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A fast-convolution based space–time Chebyshev spectral method for peridynamic models

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Abstract

Peridynamics is a nonlocal generalization of continuum mechanics theory which addresses discontinuous problems without using partial derivatives and replacing them by an integral operator. As a consequence, it finds applications in the framework of the development and evolution of fractures and damages in elastic materials.

In this paper we consider a one-dimensional nonlinear model of peridynamics and propose a suitable two-dimensional fast-convolution spectral method based on Chebyshev polynomials to solve the model. This choice allows us to gain the same accuracy both in space and time. We show the convergence of the method and perform several simulations to study the performance of the spectral scheme.

Keywords: Nonlinear peridynamics; Chebyshev spectral methods; Chebyshev polynomials; Convolution product; Fast Fourier Transform

1 Introduction

In the framework of continuum mechanic theory, peridynamics is a nonlocal version of elasticity introduced to describe the formation and evolution of fractures and damages in elastic materials. It was introduced by Silling in [34] and employs a second order in time partial integro-differential equation.

The main capability of the model is that it avoids the use of partial derivatives in space, so it can address discontinuous problems [1, 2, 12–14, 16, 17, 30].

Most standard approaches used to approximate the solution of the peridynamic equation make use of meshfree methods with the two-point Gauss quadrature (see [6, 20]) and finite difference methods with Störmer–Verlet scheme (see [36]).

Both methods need $\mathcal{O}(N^2)$ cost per time step and perform well when the nonlocality covers a small portion of the domain (see [25]). In particular, in [11, 31], the authors make a survey of the most implemented numerical methods in the peridynamic framework and propose a different approach based on spectral techniques.

Spectral methods are an important tool for the numerical solution of many applied problems, as they allow achieving high-order accuracy for smooth problems. They consist in reformulating the original problem in the frequency space, by decomposing the solution as a linear combination of a suitable basis.

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Quite recently, in [15], the authors presented a complete review on numerical methods for nonlocal and fractional models.

In [22–24, 26, 28] the authors propose a spectral discretization of the model based on the Fourier trigonometric polynomials. And since this approach can be applied only to periodic problems, the authors employ a volume penalization technique to extend the method to the aperiodic setting.

A different way to overcome the limitation of periodic solutions consists in replacing Fourier polynomials by Chebyshev polynomials (see [29]). This method happens to be very efficient in terms of computational cost per time step as it can benefit from the use of the Fast Fourier Transform algorithm.

For time integration, Störmer–Verlet scheme or Newmark- β method are commonly used in the context of wave propagation or peridynamics (see, for instance, [9, 26, 29]).

In this work, we propose a fast-convolution fully spectral method in space and time based on the implementation of Chebyshev polynomials, in order to have the same accuracy in both variables. The basic idea is to study the problem in the two-dimensional Cartesian (x, t) bounded space–time domain and to expand the unknown function in Chebyshev polynomials of the spatial variable x as well as of the time variable t . One of the advantages of this approach is that we do not need to integrate in time the semi-discrete method. Indeed, this step is substituted by a numerical procedure to solve an algebraic system. Additionally, the choice of using Chebyshev polynomials releases us from the use of periodic boundary conditions.

The paper is organized as follows: in Sect. 2 we describe the nonlinear peridynamic model we aim to study. Useful properties of Chebyshev polynomials are summarized in Sect. 3. In Sect. 4 we construct the fully spectral method to solve the peridynamic equation and prove the convergence of the proposed method. Simulations and results are shown in Sect. 5. Finally, Sect. 6 concludes the paper.

2 Statement of the problem

Peridynamics is a nonlocal version of continuum mechanics based on long-range interactions. The main motivation of the development of such a theory relies on the necessity to find an analytical description of discontinuous phenomena like fractures and cracks. The long-range interactions are parametrized thanks to the introduction of a scalar quantity $\delta > 0$, called horizon, as a measure of the nonlocality.

Let $\Omega \subset \mathbb{R}$ be the spatial domain. We consider the following nonlinear peridynamic model:

$$\partial_{tt}^2 u(x) = \int_{B_\delta(x)} f(x - x', (u(x') - u(x))) dx', \quad x \in \Omega, t > 0, \quad (1)$$

which describes the evolution of a material body, and where the unknown u represents the displacement field and $B_\delta(x) = \{x' \in \Omega : \|x - x'\| < \delta\}$ is the ball centered at x with radius δ .

We define the relative position of two particles in the reference configuration by $\xi = x' - x$, and the relative displacement by $\eta = u(x') - u(x)$.

The bivariate force function f is supposed to decompose as follows:

$$f(\xi, \eta) = C(\xi)H(\eta), \quad \text{for every } (\xi, \eta) \in \Omega \times \Omega, \quad (2)$$

where the function C is an even scalar function, called *micromodulus function* (see [35]), which vanishes when $|\xi| > \delta$, and in what follows we assume $C \in L^\infty(\mathbb{R})$, while H is an odd globally Lipschitz continuous function (see [19]), namely there is a nonnegative function $\ell \in L^1(B_\delta(0)) \cap L^\infty(B_\delta(0))$ such that for all $\xi \in \mathbb{R}$, with $|\xi| \leq \delta$ and η, η' ,

$$|H(\eta') - H(\eta)| \leq \ell(\xi) |\eta' - \eta|. \quad (3)$$

In this work, we will work with $H(\eta) = \eta^3$. This kind of power-type nonlinearity appears to be very useful from a numerical point of view, as it allows us to take advantage of the properties of Chebyshev transform and convolution products (see, for instance, [27, 29]). Moreover, from an analytical point of view, this choice is justified by the fact that the peridynamic integral operator resembles a fractional derivative (see, for instance, [21]) and in this setting the well-posedness of the model is achieved (see [10, 18]). Additionally, the dependence of H on η instead of $\xi + \eta$ is justified in this context, as we consider 1D manifolds, and, as a consequence, the conservation of the angular momentum is guaranteed.

