\lambda} = \sum_{k=0}^{n} \frac{(-1)^{\lambda k}}{(n!)^\lambda} \binom{n}{k}^\lambda \sum_{j=0}^{\lambda-1} \frac{\mathbf{B}_j(y_1,y_2,\dots,y_j)}{j!(x+k)^{\lambda-j}},$$

where $y_i = \lambda(i-1)!(H_k^{(i)} + (-1)^i H_{n-k}^{(i)})$ for $i = 1,2,\dots,\lambda-1$.

Remark 5 It is easily seen that Theorem 1 includes Chu's results, but when $M - N \geq 0$ Theorem 1 is a new result. Therefore, we say that Theorem 1 is an interesting extension of Chu's results. We also see that Theorem 1 is not obtained using the partial fraction decomposition.

Setting $\lambda = 1$ and letting $M \mapsto m$ in Theorem 1, we deduce the following result:

Corollary 6 *Suppose m is any nonnegative integer, n is any positive integer, and x is a complex number such that $x \in \mathbb{C} \setminus \{-1,-2,\dots,-n\}$. Then the following partial fraction decomposition holds*:

$$\frac{x^m}{(x+1)_n} = \sum_{j=0}^{m-n} \frac{\mathbf{B}_j(x_1,\dots,x_j)}{j!} x^{m-n-j} + \sum_{k=1}^{n} \frac{(-1)^k}{n!} \binom{n}{k} \frac{(-k)^{m+1}}{x+k},$$

where

$$x_i = (-1)^i (i-1)! \sum_{j=1}^{n} j^i, \quad i = 1,2,\dots,m-n.$$

Furthermore, taking $m = 0$ in Corollary 6, we obtain the following algebraic identity:

$$\frac{1}{(x+1)_n} = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{n!} \binom{n}{k} \frac{k}{x+k},$$

or, equivalently, the following well-known combinatorial identity (e.g., see [13]):

$$\prod_{k=1}^{n} \frac{k}{x+k} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{x}{x+k}.$$

Next we obtain the following combinatorial identities from Theorem 1:

- Setting $x = 0$, we have

$$\sum_{k=1}^{n}(-1)^{\lambda(k+1)+M}\binom{n}{k}^{\lambda}\sum_{j=0}^{\lambda-1}\frac{k^{j+M}\mathbf{B}_j(y_1,y_2,\dots,y_j)}{j!}=\begin{cases}0, & 1\le M<N,\\ (n!)^{\lambda}, & M=0,\\ -\frac{\mathbf{B}_{M-N}(x_1,\dots,x_{M-N})}{(n!)^{\lambda}(M-N)!}, & M\ge N.\end{cases}$$

- Setting $x = 0$ and letting $M \longmapsto M + 1$, we get

$$\begin{aligned}&\sum_{k=1}^{n}(-1)^{\lambda k}\binom{n}{k}^{\lambda}(-k)^{M+\lambda}\frac{\mathbf{B}_{\lambda-1}(y_1,y_2,\dots,y_{\lambda-1})}{(\lambda-1)!}\\ &\quad=\frac{\mathbf{B}_{M-N+1}(x_1,\dots,x_{M-N+1})}{(n!)^{\lambda}(M-N+1)!}\\ &\qquad+\sum_{k=1}^{n}(-1)^{\lambda(k+1)+M+1}\binom{n}{k}^{\lambda}\sum_{j=0}^{\lambda-2}\frac{k^{j+M+1}\mathbf{B}_j(y_1,y_2,\dots,y_j)}{j!},\quad M+1\ge N.\end{aligned}$$

where

$$\begin{aligned}x_i&=\lambda(-1)^i(i-1)!\sum_{j=1}^{n}j^i,\quad i=1,2,\dots,M-N+1,\\ y_i&=(i-1)!\left[\lambda\left(H_k^{(i)}+(-1)^iH_{n-k}^{(i)}\right)-\frac{\lambda+M+1}{k^i}\right],\quad i=1,2,\dots,\lambda-1.\end{aligned}$$

We can also get the following special cases of Theorem 1:

- Taking $x = 1$ gives

$$\sum_{k=1}^{n}(-1)^{\lambda k}\binom{n}{k}^{\lambda}(-k)^{\lambda+M}\sum_{j=0}^{\lambda-1}\frac{\mathbf{B}_j(y_1,y_2,\dots,y_j)}{j!(k+1)^{\lambda-j}}=\frac{1}{(n+1)^{\lambda}}-(n!)^{\lambda}\sum_{j=0}^{M-N}\frac{\mathbf{B}_j(x_1,\dots,x_j)}{j!}.$$

- Taking $x = i$ (here $i = \sqrt{-1}$) yields

$$\sum_{k=1}^{n}\frac{(-1)^{\lambda k}}{(n!)^{\lambda}}\binom{n}{k}^{\lambda}(-k)^{\lambda+M}\sum_{j=0}^{\lambda-1}\frac{\mathbf{B}_j(y_1,y_2,\dots,y_j)}{j!(i+k)^{\lambda-j}}=\frac{i^M}{(i+1)_n^{\lambda}}-\sum_{j=0}^{M-N}\frac{\mathbf{B}_j(x_1,\dots,x_j)}{j!}i^{M-N-j}.$$

2 Proof of Theorem 1

Lemma 7 *Suppose M is any nonnegative integer, λ and n are any positive integers such that $N = \lambda n$, and x is a complex number such that $x \in \mathbb{C}\setminus\{-1,-2,\dots,-n\}$. Then the following algebraic identity holds*:

$$\begin{aligned}\frac{x^M}{(x+1)_n^{\lambda}}&=\frac{\mathbf{B}_{M-N}(z_1,z_2,\dots,z_{M-N})}{(M-N)!}\\ &\quad+\sum_{k=0}^{n}\frac{(-1)^{\lambda k}(-k)^{\lambda+M}}{(n!)^{\lambda}(x+k)}\binom{n}{k}^{\lambda}\frac{\mathbf{B}_{\lambda-1}(w_1,w_2,\dots,w_{\lambda-1})}{(\lambda-1)!},\end{aligned}\tag{5}$$

where

$$z_i = (i-1)!\left((-1)^i\lambda\sum_{j=1}^{n} j^i + x^i\right), \quad i = 1,2,\dots,M-N, \tag{6}$$

$$w_i = (i-1)!\left[\lambda\left(H_k^{(i)} + (-1)^i H_{n-k}^{(i)}\right) - \frac{\lambda+M}{k^i} + \frac{1}{(x+k)^i}\right], \quad i = 1,2,\dots,\lambda-1. \tag{7}$$

Proof We first construct two polynomials $P(z)$ and $Q(z)$ of degrees M and $N+1$, respectively, which are given by

$$P(z) = z^M \quad \text{and} \quad Q(z) = (z-x)\prod_{j=0}^{n-1}(z+j+1)^\lambda$$

such that $x \neq 0,-1,-2,\dots,-n$.

