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Dynamical analysis in controlled globally coupled map lattices

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Abstract

This paper investigates a globally coupled map lattice. Rigorous proofs to the existence of chaos in the sense of both Li–Yorke and Devaney in two controlled globally coupled map lattices are presented. In addition, the existence of Li–Yorke chaos and Devaney chaos for a general discrete dynamical system in $\mathbf{R}^N$ and l^∞ is considered. For illustration, two examples are provided.

Keywords: Li–Yorke chaos; Devaney chaos; Coupled map lattice; Dynamical system

1 Introduction

Chaotification (also called anticontrol of chaos [1]) is a process of making an originally nonchaotic dynamical system chaotic, or enhancing a chaotic system to show a stronger or different type of chaos. It has been found that chaos is useful under some circumstances, such as in human-brain analysis [2], heartbeat regulation [3], and digital communications [4]. Now, it has attracted increasing attention partially due to its great potential in many nontraditional applications [5–11].

The Couple Map Lattice (CML) is a classical spatiotemporal chaos proposed by Kaneko, which spatially divides the system into various lattices and represents a kind of dynamics evolution both in time and space [6]. There are many types of coupling, e.g., one-way coupling [7], nearest-neighbor coupling [8], global coupling [9], etc. A globally coupled map lattice (GCML) is a discrete time dynamical system where elements interact with all other elements [6, 10].

There are many works on the existence of chaos in CML and GCML. In 2007, the following CML with nearest-neighbor interaction:

$$x_{m+1,n} = (1-\varepsilon)f(x_{m,n}) + 0.5\varepsilon\big(f(x_{m,n-1}) + f(x_{m,n+1})\big)$$

was studied by Tian and Chen, and it was proved that the CML is Li–Yorke chaotic under some conditions [8]. In 2011, the following GCML:

$$x^j_{k+1} = (1-\varepsilon)f\big(x^j_k\big) + \frac{\varepsilon}{L}\sum_{i=1,i\neq j}^{L+1} f\big(x^i_k\big),$$

where $f(x) = \mu x(1-x)$ is the logistic map, was studied by Khellat et al. and it was proved that Li–Yorke chaos and synchronous chaos will appear when it satisfies certain conditions [9]. In 2019, the following GCML with delays:

$$x_{n+1}(i) = (1-\varepsilon)f\bigl(x_n(i)\bigr) + \frac{\varepsilon}{N}\sum_{j=1,j\neq i}^{N+1} f\bigl(x_{n-1}(j)\bigr),$$

where $f(x) = \mu x(1-x)$, was investigated and it was proved to be chaotic in the sense of Li–Yorke using modified Marotto's theorem [11]. Recently, chaotification of the CML $x(n+1,m) = f(x(n,m), x(n,m+1))$ with general controllers, sawtooth functions, basic elementary functions, and polynomial maps are investigated and all the systems are proved to be chaotic in the sense of both Li–Yorke and Devaney by applying coupled-expanding and snap-back repeller theories [12–18].

To the best of our knowledge, there are few results on designing a controller such that the controlled GCML is chaotic in the sense of both Li–Yorke and Devaney. This fact motivates us to explore mathematically the existence of chaos in controlled GCMLs. In this paper, we consider the following original GCML:

$$x(n+1,m) = (1-\varepsilon)f\bigl(x(n,m)\bigr) + \frac{\varepsilon}{N}\sum_{j=1,j\neq m}^{N+1} f\bigl(x(n,j)\bigr), \tag{1}$$

where $\varepsilon \in (0,1)$, $n \in \mathbf{Z}$, $m = 1,2,\dots,N+1$, and $N+1$ is the number of sites in the GCML, and the objective here is to design controllers $\alpha g(\beta x(n,m))$ and $\frac{\alpha}{N}\sum_{j=1,j\neq m}^{N+1} g(\beta x(n,j))$ such that the output of the following controlled systems:

$$x(n+1,m) = (1-\varepsilon)f\bigl(x(n,m)\bigr) + \frac{\varepsilon}{N}\sum_{j=1,j\neq m}^{N+1} f\bigl(x(n,j)\bigr) + \alpha g\bigl(\beta x(n,m)\bigr), \tag{2}$$

$$x(n+1,m) = (1-\varepsilon)f\bigl(x(n,m)\bigr) + \frac{\varepsilon}{N}\sum_{j=1,j\neq m}^{N+1} f\bigl(x(n,j)\bigr) + \frac{\alpha}{N}\sum_{j=1,j\neq m}^{N+1} g\bigl(\beta x(n,j)\bigr), \tag{3}$$

where $g : I \subset \mathbf{R} \to \mathbf{R}$, α and β are constants, are chaotic in the sense of both Li–Yorke and Devaney.

The rest of the paper is organized as follows. Section 2 contains a lemma about chaos and reformulation of (1)–(3). Mathematically rigorous verification of chaos is always preferable, so, in Sect. 3, the existence of chaos in the sense of both Li–Yorke and Devaney in a general discrete dynamical system is verified by employing the snap-back repeller theory. Then, the existence of chaos in the sense of both Li–Yorke and Devaney in (2) and (3) is studied in Sect. 4, and the value of α can be made arbitrarily small if β is large enough. Two illustrative examples are provided in Sect. 5 with computer simulations.

2 Preliminaries

In this section, we introduce two basic concepts and a lemma about chaos, and then reformulate Eqs. (1)–(3).

Definition 1 ([19]) Let (X,d) be a metric space and $F: X \to X$ be a map. A subset S of X is called a scrambled set of F, if for any two different points $x, y \in S$,

$$\liminf_{n\to\infty} d\bigl(F^n(x), F^n(y)\bigr) = 0, \qquad \limsup_{n\to\infty} d\bigl(F^n(x), F^n(y)\bigr) > 0.$$

The map F is said to be chaotic in the sense of Li–Yorke if there exists an uncountable scrambled set S of F.

Definition 2 ([20]) Let (X,d) be a metric space. A map $F: V \subset X \to V$ is said to be chaotic on V in the sense of Devaney if:

(i) F is topologically transitive in V;
(ii) the periodic points of F in V are dense in V;
(iii) F has sensitive dependence on the initial conditions in V.