We denote by \mathcal{L} the peridynamic integral operator of (1) and, thanks to the assumption (2) on f , we can write it in the following way:

$$\begin{aligned} \mathcal{L}u(x, t) &= \int_{\mathbb{R}} C(x - x') u^3(x', t) dx' - 3u(x, t) \int_{\mathbb{R}} C(x - x') u^2(x', t) dx' \\ &\quad + 3u^2(x, t) \int_{\mathbb{R}} C(x' - x) u(x', t) dx' - u^3(x, t) \int_{\mathbb{R}} C(x - x') dx' \\ &= (C * u^3)(x, t) - 3u(x, t)(C * u^2)(x, t) + 3u^2(x, t)(C * u)(x, t) \\ &\quad - \beta u^3(x, t), \end{aligned} \quad (4)$$

where $*$ denotes the convolution product and

$$\beta = \int_{\mathbb{R}} C(x) dx.$$

Therefore, the peridynamic equation becomes

$$\partial_{tt}^2 u(x, t) = \mathcal{L}u(x, t). \quad (5)$$

We add the initial condition

$$u(x, 0) = u_0(x), \quad \partial_t u(x, 0) = v(x),$$

so the peridynamic model we aim to study is

$$\begin{cases} \partial_{tt}^2 u(x, t) = \mathcal{L}u(x, t), & x \in \Omega, t > 0, \\ u(x, 0) = u_0(x), \quad \partial_t u(x, 0) = v(x), & x \in \Omega. \end{cases} \quad (6)$$

Since the peridynamic operator \mathcal{L} is decomposed as the sum of convolution products, the spectral approach to discretize it represents a good framework in order to have good accuracy in the solution and to reduce the computational cost.

In particular, we can discretize the model (6) by exploiting the theory of Chebyshev polynomials.

In next section, we briefly recall the definition of Chebyshev polynomials and their main properties in relation with the convolution products and differential operators.

3 Basic properties of Chebyshev polynomials

This section is devoted to provide an overview on Chebyshev polynomials and Chebyshev collocation method.

Chebyshev polynomials of the first kind, $T_n(x)$, are explicitly defined as

$$T_n(x) = \cos(n \arccos(x)), \quad x \in [-1, 1], n \in \mathbb{N}. \quad (7)$$

Without loss of generality, it is always possible to introduce a new variable $y \in [a, b]$ and a linear map which allows scaling the polynomials from $[-1, 1]$ to $[a, b]$, so, we restrict our discussion to the normalized domain $[-1, 1]$.

Chebyshev polynomials (7) are orthogonal with respect to the weight function $w(x) = (\sqrt{1-x^2})^{-1}$, and their boundary values, as well as those of their first and second derivatives, are given by

$$\begin{aligned} T_n(\pm 1) &= (\pm 1)^n, \\ T'_n(\pm 1) &= (\pm 1)^{n+1} n^2, \\ T''_n(\pm 1) &= (\pm 1)^n n^2 \frac{n^2 - 1}{3}. \end{aligned}$$

Moreover, Chebyshev polynomials have an interpolation property: any sufficiently smooth function f defined on the interval $[-1, 1]$ can be expanded in a series of Chebyshev polynomials. The $(N + 1)$ -term interpolation of f is denoted by f^N and has the following expression:

$$f^N(x) = \sum_{n=0}^N f_n T_n(x), \quad (8)$$

where f_n are the coefficients of the expansion, whose discrete expression depends on the choice of collocation points.

If we fix a grid corresponding to the Gauss–Lobatto collocation points,

$$x_k = \cos\left(\frac{k\pi}{N}\right), \quad k = 0, \dots, N, \quad (9)$$

we can express Chebyshev coefficients f_n as follows:

$$f_n = \frac{1}{\gamma_n} \sum_{k=0}^N f(x_k) T_n(x_k) w_k, \quad (10)$$

where

$$w_k = \begin{cases} \frac{\pi}{2N}, & k = 0, N, \\ \frac{\pi}{N}, & k = 1, \dots, N-1. \end{cases}$$

and the normalization constant γ_n is given by

$$\gamma_n = \begin{cases} \pi, & n = 0, N, \\ \frac{\pi}{2}, & n = 1, \dots, N-1. \end{cases}$$

The choice of the Gauss–Lobatto points as grid points for the discretization is very useful as it can avoid Gibbs’s phenomenon at the boundaries.

Thanks to their definition, Chebyshev polynomials are strictly related to the trigonometric cosine functions and, as a consequence, the finite series (8) can be efficiently computed by the Fourier cosine transform by using the Fast Fourier Transform (FFT) algorithm.

To solve discretized problems, we need to look for the relationship between Chebyshev coefficients of a function and the coefficients of its derivative of any order.

Let f be a sufficiently smooth function approximated by f^N defined in (8), where the coefficients f_n , $n = 0, \dots, N$, are given by (10). Then, the coefficients of its first derivative f' are given by (see [7, 8])

$$f'_n = \frac{2}{c_n} \sum_{\substack{k=n+1 \\ k+n \text{ odd}}}^N k f_k, \quad (11)$$

for

$$c_n = \begin{cases} 2 & n = 0, \\ 1 & \text{otherwise,} \end{cases}$$

or equivalently, the coefficients f'_n can be computed as a matrix multiplication,

$$f'_n = \sum_{k=0}^N D_{nk} f_k, \quad (12)$$

where $D = (D_{nk})$ is a $(N+1) \times (N+1)$ derivative matrix, with the following representation:

$$D = \begin{pmatrix} 0 & 1 & 0 & 3 & 0 & \dots & n & \dots & N \\ 0 & 0 & 4 & 0 & 8 & & 0 & \dots & 0 \\ 0 & 0 & 0 & 6 & 0 & & 2n & \dots & 2N \\ 0 & 0 & 0 & 0 & 8 & & 0 & \dots & 0 \\ \vdots & 0 & 0 & 0 & 0 & \ddots & \vdots & & \vdots \\ \vdots & 0 & 0 & 0 & 0 & \ddots & 2n & & \vdots \\ \vdots & & & & & \dots & 0 & \ddots & 0 \\ \vdots & & & & & \dots & 0 & \ddots & 2N \\ 0 & & & & & \dots & 0 & \dots & 0 \end{pmatrix}. \quad (13)$$

It is an upper triangular matrix with its main diagonal terms equal to zero.

In an analogous way, we can find the coefficients for higher order derivatives, by taking the power of the matrix D ,

$$f_n^\ell = \sum_{k=0}^N (D^\ell)_{nk} f_k.$$

Let \mathcal{F}_N denote the linear map which associates to a function f its Chebyshev discrete coefficients f_n , $n = 0, \dots, N$, defined in (10), and let \mathcal{F}_N^{-1} be its inverse discrete transform defined by (8).