We next construct three contour integrals for the rational functions $P(z)/Q(z)$:

$\oint_\Gamma \frac{P(z)}{Q(z)}\,\mathrm{d}z$, where Γ is a simple closed contour which only surrounds the single pole x of $P(z)/Q(z)$;

$\oint_{\Gamma_1} \frac{P(z)}{Q(z)}\,\mathrm{d}z$, where Γ_1 is a simple closed contour which surrounds the poles $-1,-2,\dots,-n$ of $P(z)/Q(z)$;

$\oint_{\Gamma_2} \frac{P(z)}{Q(z)}\,\mathrm{d}z$, where Γ_2 is a simple closed contour which only surrounds the pole ∞ of $P(z)/Q(z)$.

In the extended complex plane, since the total sum of residues of a rational function at all finite poles and that at infinity is equal to zero [12, p. 25, Theorem 2], we have

$$\oint_{\Gamma+\Gamma_1+\Gamma_2} \frac{P(z)}{Q(z)}\,\mathrm{d}z = 0,$$

or equivalently,

$$\oint_\Gamma \frac{P(z)}{Q(z)}\,\mathrm{d}z = -\oint_{\Gamma_2} \frac{P(z)}{Q(z)}\,\mathrm{d}z - \oint_{\Gamma_1} \frac{P(z)}{Q(z)}\,\mathrm{d}z. \tag{8}$$

Below we compute the contour integrals $\oint_\Gamma \frac{P(z)}{Q(z)}\,\mathrm{d}z$, $\oint_{\Gamma_1} \frac{P(z)}{Q(z)}\,\mathrm{d}z$, and $\oint_{\Gamma_2} \frac{P(z)}{Q(z)}\,\mathrm{d}z$, respectively.

Applying Cauchy's residue theorem, we compute the contour integral $\oint_\Gamma \frac{P(z)}{Q(z)}\,\mathrm{d}z$ as follows:

$$\begin{aligned}\oint_\Gamma \frac{P(z)}{Q(z)}\,\mathrm{d}z &= 2\pi i \operatorname*{Res}_{z=x} \frac{z^M}{(z-x)\prod_{j=0}^{n-1}(z+j+1)^\lambda} = 2\pi i \lim_{z\to x} \frac{z^M}{\prod_{j=0}^{n-1}(z+j+1)^\lambda} \\ &= 2\pi i \frac{x^M}{\prod_{j=0}^{n-1}(x+j+1)^\lambda} = 2\pi i \frac{x^M}{(x+1)_n^\lambda}.\end{aligned} \tag{9}$$

We now compute the contour integral $\oint_{\Gamma_1} \frac{P(z)}{Q(z)}\,\mathrm{d}z$. By utilizing Cauchy's residue theorem, noting that the power series expansion of the logarithmic function is

$$\log(1+z) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{z^n}{n} \quad \left(|z|<1\right)$$

and using the definition of complete Bell polynomials, we obtain

$$
\begin{aligned}
\oint_{\Gamma_1} \frac{P(z)}{Q(z)}\,\mathrm{d}z &= 2\pi i \sum_{k=0}^{n-1} \operatorname*{Res}_{z=-k-1} \frac{z^M}{(z-x)\prod_{j=0}^{n-1}(z+j+1)^\lambda} \\
&= 2\pi i \sum_{k=0}^{n-1} \big[(z+k+1)^{\lambda-1}\big] \frac{z^M}{(z-x)\prod_{\substack{j=0\\ j\neq k}}^{n-1}(z+j+1)^\lambda} \\
&= 2\pi i \sum_{k=0}^{n-1} \big[z^{\lambda-1}\big] \frac{(z-k-1)^M}{(z-x-k-1)\prod_{\substack{j=0\\ j\neq k}}^{n-1}(z-k+j)^\lambda} \\
&= -2\pi i \sum_{k=0}^{n-1} \Bigg\{ \frac{(-k-1)^M}{(x+k+1)\prod_{\substack{j=0\\ j\neq k}}^{n-1}(j-k)^\lambda} \\
&\quad \times \big[z^{\lambda-1}\big] \exp\Bigg[M\log\left(1-\frac{z}{k+1}\right) - \log\left(1-\frac{z}{x+k+1}\right) \\
&\quad - \lambda \sum_{j=0, j\neq k}^{n-1} \log\left(1+\frac{z}{j-k}\right) \Bigg]\Bigg\} \\
&= -2\pi i \sum_{k=1}^{n} \Bigg\{ \frac{(-1)^{\lambda k}(-k)^{\lambda+M}}{(n!)^\lambda (x+k)} \binom{n}{k}^\lambda \\
&\quad \times \big[z^{\lambda-1}\big] \exp\Bigg[\sum_{i=1}^{\infty} (i-1)! \left(\lambda\big(H_k^{(i)} + (-1)^i H_{n-k}^{(i)}\big) - \frac{\lambda+M}{k^i} + \frac{1}{(x+k)^i} \right) \Bigg] \frac{z^i}{i!} \Bigg\} \\
&= -2\pi i \sum_{k=1}^{n} \frac{(-1)^{\lambda k}(-k)^{\lambda+M}}{(n!)^\lambda (x+k)} \binom{n}{k}^\lambda \frac{\mathbf{B}_{\lambda-1}(w_1, w_2, \ldots, w_{\lambda-1})}{(\lambda-1)!}. \qquad (10)
\end{aligned}
$$

Calculating the contour integral $\oint_{\Gamma_2} \frac{P(z)}{Q(z)}\,\mathrm{d}z$, we obtain

$$
\oint_{\Gamma_2} \frac{P(z)}{Q(z)}\,\mathrm{d}z = 2\pi i \operatorname*{Res}_{z=\infty} \frac{z^M}{(z-x)\prod_{j=0}^{n-1}(z+j+1)^\lambda} = -2\pi i \operatorname*{Res}_{t=0} \frac{t^{N-M-1}}{(1-xt)\prod_{j=0}^{n-1}(1+(j+1)t)^\lambda}.
$$

If $M-N<0$, then $t=0$ is not a pole, and so we have

$$
\oint_{\Gamma_2} \frac{P(z)}{Q(z)}\,\mathrm{d}z = -2\pi i \operatorname*{Res}_{t=0} \frac{t^{N-M-1}}{(1-xt)\prod_{j=0}^{n-1}(1+(j+1)t)^\lambda} = 0.
$$

If $M-N=0$, then $t=0$ is a single pole of order 1, so we have

$$
\oint_{\Gamma_2} \frac{P(z)}{Q(z)}\,\mathrm{d}z = -2\pi i \lim_{t\to 0} \frac{1}{(1-xt)\prod_{j=0}^{n-1}(1+(j+1)t)^\lambda} = -2\pi i.
$$

