Advances in Continuous and Discrete Models

RESEARCH Open Access

The mixed nonlinear Schrödinger equation on the half-line

Guenbo Hwang[1*]

*Correspondence:
ghwang@daegu.ac.kr
[1]Department of Mathematics and Institute of Natural Sciences, Daegu University, Gyeongsan, Gyeongbuk 38453, Korea

Abstract

We analyze the initial-boundary value problem for the mixed nonlinear Schrödinger equation posed on the half-line by using the Fokas method. Assuming that a smooth solution exists, we show that the solution can be represented in term of the solution of a matrix Riemann–Hilbert problem formulated in the complex plane. The jump matrix is defined in terms of the spectral functions obtained from the initial and boundary values. We derive a certain relation, the so-called global relation that involves all initial and boundary values. Furthermore, it can be shown that the solution for the mixed nonlinear Schrödinger equation posed on the half-line exists, which can be determined by the unique solution of the associated Riemann–Hilbert problem, as long as the spectral functions satisfy the above global relation.

MSC: 35Q15; 35Q55; 37K15

Keywords: Initial-boundary value problem; Mixed nonlinear Schrödinger equation; Fokas method; Inverse scattering transform

1 Introduction

A mixed nonlinear Schrödinger equation

$$q_t + iq_{xx} - i\alpha\left(|q|^2 q\right)_x - 2\beta^2|q|^2 q = 0, \quad \alpha, \beta \in \mathbb{R} \tag{1}$$

can be viewed as a modification of the celebrated nonlinear Schrödinger equation (NLS) equation. For $\alpha = 0$ and $\beta \neq 0$, the mixed NLS equation leads to the classical defocusing NLS equation. On the other hand, if $\alpha \neq 0$ and $\beta = 0$, it is reduced to the derivative NLS equation. The NLS equation is fundamental to the study of nonlinear wave phenomena in fluid flow, nonlinear optics, plasma physics, and various applied mathematics problems [1–6]. In the event that Alfvén waves are considered in plasma physics, the polarized nonlinear Alfvén wave propagation along the magnetic field is governed by the derivative NLS equation [7]. The mixed NLS and derivative NLS equations also describe the nonlinear self-steepening of ultrashort light pulse propagation in optical fibers [8, 9]. In addition to a variety of applications in physical settings, these NLS-type equations display a particularly rich mathematical structure, known as integrability, and subsequently, they can be solved by the inverse scattering transform (IST). For example, the IST was used to solve the NLS and derivative NLS equations under the rapid vanishing condition ($q \to 0$ as $x \to \pm\infty$), as well as the nonvanishing condition [1, 10, 11]. In a similar fashion, the IST has been applied to analyze initial value problems for the mixed NLS equation [12, 13]. Furthermore, the bright and dark soliton solutions have been investigated by using Darboux transformation [14, 15], and the long-time asymptotics for the solution has been determined by using the Deift–Zhou steepest descent method [16].

In this paper, we study the initial-boundary value problem (IBVP) for the mixed NLS equation (1) formulated on the half-line, that is, in the domain

$$\left\{(x,t) \in \mathbb{R}^2 \mid 0 \le x, 0 \le t \le T\right\} \tag{2}$$

with the initial and boundary conditions, denoted by

$$q(x,0) = q_0(x), \qquad q(0,t) = g_0(t), \qquad q_x(0,t) = g_1(t), \tag{3}$$

where the function $q_0(x)$ is assumed to be sufficiently smooth for $x > 0$ and to rapidly decay as $x \to \infty$ and the functions $g_j(t)$ ($j = 0, 1$) are assumed to be sufficiently smooth for $t > 0$ (and to decay fast as $t \to \infty$ if $T = \infty$). We use the unified transform method proposed in [17, 18], also known as the Fokas method (see also the monograph [19] and references therein), for solving Eq. (1) formulated on the half-line. The Fokas method, which is a significant extension of the IST for boundary value problems, is based on the simultaneous spectral analysis of both parts of the Lax pair. The Fokas method can be summarized as the following steps. (i) Assuming that a smooth solution $q(x,t)$ exists, it can be represented by the solution of a matrix Riemann–Hilbert problem with the jump matrix defined by the spectral functions of a spectral parameter λ, denoted by $\{a(\lambda), b(\lambda), A(\lambda), B(\lambda)\}$. The spectral functions are defined by the initial and boundary values and importantly, they satisfy a certain relation, the so-called global relation which involves all initial and boundary values. (ii) Define the spectral functions in terms of the given smooth functions $q_0(x)$ and $g_j(t)$ ($j = 0, 1$) as the initial and boundary conditions. It can be shown that if these spectral functions satisfy the global relation given in the first step, then the function $q(x,t)$ defined from the unique solution of the Riemann–Hilbert problem solves the mixed NLS equation and satisfies the initial and boundary conditions.