We have that \mathcal{F}_N satisfies the following property when it is composed with a differential operator (see [33]):

$$\frac{d^k f^N}{d\xi^k}(\xi) = \mathcal{F}_N^{-1}((-i\xi)^k \mathcal{F}_N(f^N)), \quad (14)$$

where i denotes the imaginary unit such that $i^2 = -1$. This property is equivalent to using the derivative matrix D defined in (13).

Moreover, when \mathcal{F}_N is applied to a convolution product, we find

$$f * g = \mathcal{F}_N^{-1}(\mathcal{F}_N(f)\mathcal{F}_N(g)). \quad (15)$$

In [4], the authors show the relation between the Chebyshev coefficients of $(f * g)$ in terms of the Chebyshev coefficients of f and g .

Remark 1 When we deal with functions depending both on the space and time variables, say $f(x, t)$, we can still approximate them by a finite Chebyshev series expansion in both space and time. In this context, we seek an approximation function $f^N(x, t)$ in the two variables $(x, t) \in [-1, 1]^2$ such that

$$f^N(x, t) = \sum_{j=0}^{N_x} \sum_{k=0}^{N_t} f_{jk} T_j(x) T_k(t), \quad (16)$$

where N_x and N_t represent the total number of collocation points in space and time, respectively. The coefficients f_{jk} , $j = 0, \dots, N_x$, $k = 0, \dots, N_t$, are the Chebyshev coefficients of the discrete Chebyshev expansion and, when the grid points are the Gauss–Lobatto points $(x_n, t_m) = (\cos(n\pi/N_x), \cos(m\pi/N_t))$, their expression is given by

$$f_{jk} = \frac{1}{\gamma_j \gamma_k} \sum_{n=0}^{N_x} \sum_{m=0}^{N_t} f(x_n, t_m) T_j(x_n) T_k(t_m) w_n w_m. \quad (17)$$

For the purpose of our work, we mention here only the expansion of the second order derivative in time,

$$\partial_{tt}^2 f^N(x, t) = \sum_{j=0}^{N_x} \sum_{k=0}^{N_t} \sum_{\ell=0}^{N_t} \hat{D}_{k\ell}^{(t)} f_{j\ell} T_j(x) T_k(t), \quad (18)$$

where $\hat{D} = D \cdot D$, and the superscript (t) in the derivative matrix D denotes the differentiation with respect to the temporal coordinates.

We can compactly write expression (18) as follows:

$$\partial_{tt}^2 f^N(x, t) = \sum_{j=0}^{N_x} \sum_{k=0}^{N_t} \hat{f}_{kj} T_j(x) T_k(t), \quad (19)$$

where

$$\hat{f}_{kj} = \sum_{\ell=0}^{N_t} \hat{D}_{k\ell}^{(t)} f_{j\ell}. \quad (20)$$

Even in this case we can benefit from the implementation of the Fast Fourier Transform algorithm in the two-dimensional setting to compute the coefficients f_{jk} in (16).

Additionally, the same results as in (14) and (15) hold in the two-dimensional case.

4 Chebyshev spectral method for the fully discrete problem

We develop a fast-convolution fully spectral method to solve the nonlinear peridynamic problem (6).

Without loss of generality, we assume $\Omega = [-1, 1]$ and $t \in [-1, 1]$, and we fix $N + 1 > 0$ as the total number of collocation points in both the space and time directions, and we take the Gauss–Lobatto points (x_n, t_m) as grid points for the discretization.

We look for an approximation of $u(x, t)$ in the form

$$u^N(x, t) = \sum_{j=0}^N \sum_{k=0}^N u_{jk} T_j(x) T_k(t). \quad (21)$$

Substituting $u^N(x, t)$ into (4), we find the full expression of the peridynamic operator

$$\begin{aligned} \mathcal{L}u^N(x, t) &= (C * (u^N)^3)(x, t) - 3u^N(x, t)(C * (u^N)^2)(x, t) \\ &\quad + 3(u^N)^2(x, t)(C * u^N)(x, t) - \beta(u^N(x, t))^3. \end{aligned} \quad (22)$$

If we evaluate $u^N(x, t)$ at (x_n, t_m) , we obtain the discrete form of (22), namely

$$\begin{aligned} \mathcal{L}u_{nm}^N &= (\mathcal{F}_N^{-1}(\mathcal{F}_N(C)\mathcal{F}_N((u_{nm}^N)^3))) \\ &\quad - 3(\mathcal{F}_N^{-1}(\mathcal{F}_N((u_{nm}^N)) * (\mathcal{F}_N(C)\mathcal{F}_N((u_{nm}^N)^2)))) \\ &\quad + 3(\mathcal{F}_N^{-1}(\mathcal{F}_N((u_{nm}^N)^2) * (\mathcal{F}_N(C)\mathcal{F}_N((u_{nm}^N)))))) \\ &\quad - \beta(u_{nm}^N)^3, \end{aligned} \quad (23)$$

where u_{nm}^N is equal to $u^N(x_n, t_m)$.

Moreover, thanks to the differentiation theorem for the Chebyshev transform, we have

$$\partial_{tt}^2 u^N(x_n, t_m) = \frac{2}{N} \mathcal{F}_N^{-1}(t_m^2 \mathcal{F}_N(u_{nm}^N)), \quad (24)$$

or equivalently,

$$\partial_{tt}^2 u^N(x_n, t_m) = \sum_{j=0}^N \sum_{k=0}^N \hat{u}_{kj} T_j(x_n) T_k(t_m), \quad (25)$$

with \hat{u}_{kj} as in (20), for $k, j = 0, \dots, N$.

Thus, we can consider the discrete form of the model (6), namely

$$\begin{cases} \sum_{j=0}^N \sum_{k=0}^N \hat{u}_{kj} T_j(x_n) T_k(t_m) - \mathcal{L}(u_{nm}^N, t_m) = 0, & n = 0, \dots, N, m = 1, \dots, N, \\ \sum_{j=0}^N \sum_{k=0}^N (-1)^k u_{jk} T_j(x_n) = u_0(x_n), & n = 0, \dots, N, \\ \sum_{j=0}^N \sum_{k=0}^N \sum_{\ell=0}^N (-1)^k D_{k\ell} u_{j\ell} T_j(x_n) = v(x_n), & n = 0, \dots, N, \end{cases} \quad (26)$$

where in the peridynamic operator \mathcal{L} we have explicitly shown the time dependence.