If $M-N>0$, then $t=0$ is a single pole of order $M-N+1$, and so we have

$$
\oint_{\Gamma_2} \frac{P(z)}{Q(z)}\,\mathrm{d}z = -2\pi i \operatorname*{Res}_{t=0} \frac{t^{N-M-1}}{(1-xt)\prod_{j=0}^{n-1}(1+(j+1)t)^\lambda}
$$

$$
\begin{aligned}
&= -2\pi i\left[t^{M-N}\right]\frac{1}{(1-xt)\prod_{j=0}^{n-1}(1+(j+1)t)^{\lambda}} \\
&= -2\pi i\left[t^{M-N}\right]\exp\left\{\sum_{i=1}^{\infty}\left[(i-1)!\left((-1)^{i}\lambda\sum_{j=1}^{n}j^{i}+x^{i}\right)\frac{t^{i}}{i!}\right]\right\} \\
&= -2\pi i\left[t^{M-N}\right]\sum_{k=0}^{\infty}\mathbf{B}_{k}(z_1,z_2,\dots,z_k)\frac{t^{k}}{k!} \\
&= -2\pi i\frac{\mathbf{B}_{M-N}(z_1,z_2,\dots,z_{M-N})}{(M-N)!}. \qquad (11)
\end{aligned}
$$

Therefore, by replacing (9), (10), and (11) into (8), we obtain Lemma 7. This proof is complete. □

Lemma 8 *The following recursion formula of complete Bell polynomial holds true*:

$$
\frac{\mathbf{B}_{\lambda-1}(w_1,\dots,w_{\lambda-1})}{(\lambda-1)!}=\sum_{j=0}^{\lambda-1}\frac{\mathbf{B}_{j}(y_1,\dots,y_j)}{j!(x+k)^{\lambda-j-1}}. \qquad (12)
$$

Proof Let

$$
y_i=(i-1)!\left[\lambda\left(H_k^{(i)}+(-1)^{i}H_{n-k}^{(i)}\right)-\frac{\lambda+M}{k^{i}}\right].
$$

Write $w_i=y_i+\frac{(i-1)!}{(x+k)^{i}}$ in (7). By the definition of complete Bell polynomial, we obtain

$$
\begin{aligned}
\frac{\mathbf{B}_{\lambda-1}(w_1,\dots,w_{\lambda-1})}{(\lambda-1)!} &= \left[t^{\lambda-1}\right]\exp\left(\sum_{n=1}^{\infty}w_n\frac{t^{n}}{n!}\right) \\
&= \left[t^{\lambda-1}\right]\exp\left\{\sum_{n=1}^{\infty}\left(y_n+\frac{(n-1)!}{(x+k)^{n}}\right)\frac{t^{n}}{n!}\right\} \\
&= \sum_{j=0}^{\lambda-1}\left[t^{j}\right]\exp\left\{\sum_{n=1}^{\infty}y_n\frac{t^{n}}{n!}\right\}\left[t^{\lambda-1-j}\right]\exp\left\{\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{t}{x+k}\right)^{n}\right\} \\
&= \sum_{j=0}^{\lambda-1}\frac{\mathbf{B}_{j}(y_1,\dots,y_j)}{j!}\left[t^{\lambda-1-j}\right]\exp\left\{-\log\left(1-\frac{t}{x+k}\right)\right\} \\
&= \sum_{j=0}^{\lambda-1}\frac{\mathbf{B}_{j}(y_1,\dots,y_j)}{j!}\left[t^{\lambda-1-j}\right]\sum_{n=0}^{\infty}\left(\frac{t}{x+k}\right)^{n} \\
&= \sum_{j=0}^{\lambda-1}\frac{\mathbf{B}_{j}(y_1,\dots,y_j)}{j!(x+k)^{\lambda-j-1}}.
\end{aligned}
$$

In the above process, we apply the geometric series,

$$
\frac{1}{1-z}=\sum_{n=0}^{\infty}z^{n},
$$

and the expansion of the logarithmic function,

$$\log(1+z) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{z^n}{n}.$$

This proof is complete. □

Lemma 9 *The following recursion formula of complete Bell polynomial holds true*:

$$\frac{\mathbf{B}_{M-N}(z_1,z_2,\ldots,z_{M-N})}{(M-N)!} = \sum_{j=0}^{M-N}\frac{\mathbf{B}_j(x_1,\ldots,x_j)}{j!}x^{M-N-j}. \tag{13}$$

Proof Let

$$x_i = (i-1)!(-1)^i\lambda\sum_{j=1}^{n} j^i.$$

Write $z_i = x_i + (i-1)!x^i$ in (6). By the definition of complete Bell polynomial, we obtain

$$\begin{aligned}
\frac{\mathbf{B}_{M-N}(z_1,z_2,\ldots,z_{M-N})}{(M-N)!} &= \left[t^{M-N}\right]\exp\left(\sum_{n=1}^{\infty} z_n\frac{t^n}{n!}\right) \\
&= \sum_{j=0}^{M-N}\left[t^j\right]\exp\left\{\sum_{n=1}^{\infty} x_n\frac{t^n}{n!}\right\}\left[t^{M-N-j}\right]\exp\left\{\sum_{n=1}^{\infty} x^n\frac{t^n}{n}\right\} \\
&= \sum_{j=0}^{M-N}\frac{\mathbf{B}_j(x_1,\ldots,x_j)}{j!}\left[t^{M-N-j}\right]\sum_{n=0}^{\infty}(xt)^n \\
&= \sum_{j=0}^{M-N}\frac{\mathbf{B}_j(x_1,\ldots,x_j)}{j!}x^{M-N-j}.
\end{aligned}$$

This proof is complete. □

From Lemmas 7, 8, and 9, we obtain Theorem 1 immediately. This completes the proof of Theorem 1.

3 Applications to Apostol-type polynomials

In the present section, using the contour integrals and the main result (Theorem 1), we first obtain two lemmas, and then we give some explicit formulas for Apostol-type polynomials.

Lemma 10 *Let m be a nonnegative integer and n a positive integer, also let $\mathbf{B}_m(x_1,x_2,\ldots,x_m)$ be the complete Bell polynomial. Then*

$$\mathbf{B}_m\bigl(n, n1!,\ldots,n(m-1)!\bigr) = m!\binom{n+m-1}{n-1}, \tag{14}$$

$$\mathbf{B}_m\big(-n,-n1!,\dots,-n(m-1)!\big)=(-1)^m m!\binom{n}{m},\tag{15}$$

$$\mathbf{B}_m\big(-n,n1!,\dots,(-1)^m n(m-1)!\big)=(-1)^m m!\binom{n+m-1}{n-1}.\tag{16}$$

Proof First we construct two polynomials z^{n+m} and $(z-1)^n$ of degrees $n+m$ and n, respectively.

Next, we construct the following contour integrals for the rational functions $\frac{z^{n+m}}{(z-1)^n}$:

$\oint_\Gamma \frac{z^{n+m}}{(z-1)^n}\,\mathrm{d}z$, where Γ is a simple closed contour which only surrounds the single pole $z=1$ of order n of $\frac{z^{n+m}}{(z-1)^n}$;

$\oint_{\Gamma'} \frac{z^{n+m}}{(z-1)^n}\,\mathrm{d}z$, where Γ' is a simple closed contour which only surrounds the pole $z=\infty$ of $\frac{z^{n+m}}{(z-1)^n}$.