By $\|L\|$ denote the norm of a bounded linear operator L on $\mathbf{R}^n$, that is, $\|L\| := \max\{\|Lx\| : x \in \mathbf{R}^n \text{ with } \|x\| = 1\}$. For a linear map $L: X \to X$, where $(X, \|\cdot\|)$ is a Banach space, let $\|L\|^0 := \inf\{\|Lx\| : x \in X \text{ with } \|x\| = 1\}$. If a bounded linear map $L: X \to X$ is bijective and has a bounded linear inverse map, then L is called an invertible linear map. By $B_r(x)$ and $\bar{B}_r(x)$ denote the open and closed balls of radius r, centered at $x \in X$, respectively. The following lemma about chaos is introduced.

Lemma 1 ([12]) *Let $(X, \|\cdot\|)$ be a Banach space and $F: X \to X$ be a map with a fixed point $z \in X$. Assume that*

(i) *F is continuously differential in $B_{r_0}(z)$ for some $r_0 > 0$ and $DF(z)$ is an invertible linear map satisfying $\|DF(z)\|^0 > 1$, which is equivalent to saying that there exists a positive constant $r \le r_0$ such that z is a regular expanding fixed point of F in $\overline{B}_r(z)$;*
(ii) *z is a snap-back repeller of F with $F^m(x_0) = z$ for some $x_0 \in B_r(z)$, $x_0 \ne z$, and for some positive integer m. Furthermore, F is continuously differentiable in some neighborhoods of $x_1, \dots, x_{m-1}$, respectively, satisfying that $DF(x)$ is an invertible linear map for all $x \in B_r(z)$ and for $x = x_j$, and $\|DF(x_j)\|^0 > 0$ for $1 \le j \le m-1$, where $x_j = F(x_{j-1})$.*

Then, for any neighborhood U of z, there exist an integer $n > m$ and a Cantor set $\Lambda \subset U$ such that $F^n: \Lambda \to \Lambda$ is topologically conjugate to the symbolic dynamical system $\sigma: \Sigma_2^+ \to \Sigma_2^+$. Consequently, there exists a compact and perfect invariant set $D \subset X$, containing a Cantor set, such that $x_{n+1} = F(x_n)$ $(n \ge 0)$ is chaotic on D in the sense of both Devaney and Li–Yorke, and has a dense orbit in D.

Next, we reformulate GCMLs (1)–(3) into special discrete systems.

Let $x(n) = (x(n,1), x(n,2), \dots, x(n,N+1))^T$, then GCML (1) can be written as

$$x(n+1) = \begin{pmatrix} (1-\varepsilon)f(x(n,1)) + \frac{\varepsilon}{N}\sum_{j=2}^{N+1} f(x(n,j)) \\ (1-\varepsilon)f(x(n,2)) + \frac{\varepsilon}{N}\sum_{j=1,j\ne 2}^{N+1} f(x(n,j)) \\ \vdots \\ (1-\varepsilon)f(x(n,N+1)) + \frac{\varepsilon}{N}\sum_{j=1}^{N} f(x(n,j)) \end{pmatrix} := F\bigl(x(n)\bigr). \tag{4}$$

Thus, the controlled GCMLs (2) and (3) can be written as the following general discrete dynamical systems:

$$H\bigl(\varepsilon,\alpha,\beta,x(n)\bigr)=F\bigl(x(n)\bigr)+\alpha G\bigl(\beta x(n)\bigr), \tag{5}$$

$$\widetilde{H}\bigl(\varepsilon,\alpha,\beta,x(n)\bigr)=F\bigl(x(n)\bigr)+\alpha \widetilde{G}\bigl(\beta x(n)\bigr), \tag{6}$$

respectively, where

$$G\bigl(x(n)\bigr)=\bigl(g\bigl(x(n,1)\bigr),g\bigl(x(n,2)\bigr),\ldots,g\bigl(x(n,N+1)\bigr)\bigr)^T,$$

$$\widetilde{G}\bigl(x(n)\bigr)=\frac{1}{N}\Biggl(\sum_{j=2}^{N+1}g\bigl(x(n,j)\bigr),\sum_{j=1,j\neq 2}^{N+1}g\bigl(x(n,j)\bigr),\ldots,\sum_{j=1,j\neq N+1}^{N+1}g\bigl(x(n,j)\bigr)\Biggr)^T.$$

Definition 3 The GCMLs (1)–(3) are said to be chaotic in the sense of Li–Yorke (or Devaney) on $V\subset\mathbf{R}^{N+1}$ if their induced systems (4)–(6) are chaotic in the sense of Li–Yorke (or Devaney) on $V\subset\mathbf{R}^{N+1}$, respectively.

3 Existence of chaos in a general discrete dynamical system

In this section, a general discrete dynamical system is considered, and the existence of chaos in the sense of both Li–Yorke and Devaney is investigated by employing the snap-back repeller theory.

For convenience, denote $I^N:=\underbrace{I\times I\times\cdots\times I}_{N}$ for $I\subset\mathbf{R}$.

Theorem 1 *Consider the controlled system*

$$x_{n+1}=F(x_n)+\alpha G(\beta x_n),\quad n\geq 0, \tag{7}$$

in $\mathbf{R}^N(N<\infty)$ *or* $l^\infty(N=\infty)$, *where the controller is* $\alpha G(\beta x_n)$. *Assume that*

(i) $x^*=0$ *is a fixed point of* F *and there exist positive constants* r *and* L *such that* F *is continuous in* $[-r,r]^N$ *and continuously differentiable in* $(-r,r)^N$, *satisfying*

$$\bigl\|DF(x)\bigr\|\leq L,\quad \forall x\in(-r,r)^N; \tag{8}$$

(ii) *G satisfies the following conditions*:

(iia) G *is continuous in* $[-r,r]^N\cup[a,b]^N$ *and continuously differentiable in* $(-r,r)^N\cup(a,b)^N$ *with* $r<a<b$;

(iib) $x^*=0$ *is a fixed point of* G *and there exists a point* $\xi\in(a,b)^N$ *such that* $G(\xi)=0$;

(iic) $DG(x)$ *is an invertible linear operator for each* $x\in(-r,r)^N\cup(a,b)^N$ *and there exists a constant* $\lambda>0$ *such that*

$$\bigl\|G(x)-G(y)\bigr\|\geq\lambda\|x-y\|,\quad \forall x,y\in[-r,r]^N \textit{ and } \forall x,y\in[a,b]^N. \tag{9}$$

Then, for any constants α, β *satisfying* $|\beta|>b/r$ *and*

$$|\alpha\beta|>C_0:=\max\biggl\{\frac{Lr+b}{\lambda r},\frac{Lb}{\lambda(\|\xi\|_0-a)},\frac{Lb}{\lambda(b-\|\xi\|)}\biggr\},$$

where $\|\xi\|_0 = \min\{|\xi_i| : 0 \le i \le N\}$, *system* (7) *is chaotic in the sense of both Li–Yorke and Devaney in the neighborhood of the origin.*

Proof Lemma 1 is used to prove this theorem. The proof is motivated by some ideas in the proof of Theorem 3.1 in [12] and Theorem 2.1 in [21].