After solving the above nonlinear system with respect to u_{nm}^N , we find an approximate solution of (6) having the form as in (21). In practice, in the next section, we use the FSOLVE command implemented in MATLAB software to solve the system (26). It consists in a quasi-Newton method, called Levenberg–Marquardt method.

We analyze the convergence of the proposed method in the space of functions which admit a modulus of continuity. We start by giving some definitions and recalling some standard results (see [3]).

Definition 1 A continuous function $W : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is called a *modulus of continuity* if it satisfies the following properties:

- W is increasing,
- $\lim_{z \rightarrow 0} W(z) = 0$,
- $W(z_1 + z_2) \leq W(z_1) + W(z_2)$, for $z_1, z_2 \in \mathbb{R}_+$,
- there exists a constant $c > 0$ such that $z \leq cW(z)$, for all $0 < z \leq 2$.

An example of a modulus of continuity is given by the functions $W(z) = z^\alpha$, $0 < \alpha \leq 1$.

Let B^2 be the unit ball in \mathbb{R}^2 .

Definition 2 We say that a continuous function $u(\cdot, \cdot)$ on B^2 admits a *modulus of continuity* $W(\cdot)$ if

$$|u(\cdot, \cdot)|_W = \sup_{(\bar{x}, \bar{t}) \neq (\tilde{x}, \tilde{t})} \left\{ \frac{|u(\bar{x}, \bar{t}) - u(\tilde{x}, \tilde{t})|}{W(|(\bar{x}, \bar{t}) - (\tilde{x}, \tilde{t})|)}, (\bar{x}, \bar{t}), (\tilde{x}, \tilde{t}) \in B^2 \right\} \quad (27)$$

is finite.

In (27), $|(\bar{x}, \bar{t}) - (\tilde{x}, \tilde{t})| = \max\{|\bar{t} - \tilde{t}|, |\bar{x} - \tilde{x}|\} : (\bar{x}, \bar{t}), (\tilde{x}, \tilde{t}) \in B^2, (\bar{x}, \bar{t}) \neq (\tilde{x}, \tilde{t})\}$.

We denote the class of all functions described in Definition 2 by $\mathcal{C}_W^0(B^2)$. Then, it is a Banach space with the norm

$$\|u(\cdot, \cdot)\|_{0,W} = \|u(\cdot, \cdot)\|_\infty + |u(\cdot, \cdot)|_W. \quad (28)$$

Moreover, we denote the class of k -times differentiable functions on B^2 whose k th derivatives admit W as a modulus of continuity by C_W^k . It is a Banach space with the norm

$$\|u(\cdot, \cdot)\|_{k,W} = \sum_{|s| \leq k} \left\| \frac{\partial^s u}{\partial t^s} \right\|_{\infty} + \sum_{|s| \leq k} \left\| \frac{\partial^s u}{\partial x^s} \right\|_{\infty} + \sum_{|s|=k} \left| \frac{\partial^s u}{\partial t^s} \right|_W + \sum_{|s|=k} \left| \frac{\partial^s u}{\partial x^s} \right|_W. \quad (29)$$

We can extend the previous definition on $\bar{\Omega} = [-1, 1] \times [-1, 1]$ as follows:

$$C_W^k(\bar{\Omega}) = \left\{ u \in C^k(\bar{\Omega}) : \text{for each } (\tilde{x}, \tilde{t}) \in \bar{\Omega} \text{ there exists a map } \phi : B^2 \rightarrow \bar{\Omega} \right. \\ \left. \text{such that } (\tilde{x}, \tilde{t}) \in \text{int}(\phi(B^2)) \text{ and } f \circ \phi \in C_W^k(B^2) \right\} \quad (30)$$

Since the multiplication by a C^∞ function and the composition with a C^∞ function are continuous linear transformations, it is possible to show that if

$$\phi_i : B^2 \rightarrow \bar{\Omega}, \quad i = 1, \dots, \ell,$$

are a finite collection of maps with

$$\bar{\Omega} = \bigcup_{i=1}^{\ell} \text{int}(\phi_i(B^2)),$$

then $u(\cdot, \cdot) \in C_W^k(\bar{\Omega})$ if and only if $(u \circ \phi_i)(\cdot, \cdot) \in C_W^k(B^2)$ for each $i = 1, \dots, \ell$. Moreover, the space $C_W^k(\bar{\Omega})$ is a Banach space with the norm

$$\|u(\cdot, \cdot)\|_{k,W} = \sum_{i=1}^{\ell} \|(u \circ \phi_i)(\cdot, \cdot)\|_{k,W}. \quad (31)$$

Additionally, any other choice of finitely many maps covering $\bar{\Omega}$ provides an equivalent norm for the Banach space (for more details, see [32]).

Let $\mathcal{P}(N, N, \bar{\Omega})$ be the space of all polynomials of total degree at most $2N$ on $\bar{\Omega}$, namely

$$\mathcal{P}(N, N, \bar{\Omega}) = \left\{ p(\tilde{x}, \tilde{t}) = \sum_{i=0}^N \sum_{j=0}^N b_{ij} \tilde{x}^i \tilde{t}^j : (\tilde{x}, \tilde{t}) \in \bar{\Omega}, b_{ij} \in \mathbb{R} \right\}.$$

The following result is a generalization of the Stone–Weierstrass theorem on the space $C_W^k(\bar{\Omega})$.

Theorem 1 (See [32]) *For any $u(\cdot, \cdot) \in C_W^k(\bar{\Omega})$, there exists a polynomial $p(\cdot, \cdot) \in \mathcal{P}(N, N, \bar{\Omega})$ such that*

$$\|u(\cdot, \cdot) - p(\cdot, \cdot)\|_{\infty} \leq \frac{L_0 L_1}{(2N)^k} W\left(\frac{1}{(2N)^k}\right), \quad (32)$$

where $L_1 = \|u(\cdot, \cdot)\|_{k,W}$ and L_0 is a constant depending on W , but independent of N .