By utilizing Cauchy's residue theorem, we obtain

$$\begin{aligned}\oint_\Gamma \frac{z^{n+m}}{(z-1)^n}\,\mathrm{d}z&=2\pi i\operatorname*{Res}_{z=1}\frac{z^{n+m}}{(z-1)^n}=2\pi i\big[(z-1)^{n-1}\big]z^{n+m}\\&=2\pi i\big[z^{n-1}\big](z+1)^{n+m}=2\pi i\binom{n+m}{n-1}.\end{aligned}\tag{17}$$

In the extended complex plane, we calculate the residue of the rational function $\frac{z^{n+m}}{(z-1)^n}$ at $z=\infty$.

By utilizing the power series expansion of the logarithmic function,

$$\log(1+z)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{z^n}{n}\qquad\big(|z|<1\big),$$

and combining the definition of complete Bell polynomial, we obtain

$$\begin{aligned}\oint_\Gamma \frac{z^{n+m}}{(z-1)^n}\,\mathrm{d}z&=-\oint_{\Gamma'} \frac{z^{n+m}}{(z-1)^n}\,\mathrm{d}z=-2\pi i\operatorname*{Res}_{z=\infty}\frac{z^{n+m}}{(z-1)^n}=2\pi i\operatorname*{Res}_{t=0}\frac{1}{t^{m+2}}\frac{1}{(1-t)^n}\\&=2\pi i\big[t^{m+1}\big]\exp\big[-n\log(1-t)\big]=2\pi i\big[t^{m+1}\big]\exp\left[\sum_{k=1}^{\infty}\big(n(k-1)!\big)\frac{t^k}{k!}\right]\\&=2\pi i\frac{\mathbf{B}_{m+1}(x_1,x_2,\dots,x_{m+1})}{(m+1)!}.\end{aligned}\tag{18}$$

Comparing (17) and (18) and letting $m\mapsto m-1$, we obtain the desired formula (14).

Similarly, we can obtain formula (15). By the definition of complete Bell polynomial, we have

$$\begin{aligned}\frac{\mathbf{B}_m(-n,-n1!,\dots,-n(m-1)!)}{m!}&=\big[t^m\big]\exp\left[\sum_{k=1}^{\infty}\big(-n(k-1)!\big)\frac{t^k}{k!}\right]\\&=\big[t^m\big]\exp\big[n\log(1-t)\big]=\big[t^m\big](1-t)^n=(-1)^m\binom{n}{m}.\end{aligned}$$

Applying (3) to (14), we obtain (16) directly. This completes the proof. □

Lemma 11 *The following algebraic identity holds true*:

$$\frac{x^m}{(x+1)^n}=\sum_{j=0}^{m-n}(-1)^j\binom{n+j-1}{n-1}x^{m-n-j}+\sum_{j=0}^{n-1}(-1)^{m+j}\binom{m}{j}\frac{1}{(x+1)^{n-j}}. \tag{19}$$

Proof Putting $n=1$ and letting $M\mapsto m$ and $\lambda\mapsto n$ in Theorem 1, noting (3), we get that x_i and y_i become

$$x_i=n(i-1)!,\quad i=1,2,\dots,j,$$
$$y_i=-m(i-1)!,\quad i=1,2,\dots,j,$$

respectively. Making use of (14) and (15) from Lemma 10, we obtain (19) immediately. □

• The Apostol–Bernoulli polynomials $\mathcal{B}_n^{(\alpha)}(x;\lambda)$ of order α are defined by means of the generating function (cf. Luo and Srivastava [10]):

$$\left(\frac{z}{\lambda\exp(z)-1}\right)^{\alpha}\exp(xz)=\sum_{n=0}^{\infty}\mathcal{B}_n^{(\alpha)}(x;\lambda)\frac{z^n}{n!} \tag{20}$$
$$\big(|z|<2\pi\ \text{when}\ \lambda=1;|z|<|\log\lambda|\ \text{when}\ \lambda\neq 1\big)$$

with, of course,

$$B_n^{(\alpha)}(x)=\mathcal{B}_n^{(\alpha)}(x;1)\quad\text{and}\quad\mathcal{B}_n^{(\alpha)}(\lambda):=\mathcal{B}_n^{(\alpha)}(0;\lambda), \tag{21}$$

where $\mathcal{B}_n^{(\alpha)}(\lambda)$ denote the so-called Apostol–Bernoulli numbers of order α.

Let $x\mapsto -\lambda\exp(x)$ in (19). We have

$$\begin{aligned}\frac{\lambda^m\exp(mx)}{(\lambda\exp(x)-1)^n}&=\sum_{j=0}^{m-n}\lambda^{m-n-j}\binom{n+j-1}{n-1}\exp\big((m-n-j)x\big)\\&\quad+\sum_{j=0}^{n-1}\binom{m}{j}\frac{1}{(\lambda\exp(x)-1)^{n-j}}.\end{aligned} \tag{22}$$

Multiplying both sides of (22) by x^n and noting (20), we have

$$\begin{aligned}\sum_{k=0}^{\infty}\mathcal{B}_k^{(n)}(m;\lambda)\frac{x^k}{k!}&=\sum_{k=0}^{\infty}\sum_{j=0}^{m-n}\lambda^{-n-j}\binom{n+j-1}{n-1}(m-n-j)^k\frac{x^{n+k}}{k!}\\&\quad+\sum_{k=0}^{\infty}\sum_{j=0}^{n-1}\lambda^{-m}\binom{m}{j}\mathcal{B}_k^{(n-j)}(\lambda)\frac{x^{k+j}}{k!}.\end{aligned} \tag{23}$$

Letting $k \mapsto k-n$ and $k \mapsto k-j$ in the first and second terms on the right, respectively, we obtain

$$\begin{aligned}\sum_{k=0}^{\infty}\mathcal{B}_k^{(n)}(m;\lambda)\frac{x^k}{k!} &= \sum_{k=n}^{\infty}\sum_{j=0}^{m-n}\lambda^{-n-j}n!\binom{k}{n}\binom{n+j-1}{n-1}(m-n-j)^{k-n}\frac{x^k}{k!}\\ &\quad+\sum_{j=0}^{n-1}\sum_{k=j}^{\infty}\lambda^{-m}j!\binom{k}{j}\binom{m}{j}\mathcal{B}_{k-j}^{(n-j)}(\lambda)\frac{x^k}{k!}.\end{aligned}\tag{24}$$

Comparing the coefficients of $\frac{x^k}{k!}$ on both sides of (24), we obtain the following new formula of Apostol–Bernoulli polynomials $\mathcal{B}_n^{(\alpha)}(x;\lambda)$:

$$\begin{aligned}\mathcal{B}_k^{(n)}(m;\lambda) &= \sum_{j=0}^{m-n}\lambda^{-n-j}n!\binom{k}{n}\binom{n+j-1}{n-1}(m-n-j)^{k-n}\\ &\quad+\sum_{j=0}^{n-1}\lambda^{-m}j!\binom{k}{j}\binom{m}{j}\mathcal{B}_{k-j}^{(n-j)}(\lambda).\end{aligned}\tag{25}$$