First, we will prove that the results herein hold in the case of $\beta > b/r$ and $\alpha\beta > C_0$. In this case, $[-\beta^{-1}r, \beta^{-1}r]^N, [\beta^{-1}a, \beta^{-1}b]^N \subset [-r, r]^N$. The proof is divided into four parts.

Step 1. $x^* = 0$ is an expanding fixed point of $H_{\alpha,\beta}$ in $[-\beta^{-1}r, \beta^{-1}r]^N$.

By assumptions (i) and (iib), $x^* = 0$ is a fixed point of $H_{\alpha,\beta}$. From (8) and Lemma 2.3 in [12], one has

$$\left\|F(x) - F(y)\right\| \le L\|x - y\|, \quad \forall x, y \in [-r, r]^N. \tag{10}$$

Then, it follows from (9) and (10) that for all $x, y \in [-\beta^{-1}r, \beta^{-1}r]^N$,

$$\left\|H_{\alpha,\beta}(x) - H_{\alpha,\beta}(y)\right\| \ge (\alpha\beta\lambda - L)\|x - y\|.$$

Hence, $x^* = 0$ is an expanding fixed point of $H_{\alpha,\beta}$ in $[-\beta^{-1}r, \beta^{-1}r]^N$ due to $\alpha\beta\lambda - L > 1$.

Step 2. $x^* = 0$ is a snap-back repeller of $H_{\alpha,\beta}$ in $[-\beta^{-1}r, \beta^{-1}r]^N$, that is, there exists a point $x_0 \in (-\beta^{-1}r, \beta^{-1}r)^N$ and $x_0 \ne 0$ such that $H^2_{\alpha,\beta}(x_0) = 0$. This part is divided into two subparts.

Step 2a. There exists a point $x_1 \in (\beta^{-1}a, \beta^{-1}b)^N$ such that $H_{\alpha,\beta}(x_1) = 0$.

To achieve this, consider the equation

$$H_{\alpha,\beta}(x) = 0, \tag{11}$$

in $[\beta^{-1}a, \beta^{-1}b]^N$, which can be written as

$$-\bigl(F(x) + \xi\bigr) = \alpha G(\beta x) - \xi := G_{\alpha,\beta}(x). \tag{12}$$

By (9), for all $x, y \in [\beta^{-1}a, \beta^{-1}b]^N$,

$$\left\|G_{\alpha,\beta}(x) - G_{\alpha,\beta}(y)\right\| = \left\|\alpha G(\beta x) - \alpha G(\beta y)\right\| \ge \alpha\beta\lambda\|x - y\|.$$

Hence, $G_{\alpha,\beta} : [\beta^{-1}a, \beta^{-1}b]^N \to G_{\alpha,\beta}([\beta^{-1}a, \beta^{-1}b]^N)$ is invertible, and its inverse $G^{-1}_{\alpha,\beta} : G_{\alpha,\beta}([\beta^{-1}a, \beta^{-1}b]^N) \to [\beta^{-1}a, \beta^{-1}b]^N$ is continuous and satisfies

$$\left\|G^{-1}_{\alpha,\beta}(x) - G^{-1}_{\alpha,\beta}(y)\right\| \le (\alpha\beta\lambda)^{-1}\|x - y\|, \quad \forall x, y \in G_{\alpha,\beta}\bigl(\bigl[\beta^{-1}a, \beta^{-1}b\bigr]^N\bigr). \tag{13}$$

Now, it is to show that

$$-\bigl(F(x) + \xi\bigr) \in G_{\alpha,\beta}\bigl(\bigl[\beta^{-1}a, \beta^{-1}b\bigr]^N\bigr), \quad \forall x \in \bigl[\beta^{-1}a, \beta^{-1}b\bigr]^N, \tag{14}$$

i.e., to prove that for all $x \in [\beta^{-1}a, \beta^{-1}b]^N$,

$$-F(x) \in \alpha G\bigl(\beta\bigl[\beta^{-1}a, \beta^{-1}b\bigr]^N\bigr) := \bigl\{\alpha G(\beta x) : x \in \bigl[\beta^{-1}a, \beta^{-1}b\bigr]^N\bigr\}. \tag{15}$$

In fact, it follows from (10) that

$$\left\|F(x)\right\| \le L\|x\| \le \beta^{-1}Lb, \quad \forall x \in \left[\beta^{-1}a, \beta^{-1}b\right]^N. \tag{16}$$

Obviously, $\beta^{-1}\xi \in (\beta^{-1}a, \beta^{-1}b)^N$ and $G(\beta\beta^{-1}\xi) = 0$. Hence, $0 \in \alpha G(\beta[\beta^{-1}a, \beta^{-1}b]^N)$. Consequently, for each $x \in \partial([\beta^{-1}a, \beta^{-1}b]^N)$, it follows from (9) that

$$\begin{aligned}\left\|\alpha G(\beta x)\right\| &= \left\|\alpha G(\beta x) - \alpha G(\xi)\right\| \ge \alpha\lambda\|\beta x - \xi\| \\ &\ge \alpha\lambda \min\left\{\|\xi\|_0 - a, b - \|\xi\|\right\} > \beta^{-1}Lb. \end{aligned}\tag{17}$$

Further, since $DG(\beta x)$ is invertible at each point $x \in (\beta^{-1}a, \beta^{-1}b)^N$ by assumption (iic), $G(\beta(\beta^{-1}a, \beta^{-1}b)^N)$ is open due to [22]. This, together with (16) and (17), implies that (15), i.e., (14) holds. Therefore, (12) can be written as

$$G_{\alpha,\beta}^{-1}\left(-\left(F(x) + \xi\right)\right) = x, \quad \forall x \in \left[\beta^{-1}a, \beta^{-1}b\right]^N. \tag{18}$$