In order to prove the convergence of the method and the existence of solutions of the system (26), we reformulate it as a system of algebraic inequalities in the following way:

$$\begin{cases} |\sum_{j=0}^N \sum_{k=0}^N \hat{u}_{kj} T_j(x_n) T_k(t_m) - \mathcal{L}(u_{nm}^N, t_m)| \leq \frac{\sqrt{N}}{(2N-2)^2} W\left(\frac{1}{(2N-2)^2}\right), & n = 0, \dots, N, \\ & m = 1, \dots, N, \\ |\sum_{j=0}^N \sum_{k=0}^N (-1)^k u_{jk} T_j(x_n) - u_0(x_n)| \leq \frac{\sqrt{N}}{(2N-2)^2} W\left(\frac{1}{(2N-2)^2}\right), & n = 0, \dots, N, \\ |\sum_{j=0}^N \sum_{k=0}^N \sum_{\ell=0}^N (-1)^k D_{k\ell} u_{j\ell} T_j(x_n) - v(x_n)| \leq 0, & n = 0, \dots, N, \end{cases} \quad (33)$$

where N is sufficiently large and W is a given modulus of continuity.

We can notice that

$$\lim_{N \rightarrow \infty} \frac{\sqrt{N}}{(2N-2)^2} W\left(\frac{1}{(2N-2)^2}\right) = 0,$$

so any solution $\bar{u}^N = (\bar{u}_{nm}^N)$ for $n, m = 0, \dots, N$ of the system (33) is a solution of the system (26) when N tends to infinity. As a consequence, to prove the existence of solutions of (26), it is sufficient to prove the existence of solutions for the system (33).

The following lemmas are preliminary to the convergence theorem.

Lemma 1 *Let $u \in C_W^2(\bar{\Omega})$ be a solution of the peridynamic model (6). Then there exists a function \tilde{u} such that*

$$|u(\bar{x}, \bar{t}) - \tilde{u}(\bar{x}, \bar{t})| \leq \frac{2L}{(2N-2)^2} W\left(\frac{1}{(2N-2)^2}\right), \quad (\bar{x}, \bar{t}) \in \bar{\Omega}, \quad (34)$$

for some constant $L > 0$.

Proof By Theorem 1, there exists $p(\cdot, \cdot) \in \mathcal{P}(N-2, N, \bar{\Omega})$ such that

$$\|u_{tt}(\bar{x}, \bar{t}) - p(\bar{x}, \bar{t})\|_{\infty} \leq \frac{L}{(2N-2)^2} W\left(\frac{1}{(2N-2)^2}\right), \quad (\bar{x}, \bar{t}) \in \bar{\Omega}, \quad (35)$$

for some constant $L > 0$ independent of N .

We define

$$\tilde{u}(\bar{x}, \bar{t}) = u(\bar{x}, -1) + u_t(\bar{x}, -1)(\bar{t} + 1) + \int_{-1}^{\bar{t}} \int_{-1}^{\tau} p(\bar{x}, s) ds d\tau, \quad (36)$$

and get

$$\begin{aligned} |u(\bar{x}, \bar{t}) - \tilde{u}(\bar{x}, \bar{t})| &= \left| \int_{-1}^{\bar{t}} \int_{-1}^{\tau} (u_{tt}(\bar{x}, s) - p(\bar{x}, s)) ds d\tau \right| \\ &\leq \int_{-1}^{\bar{t}} \int_{-1}^{\tau} |u_{tt}(\bar{x}, s) - p(\bar{x}, s)| ds d\tau \\ &\leq \frac{L}{(2N-2)^2} W\left(\frac{1}{(2N-2)^2}\right) \int_{-1}^{\bar{t}} \int_{-1}^{\tau} ds d\tau \\ &\leq \frac{2L}{(2N-2)^2} W\left(\frac{1}{(2N-2)^2}\right) \end{aligned}$$

□

Lemma 2 Let $u_1, u_2 \in C_W^2(\bar{\Omega})$ satisfy equation (34). Then, there is a positive constant L such that the following estimate holds:

$$|\mathcal{L}(u_1, \bar{t}) - \mathcal{L}(u_2, \bar{t})| \leq \frac{4L\beta}{(2N-2)^2} W\left(\frac{1}{(2N-2)^2}\right). \quad (37)$$

Proof Thanks to Lemma 1 and the property of the peridynamic operator (3), we find that there exists a positive constant L such that

$$\begin{aligned} |\mathcal{L}(u_1, \bar{t}) - \mathcal{L}(u_2, \bar{t})| &\leq L \int_{B_\delta(x)} C(x-x') |u_1(x', \bar{t}) - u_2(x', \bar{t})| dx' \\ &\quad + L |u_1(x, \bar{t}) - u_2(x, \bar{t})| \int_{B_\delta(x)} C(x-x') dx' \\ &\leq \frac{4L\beta}{(2N-2)^2} W\left(\frac{1}{(2N-2)^2}\right). \quad \square \end{aligned}$$

Now, we are able to prove that there exists at least one solution of (33).

Theorem 2 Let $u \in C_W^2(\bar{\Omega})$ be a solution of the peridynamic model (6). Then there exists a positive integer K such that, for any $N \geq K$, the system (33) admits a solution $\bar{u}^N = (\bar{u}_{nm}^N)$ for $n, m = 0, \dots, N$ such that

$$|u(\bar{x}_k, \bar{t}_h) - \bar{u}_{nm}^N| \leq \frac{L}{(2N-2)^2} W\left(\frac{1}{(2N-2)^2}\right), \quad h, k = 0, \dots, N, \quad (38)$$

for some positive constant L independent of N .

Proof We define

$$\bar{u}_{nm}^N = \tilde{u}(x_n, t_m), \quad n, m = 0, \dots, N, \quad (39)$$

where \tilde{u} is defined in (36) and satisfies equation (34).