Taking $\lambda = 1$ in (25), we obtain the following formula of the generalized Bernoulli polynomials at the nonnegative integers:

$$B_k^{(n)}(m) = \sum_{j=0}^{m-n}n!\binom{k}{n}\binom{n+j-1}{n-1}(m-n-j)^{k-n}+\sum_{j=0}^{n-1}j!\binom{k}{j}\binom{m}{j}B_{k-j}^{(n-j)}.\tag{26}$$

Setting $k = n$ in (26), we have

$$B_n^{(n)}(m) = \sum_{j=0}^{m-n}n!\binom{n+j-1}{n-1}+\sum_{j=0}^{n-1}j!\binom{n}{j}\binom{m}{j}B_{n-j}^{(n-j)}.\tag{27}$$

Further setting $m = n$ in (27), we have

$$B_n^{(n)}(n) = n! + \sum_{j=0}^{n-1}j!\binom{n}{j}^2 B_{n-j}^{(n-j)}.\tag{28}$$

Taking $n = 1$ in (26), we obtain the following formula of Bernoulli polynomials at the nonnegative integers:

$$B_k(m) = B_k + k\sum_{j=0}^{m-1}(m-1-j)^{k-1}.\tag{29}$$

• The Apostol–Euler polynomials $\mathcal{E}_n^{(\alpha)}(x;\lambda)$ of order α are defined by means of the generating function(cf. Luo [9]):

$$\left(\frac{2}{\lambda\exp(z)+1}\right)^{\alpha}\exp(xz) = \sum_{n=0}^{\infty}\mathcal{E}_n^{(\alpha)}(x;\lambda)\frac{z^n}{n!}\quad\bigl(|z| < \bigl|\log(-\lambda)\bigr|\bigr),\tag{30}$$

with, of course,

$$E_n^{(\alpha)}(x) = \mathcal{E}_n^{(\alpha)}(x;1) \quad \text{and} \quad \mathcal{E}_n^{(\alpha)}(\lambda) := \mathcal{E}_n^{(\alpha)}(0;\lambda), \tag{31}$$

where $\mathcal{E}_n^{(\alpha)}(\lambda)$ denote the so-called Apostol–Euler numbers of order α.

Letting $x \mapsto \lambda \exp(x)$ in (19), we have

$$\begin{aligned} \frac{\lambda^m \exp(mx)}{(\lambda \exp(x)+1)^n} &= \sum_{j=0}^{m-n} (-1)^j \binom{n+j-1}{n-1} \lambda^{m-n-j} \exp\bigl((m-n-j)x\bigr) \\ &\quad + \sum_{j=0}^{n-1} (-1)^{m+j} \binom{m}{j} \frac{1}{(\lambda \exp(x)+1)^{n-j}}. \end{aligned} \tag{32}$$

Multiplying both sides of (32) by 2^n and using (30), we have

$$\begin{aligned} \sum_{k=0}^{\infty} \mathcal{E}_k^{(n)}(m;\lambda) \frac{x^k}{k!} &= \sum_{k=0}^{\infty} 2^n \sum_{j=0}^{m-n} (-1)^j \lambda^{-n-j} \binom{n+j-1}{n-1} (m-n-j)^k \frac{x^k}{k!} \\ &\quad + \sum_{k=0}^{\infty} \sum_{j=0}^{n-1} (-1)^{m+j} 2^j \lambda^{-m} \binom{m}{j} \mathcal{E}_k^{(n-j)}(\lambda) \frac{x^k}{k!}. \end{aligned} \tag{33}$$

Comparing the coefficients of $\frac{x^k}{k!}$ on both sides of (33), we obtain the following new formula of Apostol–Euler polynomials $\mathcal{E}_n^{(\alpha)}(x;\lambda)$:

$$\begin{aligned} \mathcal{E}_k^{(n)}(m;\lambda) &= 2^n \sum_{j=0}^{m-n} (-1)^j \lambda^{-n-j} \binom{n+j-1}{n-1} (m-n-j)^k \\ &\quad + \sum_{j=0}^{n-1} (-1)^{m+j} 2^j \lambda^{-m} \binom{m}{j} \mathcal{E}_k^{(n-j)}(\lambda). \end{aligned} \tag{34}$$

Taking $\lambda = 1$ in (34), we obtain the following formula of the generalized Euler polynomials at the nonnegative integers:

$$E_k^{(n)}(m) = 2^n \sum_{j=0}^{m-n} (-1)^j \binom{n+j-1}{n-1} (m-n-j)^k + \sum_{j=0}^{n-1} (-1)^{m+j} 2^j \binom{m}{j} E_k^{(n-j)}. \tag{35}$$

Taking $k = n$ in (35), we have

$$E_n^{(n)}(m) = 2^n \sum_{j=0}^{m-n} (-1)^j \binom{n+j-1}{n-1} (m-n-j)^n + \sum_{j=0}^{n-1} (-1)^{m+j} 2^j \binom{m}{j} E_n^{(n-j)}. \tag{36}$$

Further setting $m = n$ in (36), we have

$$E_n^{(n)}(n) = \sum_{j=0}^{n-1} (-1)^{n+j} 2^j \binom{n}{j} E_n^{(n-j)}. \tag{37}$$

Taking $n = 1$ in (35), we obtain the following formula of Euler polynomials at the non-negative integers:

$$E_k(m) = (-1)^m E_k + 2\sum_{j=0}^{m-1}(-1)^j(m-j-1)^k. \tag{38}$$

• The Apostol–Genocchi polynomials of higher order are defined by means of the generating function (cf. Luo and Srivastava [11]):

$$\left(\frac{2z}{\lambda\exp(z)+1}\right)^{\alpha}\exp(xz) = \sum_{n=0}^{\infty}\mathcal{G}_n^{(\alpha)}(x;\lambda)\frac{z^n}{n!} \quad \bigl(|z| < \bigl|\log(-\lambda)\bigr|\bigr), \tag{39}$$

with, of course,

$$\begin{aligned} &G_n^{(\alpha)}(x) = \mathcal{G}_n^{(\alpha)}(x;1), \qquad \mathcal{G}_n^{(\alpha)}(\lambda) := \mathcal{G}_n^{(\alpha)}(0;\lambda), \\ &\mathcal{G}_n(x;\lambda) := \mathcal{G}_n^{(1)}(x;\lambda) \quad \text{and} \quad \mathcal{G}_n(\lambda) := \mathcal{G}_n^{(1)}(\lambda), \end{aligned} \tag{40}$$

where $\mathcal{G}_n(\lambda)$, $\mathcal{G}_n^{(\alpha)}(\lambda)$, and $\mathcal{G}_n(x;\lambda)$ denote the so-called Apostol–Genocchi numbers, Apostol–Genocchi numbers of order α, and Apostol–Genocchi polynomials, respectively.