Let $h_1(x) := G_{\alpha,\beta}^{-1}(-(F(x) + \xi))$. By assumption (iib),

$$G_{\alpha,\beta}\left(\beta^{-1}\xi\right) = \alpha G(\xi) - \xi = -\xi, \tag{19}$$

then for each $x \in [\beta^{-1}a, \beta^{-1}b]^N$,

$$h_1(x) = G_{\alpha,\beta}^{-1}\left(-\left(F(x) + \xi\right)\right) = G_{\alpha,\beta}^{-1}\left(-\left(F(x) + \xi\right)\right) - G_{\alpha,\beta}^{-1}(-\xi) + \beta^{-1}\xi, \tag{20}$$

which, together with (13) and (16), show that for each $x \in [\beta^{-1}a, \beta^{-1}b]^N$,

$$\begin{aligned}\left\|h_1(x)\right\| &\ge \beta^{-1}\|\xi\|_0 - \left\|G_{\alpha,\beta}^{-1}\left(-\left(F(x) + \xi\right)\right) - G_{\alpha,\beta}^{-1}(-\xi)\right\| \\ &\ge \beta^{-1}\left(\|\xi\|_0 - (\alpha\beta\lambda)^{-1}Lb\right) > \beta^{-1}a, \end{aligned}\tag{21}$$

$$\left\|h_1(x)\right\| \le \beta^{-1}\left(\|\xi\| + (\alpha\beta\lambda)^{-1}Lb\right) < \beta^{-1}b. \tag{22}$$

It follows from (21) and (22) that h_1 maps $[\beta^{-1}a, \beta^{-1}b]^N$ into itself. In addition, from (10), (13), and (20), we have

$$\left\|h_1(x) - h_1(y)\right\| \le (\alpha\beta\lambda)^{-1}\left\|F(x) - F(y)\right\| \le (\alpha\beta\lambda)^{-1}L\|x - y\|, \quad \forall x, y \in \left[\beta^{-1}a, \beta^{-1}b\right]^N.$$

Hence, h_1 is contractive on $[\beta^{-1}a, \beta^{-1}b]^N$ due to $(\alpha\beta\lambda)^{-1}L < 1$. By the Banach contractive mapping principle, there exists a unique point $x_1 \in [\beta^{-1}a, \beta^{-1}b]^N$ such that $h_1(x_1) = x_1$. It follows from (21) and (22) that $x_1 \in (\beta^{-1}a, \beta^{-1}b)^N$. Therefore, x_1 solves Eq. (11).

Step 2b. There exists a point $x_0 \in (-\beta^{-1}r, \beta^{-1}r)^N$ and $x_0 \neq 0$ such that $H_{\alpha,\beta}(x_0) = x_1$.

Consider the following equation:

$$H_{\alpha,\beta}(x) = x_1, \tag{23}$$

in $[-\beta^{-1}r, \beta^{-1}r]^N$, which can be written as

$$-F(x) + x_1 = \alpha G(\beta x) := \widehat{G}_{\alpha,\beta}(x). \tag{24}$$

Then, it follows from (9) that for all $x, y \in [-\beta^{-1}r, \beta^{-1}r]^N$,

$$\left\| \widehat{G}_{\alpha,\beta}(x) - \widehat{G}_{\alpha,\beta}(y) \right\| = \left\| \alpha G(\beta x) - \alpha G(\beta y) \right\| \geq \alpha\beta\lambda \|x - y\|.$$

By assumption (ii), it can be easily verified that $\widehat{G}_{\alpha,\beta} : [-\beta^{-1}r, \beta^{-1}r]^N \to \widehat{G}_{\alpha,\beta}([-\beta^{-1}r, \beta^{-1}r]^N)$ is invertible, and that its inverse $\widehat{G}^{-1}_{\alpha,\beta} : \widehat{G}_{\alpha,\beta}([-\beta^{-1}r, \beta^{-1}r]^N) \to [-\beta^{-1}r, \beta^{-1}r]^N$ is continuous and satisfies

$$\left\| \widehat{G}^{-1}_{\alpha,\beta}(x) - \widehat{G}^{-1}_{\alpha,\beta}(y) \right\| \leq (\alpha\beta\lambda)^{-1} \|x - y\|, \quad \forall x, y \in \widehat{G}\left(\left[-\beta^{-1}r, \beta^{-1}r\right]^N\right).$$

Next, it is to show that

$$-F(x) + x_1 \in \widehat{G}_{\alpha,\beta}\left(\left[-\beta^{-1}r, \beta^{-1}r\right]^N\right) := \left\{\alpha G(\beta x) : x \in \left[-\beta^{-1}r, \beta^{-1}r\right]^N\right\}. \tag{25}$$

In fact, it follows from $x_1 \in (\beta^{-1}a, \beta^{-1}b)^N$ and (10) that

$$\left\| -F(x) + x_1 \right\| \leq L\|x\| + \|x_1\| \leq \beta^{-1}Lr + \beta^{-1}b < \alpha\lambda r, \quad \forall x \in \left[-\beta^{-1}r, \beta^{-1}r\right]^N. \tag{26}$$

Obviously, $0 \in (-\beta^{-1}r, \beta^{-1}r)^N$ and $\alpha G(\beta 0) = 0$. Hence, $0 \in \alpha G(\beta[-\beta^{-1}r, \beta^{-1}r]^N)$. Consequently, for each $x \in \partial([-\beta^{-1}r, \beta^{-1}r]^N)$, it follows from (9) that

$$\left\| \alpha G(\beta x) \right\| = \left\| \alpha G(\beta x) - \alpha G(\beta 0) \right\| \geq \alpha\lambda \|\beta x\| = \alpha\lambda r. \tag{27}$$

Further, since $DG(\beta x)$ is invertible at each point $x \in (\beta^{-1}a, \beta^{-1}b)^N$ by assumption (iic), $G(\beta(\beta^{-1}a, \beta^{-1}b)^N)$ is open due to [22]. Hence, it follows from (26) and (27) that (25), i.e., (24) holds. Hence, (24) can be written as

$$\widehat{G}^{-1}_{\alpha,\beta}\left(-F(x) + x_1\right) = x, \quad \forall x \in \left[-\beta^{-1}r, \beta^{-1}r\right]^N.$$