By the definition of \tilde{u} , we find that it is a polynomial of degree at most $2N$. Thus, its second derivatives at Gauss–Lobatto nodes (x_n, t_m) , $n, m = 0, \dots, N$ are given by

$$\tilde{u}_{tt}(x_n, t_m) = \sum_{j=0}^N \sum_{k=0}^N \sum_{\ell=0}^N \hat{D}_{k\ell}^{(t)} u_{j\ell} T_j(x_n) T_k(t_m). \quad (40)$$

Using the relations (6), (34), (37), and (40), we get

$$\begin{aligned} &\left| \sum_{j=0}^N \sum_{k=0}^N \sum_{\ell=0}^N \hat{D}_{k\ell}^{(t)} u_{j\ell} T_j(x_n) T_k(t_m) - \mathcal{L}(\bar{u}_{nm}^N, t_m) \right| \\ &= |\tilde{u}_{tt}(x_n, t_m) - \mathcal{L}(\bar{u}_{nm}^N, t_m)| \\ &\leq |\tilde{u}_{tt}(x_n, t_m) - u_{tt}(x_n, t_m)| + |u_{tt}(x_n, t_m) - \mathcal{L}(\bar{u}_{nm}^N, t_m)| \\ &= |p(x_n, t_m) - u_{tt}(x_n, t_m)| + |\mathcal{L}(u(x_n, t_m)) - \mathcal{L}(\bar{u}_{nm}^N, t_m)| \\ &\leq \frac{L(1+4\beta)}{(2N-2)^2} W\left(\frac{1}{(2N-2)^2}\right). \end{aligned} \quad (41)$$

Moreover, we find an analogous estimate for the initial conditions:

$$\begin{aligned} |\tilde{u}(x_n, -1) - u_0(x_n)| &\leq |\tilde{u}(x_n, -1) - u(x_n, -1)| + |u(x_n, -1) - u_0(x_n)| \\ &\leq \frac{2L}{(2N-2)^2} W\left(\frac{1}{(2N-2)^2}\right) \end{aligned} \quad (42)$$

and, by equation (36),

$$|\tilde{u}_t(x_n, -1) - v(x_n)| \leq |\tilde{u}_t(x_n, -1) - u_t(x_n, -1)| + |u_t(x_n, -1) - v(x_n)| \leq 0. \quad (43)$$

Therefore, if we choose K such that

$$\max\{L(4\beta + 1), 2L\} \leq \sqrt{N}, \quad (44)$$

we have that \tilde{u}_{nm}^N , $n, m = 0, \dots, N$ defined in (39) satisfies (33) for $N \geq K$, and this concludes the proof. \square

Finally, we prove that the solution of the system (33) converges to the solution of the peridynamic model (6).

Theorem 3 *Let K be the index defined in (44) and $\tilde{u}^N = (\tilde{u}_{nm}^N)_{n,m=0}^N$ for $N \geq K$ be the sequence of solutions of (33) given by (39), and let $u^N(\cdot, \cdot)$ for $N \geq K$ be its interpolating polynomial*

$$u^N(\bar{x}, \bar{t}) = \sum_{i=0}^N \sum_{j=0}^N \bar{a}_{ij}^N T_i(\bar{x}) T_j(\bar{t}), \quad (45)$$

with

$$\bar{a}_{ij}^N = \frac{1}{\gamma_i \gamma_j} \sum_{n=0}^N \sum_{m=0}^N \tilde{u}_{nm}^N T_i(x_n) T_j(t_m) w_n w_m.$$

Assume that, for any $\bar{x} \in [-1, 1]$, the sequence $\{u^N(\bar{x}, -1), u_t^N(\bar{x}, -1), u_{tt}^N(\cdot, \cdot)\}_{N=K}^\infty$ has a subsequence $\{u^{N_i}(\bar{x}, -1), u_t^{N_i}(\bar{x}, -1), u_{tt}^{N_i}(\cdot, \cdot)\}_{i=0}^\infty$ uniformly converging to

$$(\varphi_1(\bar{x}), \varphi_2(\bar{x}), \varphi_3(\cdot, \cdot)),$$

where $\varphi_1, \varphi_2 \in C^2([-1, 1])$ and $\varphi_3 \in C^2(\bar{\Omega})$. Then

$$\lim_{i \rightarrow \infty} u^{N_i}(\bar{x}, \bar{t}) = \tilde{u}(\bar{x}, \bar{t}), \quad (\bar{x}, \bar{t}) \in \bar{\Omega} \quad (46)$$

is a solution of the peridynamic model (6).

Proof Due to our assumptions, we have

$$\tilde{u}(\bar{x}, \bar{t}) = \varphi_1(\bar{x}) + \varphi_2(\bar{x})(\bar{t} + 1) + \int_{-1}^{\bar{t}} \int_{-1}^{\tau} \varphi_3(\bar{x}, s) ds d\tau. \quad (47)$$

By contradiction, assume that there is an $n \in \{1, \dots, N\}$ such that $\tilde{u}(\bar{x}_n, \cdot)$ does not satisfy (6). Hence, there is a $y \in (-1, 1)$ such that

$$\tilde{u}_{tt}(\bar{x}_n, y) - \mathcal{L}(\tilde{u}(\bar{x}_n, y), \bar{t}_m) \neq 0.$$

Since the Gauss–Lobatto nodes $\{\bar{t}_m\}_{m=0}^N$ are dense in $[-1, 1]$ for $N \rightarrow \infty$, there is a subsequence $\{\bar{t}_{\ell_{N_i}}\}_{i=1}^\infty$ such that $\lim_{i \rightarrow \infty} \bar{t}_{\ell_{N_i}} = y$ and $0 < \ell_{N_i} < N_i$.

We have

$$\begin{aligned} 0 &\neq \tilde{u}_{tt}(\bar{x}_n, y) - \mathcal{L}(\tilde{u}(\bar{x}_n, y), \bar{t}_m) \\ &= \lim_{i \rightarrow \infty} (\tilde{u}_{tt}(\bar{x}_n, \bar{t}_{\ell_{N_i}}) - \mathcal{L}(\tilde{u}(\bar{x}_n, \bar{t}_{\ell_{N_i}}), \bar{t}_m)) \\ &\leq \lim_{i \rightarrow \infty} \frac{\sqrt{N_i}}{(2N_i - 2)^2} W\left(\frac{1}{(2N_i - 2)^2}\right) = 0. \end{aligned}$$

Therefore, $\tilde{u}(\bar{x}, \bar{t})$ satisfies the model (6) for all $\bar{t} \in [-1, 1]$ and $\bar{x} = \bar{x}_n$, $n = 1, \dots, N$.

Using the same argument, we can prove that $\tilde{u}(\bar{x}_n, -1) = u_0(\bar{x}_n)$ and $\tilde{u}_t(\bar{x}_n, -1) = v(\bar{x}_n)$ for $n = 0, \dots, N$, and this completes the proof. \square

5 Numerical tests

In what follows, we make some simulations to validate the proposed method and study the properties of the solution of the peridynamic model (6). All our codes have been written in MATLAB using an Intel(R) Core(TM) i7-5500U CPU @ 2.40 GHz computer.