Letting $x \mapsto \lambda\exp(x)$ in (19), we have

$$\begin{aligned} \frac{\lambda^m\exp(mx)}{(\lambda\exp(x)+1)^n} = {}& \sum_{j=0}^{m-n}(-1)^j\lambda^{m-n-j}\binom{n+j-1}{n-1}\exp\bigl((m-n-j)x\bigr) \\ & + \sum_{j=0}^{n-1}(-1)^{m+j}\binom{m}{j}\frac{1}{(\lambda\exp(x)+1)^{n-j}}. \end{aligned} \tag{41}$$

Multiplying both sides of (41) by $2^n x^n$ and noting (39), we have

$$\begin{aligned} \sum_{k=0}^{\infty}\mathcal{G}_k^{(n)}(m;\lambda)\frac{x^k}{k!} = {}& \sum_{k=0}^{\infty}2^n\sum_{j=0}^{m-n}(-1)^j\lambda^{-n-j}\binom{n+j-1}{n-1}(m-n-j)^k\frac{x^{n+k}}{k!} \\ & + \sum_{k=0}^{\infty}\sum_{j=0}^{n-1}(-1)^{m+j}\lambda^{-m}2^j\binom{m}{j}\mathcal{G}_k^{(n-j)}(\lambda)\frac{x^{k+j}}{k!}. \end{aligned} \tag{42}$$

Let $k \mapsto k-n$ and $k \mapsto k-j$ in the first and second terms on the right, respectively, we have

$$\begin{aligned} \sum_{k=0}^{\infty}\mathcal{G}_k^{(n)}(m;\lambda)\frac{x^k}{k!} = {}& \sum_{k=n}^{\infty}2^n\sum_{j=0}^{m-n}(-1)^j\lambda^{-n-j}n!\binom{k}{n}\binom{n+j-1}{n-1}(m-n-j)^{k-n}\frac{x^k}{k!} \\ & + \sum_{j=0}^{n-1}\sum_{k=j}^{\infty}(-1)^{m+j}2^j\lambda^{-m}j!\binom{k}{j}\binom{m}{j}\mathcal{G}_{k-j}^{(n-j)}(\lambda)\frac{x^k}{k!}. \end{aligned} \tag{43}$$

Comparing the coefficients of $\frac{x^k}{k!}$ on both sides of (43), we obtain the following new formula of Apostol–Genocchi polynomials $\mathcal{G}_n^{(\alpha)}(x;\lambda)$:

$$\begin{aligned}\mathcal{G}_k^{(n)}(m;\lambda) &= 2^n n! \sum_{j=0}^{m-n} (-1)^j \lambda^{-n-j} \binom{k}{n} \binom{n+j-1}{n-1} (m-n-j)^{k-n} \\ &\quad + \sum_{j=0}^{n-1} (-1)^{m+j} 2^j \lambda^{-m} j! \binom{k}{j} \binom{m}{j} \mathcal{G}_{k-j}^{(n-j)}(\lambda). \end{aligned} \tag{44}$$

Taking $\lambda = 1$ in (44), we obtain the following formula of the generalized Genocchi polynomials at the nonnegative integers:

$$\begin{aligned} G_k^{(n)}(m) &= 2^n n! \sum_{j=0}^{m-n} (-1)^j \binom{k}{n} \binom{n+j-1}{n-1} (m-n-j)^{k-n} \\ &\quad + \sum_{j=0}^{n-1} (-1)^{m+j} 2^j j! \binom{k}{j} \binom{m}{j} G_{k-j}^{(n-j)}. \end{aligned} \tag{45}$$

Taking $k = n$ in (45), we have

$$G_n^{(n)}(m) = 2^n n! \sum_{j=0}^{m-n} (-1)^j \binom{n+j-1}{n-1} + \sum_{j=0}^{n-1} (-1)^{m+j} 2^j j! \binom{n}{j} \binom{m}{j} G_{n-j}^{(n-j)}. \tag{46}$$

Further setting $m = n$ in (46), we have

$$G_n^{(n)}(n) = 2^n n! + \sum_{j=0}^{n-1} (-1)^{n+j} 2^j j! \binom{n}{j}^2 G_{n-j}^{(n-j)}. \tag{47}$$

Taking $n = 1$ in (45), we obtain the following formula of Genocchi polynomials at the nonnegative integers:

$$G_k(m) = (-1)^m G_k + 2k \sum_{j=0}^{m-1} (-1)^j (m-j-1)^{k-1}. \tag{48}$$

• The generalized Apostol-type polynomials

$$\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu;\nu) \quad (\alpha, \lambda, \mu, \nu \in \mathbb{C})$$

of (real or complex) order α are defined by means of the following generating function (cf. Luo and Srivastava [11]):

$$\left(\frac{2^\mu z^\nu}{\lambda \exp(z) + 1} \right)^\alpha \exp(xz) = \sum_{n=0}^{\infty} \mathcal{F}_n^{(\alpha)}(x;\lambda;\mu;\nu) \frac{z^n}{n!} \quad \left(|z| < \left|\log(-\lambda)\right|; 1^\alpha := 1 \right), \tag{49}$$

with, of course, we have

$$\mathcal{B}_n^{(\alpha)}(x;\lambda) = (-1)^\alpha \mathcal{F}_n^{(\alpha)}(x;-\lambda;0;1), \tag{50}$$

$$\mathcal{E}_n^{(\alpha)}(x;\lambda) = \mathcal{F}_n^{(\alpha)}(x;\lambda;1;0) \tag{51}$$

and

$$\mathcal{G}_n^{(\alpha)}(x;\lambda) = \mathcal{F}_n^{(\alpha)}(x;\lambda;1;1). \tag{52}$$

Letting $x \mapsto \lambda\exp(x)$ in (19), we have

$$\begin{aligned}\frac{\lambda^m \exp(mx)}{(\lambda\exp(x)+1)^n} &= \sum_{j=0}^{m-n}(-1)^j\lambda^{m-n-j}\binom{n+j-1}{n-1}\exp\bigl((m-n-j)x\bigr)\\ &\quad+\sum_{j=0}^{n-1}(-1)^{m+j}\binom{m}{j}\frac{1}{(\lambda\exp(x)+1)^{n-j}}.\end{aligned}\tag{53}$$

Multiplying both sides of (53) by $2^{n\mu}x^{n\nu}$ and noting (49), we get

$$\begin{aligned}\sum_{k=0}^{\infty}\mathcal{F}_k^{(n)}(m;\lambda;\mu;\nu)\frac{x^k}{k!} &= \sum_{k=0}^{\infty}2^{n\mu}\sum_{j=0}^{m-n}(-1)^j\lambda^{-n-j}\binom{n+j-1}{n-1}(m-n-j)^k\frac{x^{k+n\nu}}{k!}\\ &\quad+\sum_{k=0}^{\infty}\sum_{j=0}^{n-1}(-1)^{m+j}\lambda^{-m}\binom{m}{j}2^{j\mu}\mathcal{F}_k^{(n-j)}(\lambda;\mu;\nu)\frac{x^{k+j\nu}}{k!}.\end{aligned}\tag{54}$$