Let

$$h_2(x) := \widehat{G}^{-1}_{\alpha,\beta}\left(-F(x) + x_1\right), \quad \forall x \in \left[-\beta^{-1}r, \beta^{-1}r\right]^N. \tag{28}$$

It follows from (10) and (13) that for all $x, y \in [-\beta^{-1}r, \beta^{-1}r]^N$, one has

$$\begin{aligned}
&\left\| h_2(x) \right\| \leq (\alpha\beta\lambda)^{-1}\left(\left\| F(x) \right\| + \|x_1\|\right) \leq (\alpha\beta\lambda)^{-1}\left(L\beta^{-1}r + \beta^{-1}b\right) < \beta^{-1}r, \\
&\left\| h_2(x) - h_2(y) \right\| \leq (\alpha\beta\lambda)^{-1}\left\| F(x) - F(y) \right\| \leq (\alpha\beta\lambda)^{-1}L\|x - y\|.
\end{aligned}$$

Hence, h_2 maps $[-\beta^{-1}r, \beta^{-1}r]^N$ into itself and is contractive in $[-\beta^{-1}r, \beta^{-1}r]^N$ because $(\alpha\beta\lambda)^{-1}L < 1$. Again, by the Banach contractive mapping principle, there exists a unique point $x_0 \in [-\beta^{-1}r, \beta^{-1}r]^N$ such that $h_2(x_0) = x_0$. One can also show that $x_0 \in (-\beta^{-1}r, \beta^{-1}r)^N$. Hence, it follows from (24) and (28) that x_0 solves Eq. (23). Further, it follows from (24) that $x_0 \neq 0$. Otherwise, suppose that $x_0 = 0$, then by (24), $-F(0) + x_1 = \alpha G(\beta 0)$, hence, it follows from assumptions (i) and (iib) that $x_1 = 0$, which contradicts $x_1 \in (-\beta^{-1}a, -\beta^{-1}b)^N$.

Based on the above discussions, there exists $x_0 \in (-\beta^{-1}r, \beta^{-1}r)^N$, $x_0 \neq 0$, such that $H^2_{\alpha,\beta}(x_0) = 0$. Hence, $x^* = 0$ is a snap-back repeller of $H_{\alpha,\beta}$ in $[-\beta^{-1}r, \beta^{-1}r]^N$.

Step 3. $\|DH_{\alpha,\beta}(0)\|^0 > 1$ and $\|DH_{\alpha,\beta}(x_1)\|^0 > 0$.

Let $\Omega_\beta = (-\beta^{-1}r, \beta^{-1}r)^N \cup (\beta^{-1}a, \beta^{-1}b)^N$. Since $\Omega_\beta \subset [-r,r]^N$, $H_{\alpha,\beta}$ is continuously differential in Ω_β. Obviously, $DH_{\alpha,\beta}(x) = DF(x) + \alpha\beta DG(\beta x)$ for all $x \in \Omega_\beta$. It follows from (9) and Lemma 2.2 in [12] that

$$\|DG(x)\|^0 \geq \lambda, \quad \forall x \in (-r,r)^N, \forall x \in (a,b)^N. \tag{29}$$

Hence, it follows from (8) that for any fixed $x \in \Omega_\beta$ and for each $y \in \mathbf{R}^N$,

$$\begin{aligned}\|DH_{\alpha,\beta}(x)y\| &\geq \alpha\beta\|DG(\beta x)y\| - \|DF(x)y\| \\ &\geq \left(\alpha\beta\|DG(\beta x)\|^0 - \|DF(x)\|\right)\|y\| \geq (\alpha\beta\lambda - L)\|y\|.\end{aligned} \tag{30}$$

This implies that for any $x \in \Omega_\beta$, $\|DH_{\alpha,\beta}(x)\|^0 \geq \alpha\beta\lambda - L > 1$. Thus, $\|DH_{\alpha,\beta}(0)\|^0 > 1$ and $\|DH_{\alpha,\beta}(x_1)\|^0 > 0$.

Step 4. $DH_{\alpha,\beta}(x)$ is invertible for each $x \in \Omega_\beta$

It follows from assumption (iic), (8), and (30) that $DH_{\alpha,\beta}(x)$ is bounded and injective for each $x \in \Omega_\beta$. Next, we will prove that $DH_{\alpha,\beta}(x)$ is also surjective. In fact, for any fixed $x \in \Omega_\beta$ and for any $z \in \mathbf{R}^N$, the equation

$$DH_{\alpha,\beta}(x)y = z \tag{31}$$

can be written as

$$(\alpha\beta)^{-1}\left(-DF(x)y + z\right) = DG(\beta x)y. \tag{32}$$

By assumption (iic), (32) can be rewritten as $(\alpha\beta)^{-1}(DG(\beta x))^{-1}(-DF(x)y + z) = y$. Let

$$h_3(y) := (\alpha\beta)^{-1}\left(DG(\beta x)\right)^{-1}\left(-DF(x)y + z\right), \qquad K := \|z\|(\alpha\beta\lambda - L)^{-1}.$$

Then, it follows from (8) and (29) that for each $y, y_1, y_2 \in [-K,K]^N$,

$$\begin{aligned}\|h_3(y)\| &\leq (\alpha\beta)^{-1}\left\|\left(DG(\beta x)\right)^{-1}\right\|\left(\|DF(x)\|\|y\| + \|z\|\right) \leq (\alpha\beta\lambda)^{-1}\left(L\|y\| + \|z\|\right) \leq K, \\ \|h_3(y_1) - h_3(y_2)\| &\leq (\alpha\beta)^{-1}\left\|\left(DG(\beta x)\right)^{-1}\right\|\|DF(x)\|\|y_1 - y_2\| \leq (\alpha\beta\lambda)^{-1}L\|y_1 - y_2\|.\end{aligned}$$

Hence, h_3 maps $[-K,K]^N$ into itself and is contractive on $[-K,K]^N$ due to $(\alpha\beta\lambda)^{-1}L < 1$. Consequently, there exists a unique point $y \in [-K,K]^N$ such that $h_3(y) = y$. This implies that $DH_{\alpha,\beta}(x)$ is surjective. Hence, $DH_{\alpha,\beta}(x)$ has an inverse $(DH_{\alpha,\beta}(x))^{-1}$. Further, it follows from (30) that $(DH_{\alpha,\beta}(x))^{-1}$ is a bounded linear operator. Hence, $DH_{\alpha,\beta}(x)$ is invertible for each $x \in \Omega_\beta$ and $\alpha\beta > C_0$.