5.1 Validation of the two-dimensional Chebyshev scheme

The validation of the spectral Chebyshev method is made by comparing the obtained approximated solution with the solution of a benchmark problem.

We consider a bar on the spatial domain $[-1, 1]$ and let the solution evolve in the time interval $[-1, 1]$, so that the computational domain is given by $\bar{\Omega} = [-1, 1] \times [-1, 1]$.

We fix $N > 0$ and discretize $\bar{\Omega}$ by using the Gauss–Lobatto mesh points $(x_n, t_m) = (\cos(n\pi/N), \cos(m\pi/N))$, for $n, m = 0, \dots, N$. We take $u_0(x) = e^{-x^2}$, $v(x) = 0$ as initial conditions for $t = -1$, $\delta = 0.1$ as the size of the horizon, and $C(x) = e^{-x^2}$ as the micromodulus function.

Figure 1 depicts the evolution of the solution, computed by our method, corresponding to the initial condition $u_0(x) = e^{-x^2}$ on the domain $\bar{\Omega}$. To evaluate the convergence of the fully-discrete scheme, we use the relative error E^m , defined as

$$E^m = \frac{\sum_{n=0}^N |u_{nm}^N - u^*(x_n, t_m)|^2}{\sum_{n=0}^N |u^*(x_n, t_m)|^2},$$

where u^* is the reference solution.

We notice that finding an exact solution of a nonlinear problem is a not trivial issue. In this work we determine u^* using our method with a finer mesh.

Table 1 shows the relative error E^m between the exact and numerical solutions for different values of the total number of mesh points N at time $t_m = 1$. We find that the rate of convergence of the scheme is compatible with the theoretical result.

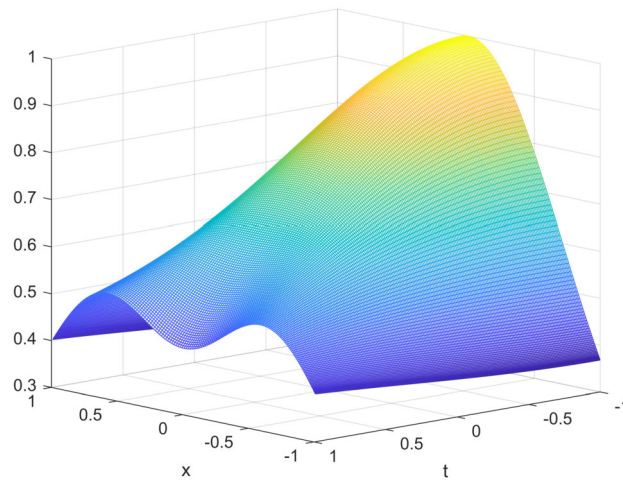


Figure 1 With reference to Sect. 5.1, the evolution of the solution of the problem. The parameters for the simulation are $\delta = 0.1$, $N = 1600$

Table 1 With reference to Sect. 5.1, the relative error between the exact solution and its numerical approximation, related to the initial condition $u_0(x) = e^{-x^2}$, at time $t_m = 1$ as function of the number of discretization points

N	E^m	Convergence rate
100	6.8594×10^{-2}	—
200	1.2508×10^{-2}	2.4552
400	2.1895×10^{-3}	2.4847
800	5.3782×10^{-4}	2.3499
1600	2.6481×10^{-5}	2.7217

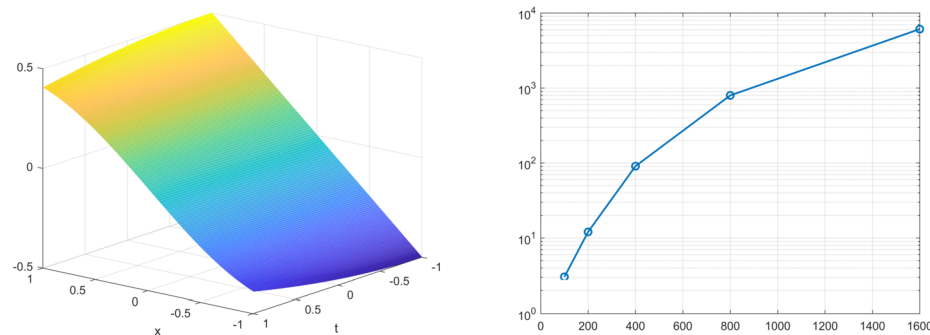


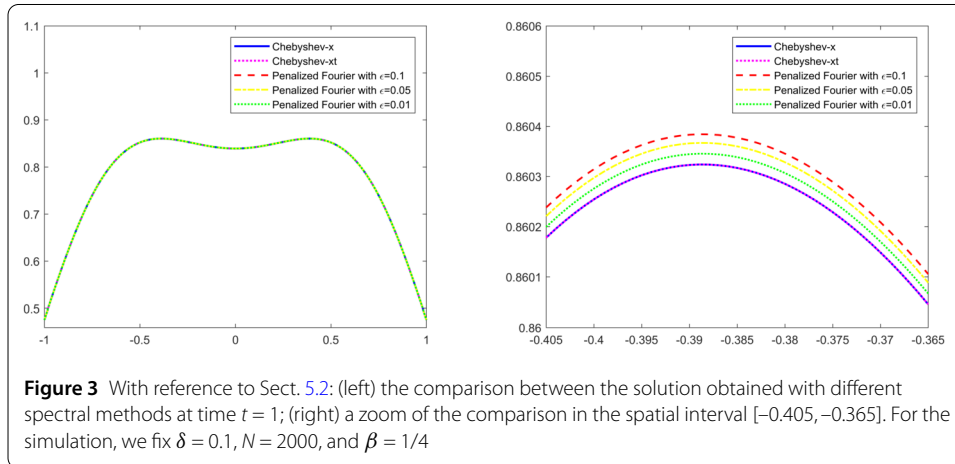
Figure 2 With reference to Sect. 5.1: (left) the evolution of the solution corresponding the the initial displacement $u_0(x) = x/2$; (right) the behavior of the CPU cost depending of the total number of collocation points. For the simulation, we fix $\delta = 0.1$

Additionally, we analyze the performance of the method in terms of the computational cost required to complete the simulation.