Letting $k \mapsto k-n\nu$ and $k \mapsto k-j\nu$ in the first and second terms on the right, respectively, we have

$$\begin{aligned}\sum_{k=0}^{\infty}\mathcal{F}_k^{(n)}(m;\lambda;\mu;\nu)\frac{x^k}{k!} &= \sum_{k=n\nu}^{\infty}2^{n\mu}\sum_{j=0}^{m-n}(-1)^j\lambda^{-n-j}(n\nu)!\binom{k}{n\nu}\binom{n+j-1}{n-1}(m-n-j)^{k-n\nu}\frac{x^k}{k!}\\ &\quad+\sum_{j=0}^{n-1}\sum_{k=j\nu}^{\infty}(-1)^{m+j}(j\mu)!\lambda^{-m}2^{j\mu}\binom{k}{j\mu}\binom{m}{j}\mathcal{F}_{k-j\nu}^{(n-j)}(\lambda;\mu;\nu)\frac{x^k}{k!}.\end{aligned}\tag{55}$$

Comparing the coefficients of $\frac{x^k}{k!}$ on both sides of (55), we obtain the following new formula of Apostol-type polynomials $\mathcal{F}_k^{(n)}(m;\lambda;\mu;\nu)$:

$$\begin{aligned}\mathcal{F}_k^{(n)}(m;\lambda;\mu;\nu) &= 2^{n\mu}\sum_{j=0}^{m-n}(-1)^j\lambda^{-n-j}(n\nu)!\binom{k}{n\nu}\binom{n+j-1}{n-1}(m-n-j)^{k-n\nu}\\ &\quad+\sum_{j=0}^{n-1}(-1)^{m+j}(j\mu)!\lambda^{-m}2^{j\mu}\binom{k}{j\mu}\binom{m}{j}\mathcal{F}_{k-j\nu}^{(n-j)}(\lambda;\mu;\nu).\end{aligned}\tag{56}$$

Taking $\lambda = 1$ in (56), we obtain the following formula of Apostol-type polynomials at the nonnegative integers;

$$\begin{aligned}F_k^{(n)}(m;\mu;\nu) &= 2^{n\mu}\sum_{j=0}^{m-n}(-1)^j(n\nu)!\binom{k}{n\nu}\binom{n+j-1}{n-1}(m-n-j)^{k-n\nu}\\ &\quad+\sum_{j=0}^{n-1}(-1)^{m+j}(j\mu)!2^{j\mu}\binom{k}{j\mu}\binom{m}{j}F_{k-j\nu}^{(n-j)}(\mu;\nu).\end{aligned}\tag{57}$$

Taking $n = 1$ in (57), we obtain the following formula of the Apostol-type polynomials at the nonnegative integers:

$$F_k(m;\mu;\nu) = (-1)^m F_k(\mu;\nu) + 2^\mu \nu! \binom{k}{\nu} \sum_{j=0}^{m-1} (-1)^j (m-j-1)^{k-\nu}. \tag{58}$$

4 Applications to the harmonic Stirling numbers of the second kind

The number $S(n,k)$ of k-partitions is called a Stirling number of the second kind. Hence $S(n,k) > 0$ for $1 \leq k \leq n$ and

$$S(n,k) = 0 \quad \text{if } 1 \leq n < k.$$

We put $S(0,0) = 1$ and $S(0,k) = 0$ for $k \geq 1$.

Theorem 12 ([6, p. 204, Theorem A]) *The following explicit representation formulas hold true*:

$$S(n,k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n \tag{59}$$

and

$$S(n,k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n. \tag{60}$$

Theorem 13 ([6, p. 207, [2d], Theorem C]) *The rational generating function for the Stirling numbers of the second kind is*

$$\sum_{n=0}^{\infty} S(n,k) u^n = \frac{u^k}{(1-u)(1-2u)(1-3u)\cdots(1-ku)}, \quad k \geq 1. \tag{61}$$

Below we further generalize the above classical Stirling numbers of the second kind and obtain some new formulas using our main result Theorem 1.

Definition 14 The harmonic Stirling numbers of the second kind are defined by means of the following generating function:

$$\sum_{n=0}^{\infty} HS(n,k) u^n = \frac{u^k}{(1-u)(2-u)\cdots(k-u)}, \quad k \geq 1. \tag{62}$$

Definition 15 The harmonic Stirling numbers of the second kind of higher order are defined by means of the following generating function:

$$\sum_{n=0}^{\infty} HS^{(r)}(n,k) u^n = \frac{u^k}{[(1-u)(2-u)\cdots(k-u)]^r}, \quad k,r \geq 1. \tag{63}$$

Definition 16 The generalized harmonic Stirling numbers of the second kind are defined by means of the following generating function:

$$\sum_{n=0}^{\infty} HS(n,k;m)u^n = \frac{u^m}{(1-u)(2-u)\cdots(k-u)}, \quad m \geq 0, k \geq 1. \tag{64}$$

Definition 17 The generalized harmonic Stirling numbers of the second kind of higher order are defined by means of the following generating function:

$$\sum_{n=0}^{\infty} HS^{(r)}(n,k;m)u^n = \frac{u^m}{[(1-u)(2-u)\cdots(k-u)]^r}, \quad m \geq 0, k, r \geq 1. \tag{65}$$

For other generalized Stirling numbers of the second kind, we refer to [1–4] and the references therein.

Letting $M \mapsto m$, $n \mapsto k$, $\lambda \mapsto r$, $x \mapsto -x$ in Theorem 1 and using (3), we obtain

$$\begin{aligned} \frac{x^m}{[(x-1)(x-2)\cdots(x-k)]^r} &= \sum_{n=0}^{m-rk} \frac{\mathbf{B}_{m-rk-n}(x_1,\ldots,x_{m-rk-n})}{(m-rk-n)!}x^n \\ &\quad + \sum_{j=1}^{k} \frac{(-1)^{r(k-j)}}{k!^r}\binom{k}{j}^r j^{m+r} \sum_{l=0}^{r-1} \frac{(-1)^l \mathbf{B}_l(y_1,y_2,\ldots,y_l)}{l!(x-j)^{r-l}}, \end{aligned} \tag{66}$$

where

$$x_s = r(s-1)!\sum_{j=1}^{k} j^s, \quad s = 1,2,\ldots,m-rk, \tag{67}$$

$$y_s = (s-1)!\left[r\left(H_j^{(s)} + (-1)^s H_{k-j}^{(s)}\right) - \frac{m+r}{j^s}\right], \quad s = 1,2,\ldots,r-1. \tag{68}$$

By (65) and the binomial theorem,

$$(1-z)^{-r} = \sum_{n=0}^{\infty} \binom{r+n-1}{r-1} z^n \quad \left(|z| < 1\right),$$

we have

$$\begin{aligned} &\sum_{n=0}^{\infty} (-1)^{rk} HS^{(r)}(n,k;m)x^n \\ &\quad = \sum_{n=0}^{\infty} \frac{\mathbf{B}_{m-rk-n}(x_1,\ldots,x_{m-rk-n})}{(m-rk-n)!}x^n \\ &\qquad + \sum_{n=0}^{\infty}\sum_{j=1}^{k} \frac{(-1)^{r(k-j+1)}}{k!^r}\binom{k}{j}^r j^{m-n} \sum_{l=0}^{r-1} j^l \binom{n+r-l-1}{r-l-1} \frac{\mathbf{B}_l(y_1,y_2,\ldots,y_l)}{l!}x^n. \end{aligned} \tag{69}$$