By the above discussions and Lemma 1, the theorem holds in the case of $\beta > b/r$ and $\alpha\beta > C_0$.

Secondly, we will prove that the theorem holds in the other three cases: Case I: $\beta > b/r$, $\alpha\beta < -C_0$; Case II: $\beta < -b/r$, $\alpha\beta > C_0$; and Case III: $\beta < -b/r$, $\alpha\beta < -C_0$.

The proof is similar to that of $\beta > b/r$ and $\alpha\beta > C_0$. The difference between their proofs is as follows. In Case I, α and $\alpha\beta$ in the above inequalities are replaced by $-\alpha$ and $-\alpha\beta$, respectively. In Case II, α, β in inequalities, $[-\beta^{-1}r, \beta^{-1}r]$, and $[\beta^{-1}a, \beta^{-1}b]$ are replaced by $-\alpha$, $-\beta$, $[\beta^{-1}r, -\beta^{-1}r]$, and $[\beta^{-1}b, \beta^{-1}a]$, respectively. In Case III, β, $\alpha\beta$ in inequalities, $[-\beta^{-1}r, \beta^{-1}r]$, and $[\beta^{-1}a, \beta^{-1}b]$ are replaced by $-\beta$, $-\alpha\beta$, $[\beta^{-1}r, -\beta^{-1}r]$, and $[\beta^{-1}b, \beta^{-1}a]$, respectively. The other proofs are similar to that of $\beta > b/r$, $\alpha\beta > C_0$. Hence, the details are omitted.

By all the above discussions, the theorem holds. The proof is now complete. □

Remark 1 All the assumptions (i), (iia), (iib), and (iic) in Theorem 1 are necessary conditions of making (7) chaotic.

Remark 2 In [21], the controlled systems $x_{n+1} = f(x_n) + g(\mu x_n)$ and $x_{n+1} = f(x_n) + \mu g(x_n)$ were considered, and they have been proved to be chaotic in the case of $\mu > \mu_0 > 0$ and under some other conditions. First, systems in [21] are special cases of Theorem 1 herein. Secondly, compared with the range of values of μ, the ranges of α, β become wider, because they could be positive or negative, and the value of α can be made arbitrarily small if β is large enough.

4 Existence of chaos in the controlled GCMLs (2) and (3)

Theorem 2 *Consider the controlled GCML* (2). *Assume that*

(i) $f(0) = 0$ *and there exists a positive constant* r *such that* f *is continuously differentiable in* $[-r, r]$;

(ii) g *is continuously differentiable in* $[-r, r] \cup [a, b]$ *with* $r < a < b$, $g'(x) \neq 0$ *for all* $x \in [-r, r] \cup [a, b]$, $g(0) = 0$, *and there exists a point* $\xi \in (a, b)$ *such that* $g(\xi) = 0$.

Then, for any constants α, β *satisfying* $|\beta| > b/r$ *and*

$$|\alpha\beta| > C_0 := \max\left\{\frac{Lr + b}{\lambda r}, \frac{Lb}{\lambda(\xi - a)}, \frac{Lb}{\lambda(b - \xi)}\right\},$$

where $L := \max\{|f'(x)| : x \in [-r, r]\}$ *and* $\lambda := \min\{|g'(x)| : x \in [-r, r] \cup [a, b]\}$, (2) *is chaotic in the sense of both Li–Yorke and Devaney in the neighborhood of the origin.*

Proof Theorem 1 is used to prove this theorem. The induced system of the controlled GCML (2) is (5).

It is clear that $O := (\underbrace{0, 0, \ldots, 0}_{N+1})^T \in \mathbf{R}^{N+1}$ is a fixed point of F, F is continuously differentiable in $[-r, r]^{N+1}$, and its Jacobian matrix is

$$DF(x) = \begin{pmatrix} (1-\varepsilon)f'(x(1)) & \frac{\varepsilon}{N}f'(x(2)) & \cdots & \frac{\varepsilon}{N}f'(x(N+1)) \\ \frac{\varepsilon}{N}f'(x(1)) & (1-\varepsilon)f'(x(2)) & \cdots & \frac{\varepsilon}{N}f'(x(N+1)) \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\varepsilon}{N}f'(x(1)) & \frac{\varepsilon}{N}f'(x(2)) & \cdots & (1-\varepsilon)f'(x(N+1)) \end{pmatrix}.$$

Hence, one has

$$\begin{aligned} DF(x)z = \Big(&(1-\varepsilon)f'\big(x(1)\big)z(1) + \frac{\varepsilon}{N}f'\big(x(2)\big)z(2) + \cdots + \frac{\varepsilon}{N}f'\big(x(N+1)\big)z(N+1), \\ &\ldots, \frac{\varepsilon}{N}f'\big(x(1)\big)z(1) + \frac{\varepsilon}{N}f'\big(x(2)\big)z(2) + \cdots + (1-\varepsilon)f'\big(x(N+1)\big)z(N+1)\Big)^T, \end{aligned}$$

where $z = \{z(j)\}_{j=1}^{N+1} \in \mathbf{R}^{N+1}$. Thus,

$$\|DF(x)\| \le L, \quad \forall x \in [-r,r]^{N+1}.$$

It follows from assumption (ii) that $G(O) = G(\xi^*) = 0$, where $\xi^* := (\underbrace{\xi,\xi,\ldots,\xi}_{N+1})^T \in (a,b)^{N+1}$. Again by assumption (ii), $G(x)$ is continuously differentiable in $[-r,r]^{N+1} \cup [a,b]^{N+1}$ and satisfies

$$DG(x) = \operatorname{diag}\{g'(x(1)), g'(x(2)), \ldots, g'(x(N+1))\}.$$

By assumption (ii), $DG(x)$ is invertible and its inverse matrix is

$$(DG(x))^{-1} = \operatorname{diag}\{(g'(x(1)))^{-1}, (g'(x(2)))^{-1}, \ldots, (g'(x(N+1)))^{-1}\},$$

which implies that $\|(DG(x))^{-1}\| \le 1/\lambda$ for all $x \in [-r,r]^{N+1} \cup [a,b]^{N+1}$. Hence, $DG(x)$ is an invertible linear map. Further, for any $x, y \in [-r,r]^{k+1}$ or $[a,b]^{k+1}$, by the mean value theorem, we have that

$$\|G(x) - G(y)\| = \max\{|g(x(j)) - g(y(j))| : 1 \le j \le N+1\} \ge \lambda\|x-y\|.$$