We consider the same setting as before and fix $u_0(x) = x/2$ as the initial displacement. The solution of the problem is plotted in the left panel of Fig. 2. In Table 2 and in the right panel of Fig. 2, we find that the method seems very competitive in terms of CPU cost. This is because the method exploits the properties of the Fast Fourier Transform algorithm.

Table 2 With reference to Sect. 5.1, the execution time of the Chebyshev spectral method as a function of the total number of collocation points, N

N	CPU time [s]
100	3.0316×10^0
200	1.2051×10^1
400	9.0659×10^1
800	7.9556×10^2
1600	6.0924×10^3



5.2 A comparison between Chebyshev–Newmark- β and the two-dimensional Chebyshev methods

In [29], the authors propose a spectral Chebyshev method for the spatial domain coupled with the Newmark- β integrator to approximate the solution of the peridynamic model (6). They showed good accuracy and performance in terms of CPU cost with respect to other spectral methods.

In this section, we make a comparison with our two-dimensional Chebyshev method and the Chebyshev–Newmark- β method of [29].

We clearly expect to find the same accuracy in space, as the spatial discretization method is practically the same. So, the aim of the comparison is to study the performance of the two methods in terms of CPU cost to complete the simulation.

We make some tests similar to those made in [29, Sect. 4.2]. We work on $\bar{\Omega} = [-1, 1] \times [-1, 1]$, take $u_0(x) = e^{-x^2}$ as the initial displacement, $\delta = 0.1$, $N = 2000$, and $\beta = 1/4$.

The solution of the problem at $t = 1$ and its zoom on a small portion of the spatial domain is shown in Fig. 3. As expected, we find a good agreement between the solution obtained with the two methods, and both of them are more accurate with respect to the penalized Fourier method. (We refer the reader to [26] for a detailed description of the penalized Fourier spectral method).

Moreover, using the same setting, we analyze the methods in terms of time required to complete the simulation. When we deal with the Chebyshev–Newmark- β method we have to vary both the space and the time step size, while this is implicitly done with the two-dimensional Chebyshev method if we fix $N > 0$ as total number of grid points for space and time variables. In what follows, we fix $\Delta x = \Delta t = 2/N$.

Table 3 With reference to Sect. 5.2, the execution time of the Chebyshev–Newmark- β method and the two-dimensional Chebyshev method as a function of the number of discretization points, N , for $\beta = 1/4$ and $\delta = 0.1$

N	CPU time [s]	
	Chebyshev–Newmark	2D Chebyshev
128	7.8416×10^0	3.6781×10^0
256	9.0142×10^1	1.3984×10^1
512	5.1257×10^2	1.0483×10^2
1024	4.6573×10^3	9.1195×10^2
2048	7.2561×10^4	6.9763×10^3

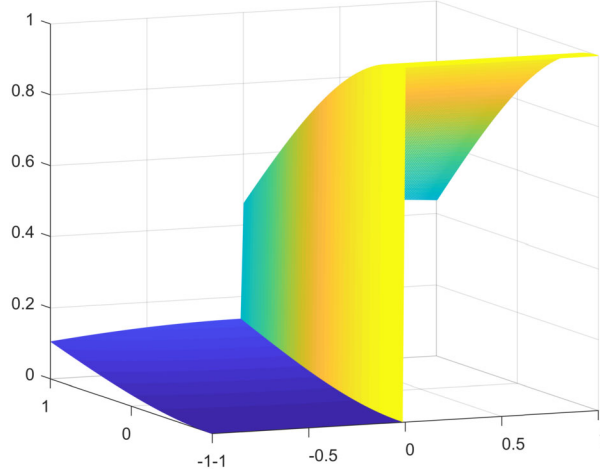


Figure 4 With reference to Sect. 5.3, the evolution of the solution corresponding to a singular initial displacement. The parameters for the simulation are $\delta = 0.1$ and $N = 1000$

We summarize the results in Table 3. We find a better result for the two-dimensional Chebyshev method, and this should depend on the fact that our method does not require any direct time integration as it is incorporated in the part of the algorithm in which we exploit the Fast Fourier Transform algorithm. Instead, the computational cost to solve the algebraic system (26) is compensated by the cost to solve the system derived by the implementation of the implicit Newmark- β method.

5.3 The case of a discontinuous initial datum

We now study the performance of Chebyshev spectral method applied to a problem with a discontinuous initial condition. We consider the same setting as in the previous sections and take $u_0(x) = \chi_{[0,1]}(x)$ as initial displacement, where $\chi_{[0,1]}(x)$ denotes the indicator function, which is identically one on the interval $[0, 1]$, and is zero elsewhere. We plot the dynamic of the solution in Fig. 4, while the error study is summarized in Table 4.

We can notice the loss of one order of convergence due to the presence of a singularity in the initial displacement. This is in accordance with the results in [26, 29].

6 Conclusion and future works

In this work, we propose a two-dimensional fast-convolution spectral method based on the implementation of Chebyshev polynomials to approximate the solution of a one-dimensional nonlinear peridynamic model having a power-type nonlinearity in the bi-

Table 4 With reference to Sect. 5.3, the relative error and the convergence rate, related to the initial displacement $u_0(x) = \chi_{[0,1]}(x)$, at time $t_m = 1$ as a function of the number of discretization points

N	E^m	Convergence rate
100	4.1382×10^{-1}	—
200	1.4162×10^{-1}	1.5470
400	6.9453×10^{-2}	1.2874
800	3.6519×10^{-2}	1.1535
1600	2.6514×10^{-3}	1.6528

variate force function. The method is very accurate as it can exploit the benefits of the Fast Fourier Transform algorithm. Moreover, the idea to deal with the problem in a two-dimensional domain allows us to obtain the same accuracy in both space and time variables without requiring the implementation of a numerical scheme to integrate the discrete method in time. We prove the convergence of the proposed method and perform some simulations to validate the Chebyshev scheme and study the properties of the solutions.

In the future, we plan to extend the method to higher-dimensional problems and we aim to couple the approach with techniques based on mimetic and virtual element methods (see, for example, [5]).

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Availability of data and materials

The datasets used and/or analyzed during the current study are available from the corresponding author on reasonable request.

Declarations

Ethics approval and consent to participate

The authors approve the ethics of the journal and give the consent to participate.

Consent for publication

The authors consent to the publication.

Competing interests

The authors declare that they have no competing interests.

Author contribution

The authors declare that they gave their individual contributions in every section of the manuscript. All authors read and approved the final manuscript.

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