Comparing the coefficients of x^n on both sides of (69), we obtain the formula of the generalized harmonic Stirling numbers of the second kind of higher order $HS^{(r)}(n,k;m)$:

$$\begin{aligned} &HS^{(r)}(n,k;m) \\ &\quad = (-1)^{rk}\frac{\mathbf{B}_{m-rk-n}(x_1,\ldots,x_{m-rk-n})}{(m-rk-n)!} \\ &\qquad + \frac{1}{k!^r}\sum_{j=1}^{k}(-1)^{r(j-1)}\binom{k}{j}^r j^{m-n}\sum_{l=0}^{r-1}j^l\binom{n+r-l-1}{r-l-1}\frac{\mathbf{B}_l(y_1,y_2,\ldots,y_l)}{l!}, \end{aligned} \tag{70}$$

where x_s and y_s are given by (67) and (68), respectively.

When $m-rk-n<0$ in (70),

$$HS^{(r)}(n,k;m) = \frac{1}{k!^r}\sum_{j=1}^{k}(-1)^{r(j+1)}\binom{k}{j}^r j^{m-n}\sum_{l=0}^{r-1}j^l\binom{n+r-l-1}{r-l-1}\frac{\mathbf{B}_l(y_1,y_2,\ldots,y_l)}{l!}. \tag{71}$$

When $m-rk-n=0$ in (70),

$$\begin{aligned} &HS^{(r)}(n,k;m) \\ &\quad = (-1)^{rk} + \frac{1}{k!^r}\sum_{j=1}^{k}(-1)^{r(j+1)}\binom{k}{j}^r j^{rk}\sum_{l=0}^{r-1}j^l\binom{n+r-l-1}{r-l-1}\frac{\mathbf{B}_l(y_1,y_2,\ldots,y_l)}{l!}, \end{aligned} \tag{72}$$

where y_s is given by (68).

Taking $r=1$, $m=k$ in (70), we obtain a formula of the harmonic Stirling numbers of the second kind $HS(n,k)$:

$$HS(n,k) = \frac{1}{k!}\sum_{j=1}^{k}(-1)^{j-1}\binom{k}{j}j^{k-n}. \tag{73}$$

Taking $m=k$ in (70), we obtain a formula of the harmonic Stirling numbers of the second kind of higher order $HS^{(r)}(n,k)$:

$$HS^{(r)}(n,k) = \frac{1}{k!^r}\sum_{j=1}^{k}(-1)^{r(j-1)}\binom{k}{j}^r j^{k-n}\sum_{l=0}^{r-1}j^l\binom{n+r-l-1}{r-l-1}\frac{\mathbf{B}_l(y_1,y_2,\ldots,y_l)}{l!}, \tag{74}$$

where y_s is given by (68).

Taking $r=1$ in (70), we obtain a formula of the generalized harmonic Stirling numbers of the second kind $HS(n,k;m)$:

$$HS(n,k;m) = (-1)^k\frac{\mathbf{B}_{m-k-n}(x_1,\ldots,x_{m-k-n})}{(m-k-n)!} + \frac{1}{k!}\sum_{j=1}^{k}(-1)^{j-1}\binom{k}{j}j^{m-n}, \tag{75}$$

where

$$x_s = (s-1)!\sum_{j=1}^{k}j^s, \quad s=1,2,\ldots,m-k.$$

When $m-k<n$ in (75),

$$HS(n,k;m)=\frac{1}{k!}\sum_{j=1}^{k}(-1)^{j-1}\binom{k}{j}j^{m-n}=(-1)^{k-1}S(m-n,k)\equiv 0. \tag{76}$$

When $m-k=n$ in (75),

$$HS(n,k)=(-1)^{k}+\frac{1}{k!}\sum_{j=1}^{k}(-1)^{j+1}\binom{k}{j}j^{k}=(-1)^{k}+(-1)^{k-1}S(k,k)\equiv 0. \tag{77}$$

When $m-k>n$ in (75),

$$HS(n,k;m)=(-1)^{k-1}S(m-n,k)+(-1)^{k}\frac{\mathbf{B}_{m-k-n}(x_1,\ldots,x_{m-k-n})}{(m-k-n)!}. \tag{78}$$

Definition 18 (Knuth, [8, p. 264, (7)]) The generalized Stirling numbers of the second kind $S(-n,k)$ are defined by means of the following sum:

$$S(-n,k):=\frac{1}{k!}\sum_{j=1}^{k}(-1)^{k-j}\binom{k}{j}j^{-n}\quad (n,k\in\mathbb{N}). \tag{79}$$

When $m=0$ in (75),

$$HS(n,k;0)=\frac{1}{k!}\sum_{j=1}^{k}(-1)^{j-1}\binom{k}{j}j^{-n}=(-1)^{k-1}S(-n,k). \tag{80}$$

When $m<n$ in (75),

$$HS(n,k;m)=\frac{1}{k!}\sum_{j=1}^{k}(-1)^{j-1}\binom{k}{j}j^{m-n}=(-1)^{k-1}S(m-n,k). \tag{81}$$

5 Conclusions

We change the form of (62) as follows:

$$\sum_{n=0}^{\infty}HS(n,k)u^{n}=\frac{1}{k!}\frac{u^{k}}{(1-\frac{u}{1})(1-\frac{u}{2})\cdots(1-\frac{u}{k})},\quad k\geq 1.$$

Comparing the above form and the definition of the classical Stirling numbers of the second kind (61), we call the coefficients in (62) the harmonic Stirling numbers of the second kind.

As is well known, the basic (or q-) series and basic (or q-) polynomials, especially the basic (or q-) hypergeometric functions and basic (or q-) hypergeometric polynomials, are known to have widespread applications, particularly in several areas of number theory and combinatorial analysis, in particular the theory of partitions.

Recently, Srivastava [19–21] published a survey-cum-expository article on the q-calculus and the fractional q-calculus in geometric function theory of complex analysis, and gave a survey of some recent developments on higher transcendental functions of analytic number theory and applied mathematics, as well as some operators of fractional calculus, related special functions, and integral transformations. In [19, p. 340], professor Srivastava pointed out an important demonstrated observation that any (p,q)-variation of the proposed q-results would be trivially inconsequential, because the additional parameter p is obviously redundant. Hence we suggest the corresponding basic (or q-) extensions of the results of this paper.

In this concluding section, we mention that for a general improper rational function, with m and n being the degrees of the numerator and denominator polynomials, $p(z)$ and $q(z)$, of this improper rational function, respectively, decomposing such a function into a polynomial plus a proper rational fraction is usually very difficult. However, can we decompose a general improper rational function $p(z)/q(z)$ into partial fractions?

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