By summarizing the above discussions, F and G satisfy all the assumptions of Theorem 1. Thus, the theorem herein holds. The proof is complete. □

Theorem 3 *Consider the controlled GCML* (3). *Suppose that f and g satisfy all assumptions of Theorem* 2, *then for any constants α, β satisfying $|\beta| > b/r$ and*

$$|\alpha\beta| > C_0 := \max\left\{\frac{(2N-1)(Lr+b)}{\lambda r}, \frac{(2N-1)Lb}{\lambda(\xi-a)}, \frac{(2N-1)Lb}{\lambda(b-\xi)}\right\},$$

Equation (3) *is chaotic in the sense of both Li–Yorke and Devaney in the neighborhood of the origin.*

Proof Theorem 1 is used to prove this theorem. The induced system of (3) is (6).

Obviously, $\widetilde{G}(x)$ is continuously differentiable in $[-r,r]^{N+1} \cup [a,b]^{N+1}$, and its Jacobian matrix is

$$D\widetilde{G}(x) = \frac{1}{N}\begin{pmatrix} 0 & g'(x(2)) & \cdots & g'(x(N+1)) \\ g'(x(1)) & 0 & \cdots & g'(x(N+1)) \\ \cdots & \cdots & \cdots & \cdots \\ g'(x(1)) & g'(x(2)) & \cdots & 0 \end{pmatrix}.$$

Thus, for any $z = \{z(j)\}_{j=1}^{N+1} \in \mathbf{R}^{N+1}$, one has

$$D\widetilde{G}(x)z = \frac{1}{N}\big(g'(x(2))z(2) + \cdots + g'(x(N+1))z(N+1),$$
$$\cdots, g'(x(1))z(1) + \cdots + g'(x(N))z(N)\big)^T.$$

Therefore,

$$\|D\widetilde{G}(x)\| \le \max\{|g'(x)| : x \in [-r,r] \cup [a,b]\}.$$

Further, $D\widetilde{G}(x)$ is invertible and its inverse is

$$\begin{aligned}&(D\widetilde{G}(x))^{-1}\\&\quad=\begin{pmatrix}-(N-1)(g'(x(1)))^{-1} & (g'(x(1)))^{-1} & \cdots & (g'(x(1)))^{-1}\\ (g'(x(2)))^{-1} & -(N-1)(g'(x(2)))^{-1} & \cdots & (g'(x(2)))^{-1}\\ \cdots & \cdots & \cdots & \cdots\\ (g'(x(N+1)))^{-1} & (g'(x(N+1)))^{-1} & \cdots & -(N-1)(g'(x(N+1)))^{-1}\end{pmatrix},\end{aligned}$$

which implies that for all $x \in [-r,r]^{N+1} \cup [a,b]^{N+1}$ and for any $z = \{z(j)\}_{j=1}^{N+1} \in \mathbf{R}^{N+1}$,

$$\begin{aligned}(D\widetilde{G}(x))^{-1}z = \bigl(&-(N-1)(g'(x(1)))^{-1}z(1) + (g'(x(1)))^{-1}z(2) + \cdots + (g'(x(1)))^{-1}z(N+1),\\ &\cdots, (g'(x(N+1)))^{-1}z(1) + \cdots - (N-1)(g'(x(N+1)))^{-1}z(N+1)\bigr)^{T},\end{aligned}$$

thus,

$$\|(D\widetilde{G}(x))^{-1}\| \le (2N-1)/\lambda.$$

Hence, $D\widetilde{G}(x)$ is an invertible linear map. By the compatibility of the matrix norm,

$$1 = \|D\widetilde{G}(x)(D\widetilde{G}(x))^{-1}\| \le \|D\widetilde{G}(x)\|\,\|(D\widetilde{G}(x))^{-1}\|,$$

which together with the above inequality implies that for all $x \in [-r,r]^{N+1} \cup [a,b]^{N+1}$,

$$\|D\widetilde{G}(x)\| \ge \bigl(\|(D\widetilde{G}(x))^{-1}\|\bigr)^{-1} \ge \lambda/(2N-1),$$

which implies that for any $x, y \in [-r,r]^{N+1}$ or $[a,b]^{N+1}$,

$$\|\widetilde{G}(x) - \widetilde{G}(y)\| \ge \lambda/(2N-1)\|x-y\|.$$

Further, $\widetilde{G}(\xi) = \widetilde{G}(0) = 0$. Hence, $\widetilde{G}(x)$ satisfies assumption (ii) of Theorem 1. By the above discussions in Theorem 2, F herein satisfies assumption (i) of Theorem 1.

By summarizing the above discussions, F and $\widetilde{G}$ satisfy all the assumptions of Theorem 1. Thus, all the results herein hold. The proof is complete. □

5 Examples

In this section, two examples with computer simulations are given to illustrate the results of Theorems 2 and 3.

Example 1 Consider the controlled GCML (2), where

$$f(x) = \frac{1}{2}x\sin^2 x, \quad x \in \mathbf{R},$$

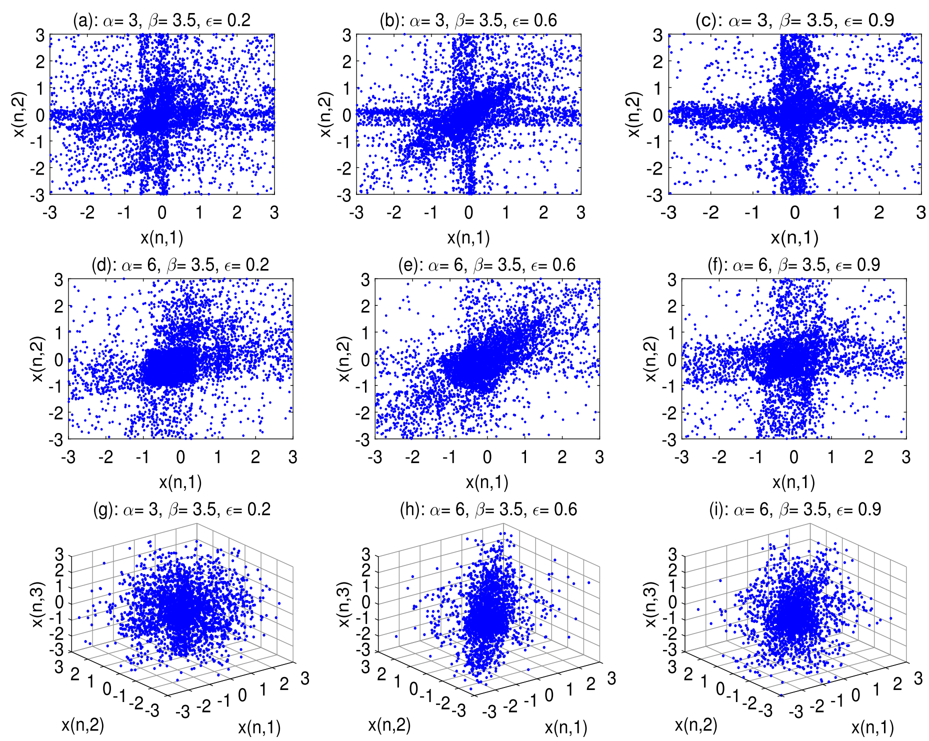

Figure 1 Computer simulations for the controlled GCML (2). In 2D graphs, $N = 1$, the initial value is taken as $x(1) = 0.1$, $x(2) = -0.1$, and $n = 0, 1, \dots, 10{,}000$. In 3D graphs, $N = 2$, the initial value is $x(1) = 0.1$, $x(2) = -0.1$, $x(3) = 2$, and $n = 0, 1, \dots, 5000$

and

$$g(x) = \begin{cases} 2x, & x \in [-1, 1], \\ (x - \frac{5}{2})(x + \frac{5}{2}), & x \in [2, 3], \\ \frac{1}{10}\cos x, & \text{else}. \end{cases}$$

Obviously, f and g satisfy all the assumptions of Theorem 2 with $r = 1$, $a = 2$, $b = 3$, $\xi = 5/2$, $L < 1$, and $\lambda = 2$. Thus, by Theorem 1, for any constants $|\beta| > 3$ and $|\alpha\beta| > 3$, (2) is chaotic in the sense of both Li–Yorke and Devaney in the neighborhood of the origin.

For computer simulation, we take $N = 1, 2$, $\beta = 3.5$, $\alpha = 3, 6$, and $\varepsilon = 0.2, 0.6, 0.9$, respectively. The simulation results in the two-dimensional plane $(x(\cdot, 1), x(\cdot, 2))$ and three-dimensional space $(x(\cdot, 1), x(\cdot, 2), x(\cdot, 3))$ that are shown in Fig. 1, which indicates that the controlled GCML (2) has very complicated dynamical behaviors.

Example 2 Consider the controlled GCML (3), where f and g are given in Example 1.

Similar to the discussions in Example 1, by Theorem 3, for any constants $|\beta| > 3$ and $|\alpha\beta| > 3(2N - 1)$, (3) is chaotic in the sense of both Li–Yorke and Devaney.

In order to compare the dynamical behaviors of the GCMLs (2) and (3) intuitively, we take all the values of α, β, ε, n, N, and the initial values are the same as those in Example 1. The simulation results are shown in Fig. 2. Obviously, the dynamical behaviors of (2) and (3) are similar in the 2D graphs, while they are different in the 3D graphs.

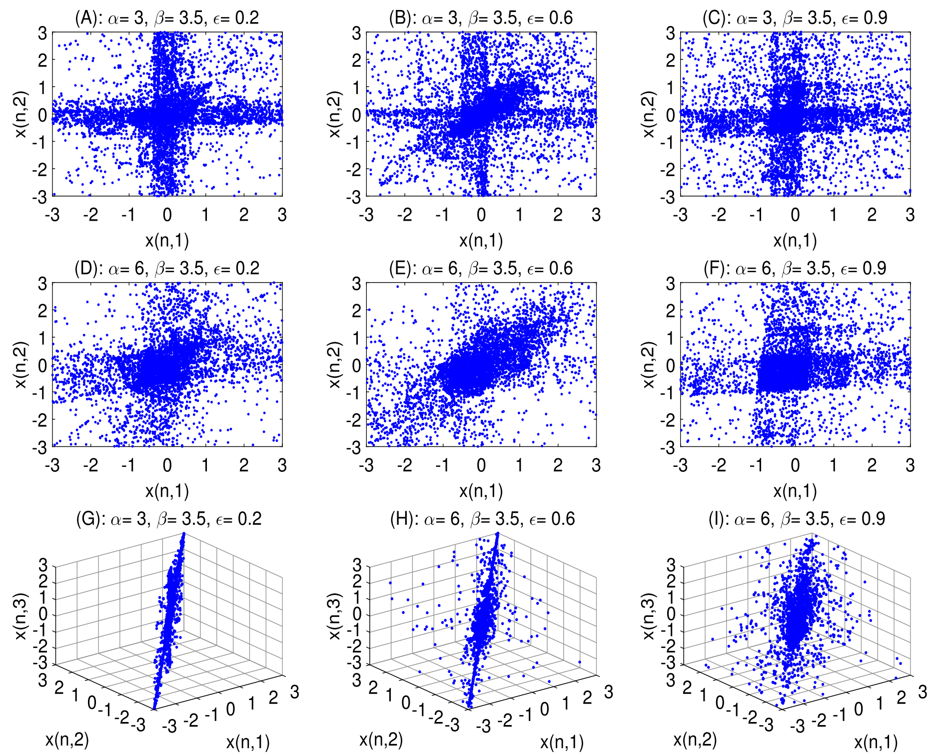

Figure 2 Computer simulations for the controlled GCML (3). All the values of α, β, ε, n, N, and the initial values are the same as those in Fig. 1

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Competing interests
The authors declare that they have no competing interests.

Author contribution
WL contributed to the idea of this paper, wrote the manuscript, and revised it. YY and TJ proved the theorems and wrote the paper. All authors read and approved the final manuscript.

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