The Fokas method has several important advantages in analyzing boundary value problems. First, the jump matrix of the Riemann–Hilbert problem defined by the spectral functions has an explicit exponential (x,t)-dependence. Thus, it makes it possible to study the long-time asymptotic behavior of the solution by using the nonlinear steepest descent method presented in [20]. The steepest descent method with the Fokas method has been studied to determine the long-time asymptotics of the solutions for boundary values problems of integrable systems, such as the modified Korteweg–de Vries equation [21], the NLS equation [22], the derivative NLS equation [23], and Kundu–Eckhaus equation [24]. Next, the spectral functions satisfy the global relation, which can be used to establish the existence theorem for the unique solution of IBVPs. In addition, the global relation provides a constraint on the initial and boundary values, so that we can characterize unknown boundary values [25–27].

The outline of the paper is as follows. In Sect. 2, we analyze the Lax pair for the mixed NLS equation and we then define the appropriate eigenfunctions and the spectral functions that will be used to formulate a matrix Riemann–Hilbert problem for the mixed NLS equation posed on the half-line. We derive the global relation in terms of the spectral functions. In Sect. 3, the spectral functions defined by the given initial and boundary values are investigated with the corresponding Riemann–Hilbert problems as inverse problems. In Sect. 4, we establish the existence theorem for the solution of the mixed NLS equation posed on the half-line under the assumption that the spectral functions satisfy the global relation. We end with concluding remarks in Sect. 5.

2 Spectral analysis

2.1 Lax pair and eigenfunctions

The mixed NLS equation admits an overdetermined linear system, known as the Lax pair [16]

$$\psi_x + if_1(\lambda)\sigma_3\psi = U_1\psi, \tag{4a}$$

$$\psi_t + if_2(\lambda)\sigma_3\psi = U_2\psi, \tag{4b}$$

where $\lambda \in \mathbb{C}$ is a spectral parameter, $\psi(x,t,k)$ is a 2×2 matrix-valued eigenfunction and

$$f_1(\lambda) = \lambda(\alpha\lambda - 2\beta), \qquad f_2(\lambda) = 2\lambda^2(\alpha\lambda - 2\beta)^2, \tag{5a}$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad U_1 = (\alpha\lambda - \beta)Q, \qquad Q = \begin{pmatrix} 0 & q \\ \bar{q} & 0 \end{pmatrix}, \tag{5b}$$

$$\begin{aligned} U_2 = {}& -i\big(\alpha^2\lambda^2 - 2\alpha\beta\lambda + \beta^2\big)Q^2\sigma_3 + \big(2\alpha^2\lambda^3 - 6\alpha\beta\lambda^2 + 4\beta^2\lambda\big)Q \\ & + i(\alpha\lambda - \beta)Q_x\sigma_3 + \big(\alpha^2\lambda - \alpha\beta\big)Q^3. \end{aligned} \tag{5c}$$

Throughout the paper, we consider the case of $\alpha\beta \neq 0$. Letting $\Psi = \psi e^{i(f_1(\lambda)x + f_2(\lambda)t)\sigma_3}$, we find the modified Lax pair

$$\Psi_x + if_1(\lambda)[\sigma_3, \Psi] = U_1\Psi, \tag{6a}$$

$$\Psi_t + if_2(\lambda)[\sigma_3, \Psi] = U_2\Psi. \tag{6b}$$

Let $\hat{\sigma}_3$ denote the matrix commutator and then $e^{\hat{\sigma}_3}$ can be easily computed as

$$\hat{\sigma}_3 A = [\sigma_3, A] = \sigma_3 A - A\sigma_3, \qquad e^{\hat{\sigma}_3}A = e^{\sigma_3}Ae^{-\sigma_3},$$

where A is a 2×2 matrix. We seek solutions of the Lax pair which are bounded and approach the 2×2 identity matrix as $\lambda \to \infty$. In this respect, we expand Ψ as

$$\Psi(x,t,\lambda) = \Psi^{(0)}(x,t) + \frac{\Psi^{(1)}(x,t)}{\lambda} + O\big(1/\lambda^2\big) \quad (\lambda \to \infty). \tag{7}$$

Substituting Eq. (7) into Eqs. (6a)–(6b), one can find [16]

$$\Psi^{(0)}(x,t) = e^{i\int_{(x_0,t_0)}^{(x,t)} \Delta\sigma_3}$$

with the closed differential one-form defined by

$$\Delta(x,t) = \Delta_1\,dx + \Delta_2\,dt = \frac{1}{2}\alpha|q|^2\,dx + \left[\frac{3}{4}\alpha^2|q|^4 - \frac{i}{2}(\bar{q}q_x - \bar{q}_x q)\right]dt \tag{8}$$

and the off-diagonal part of $\Psi^{(1)}$, denoted by $\Psi^{(1,O)}$, as

$$\Psi^{(1,O)} = \frac{1}{2i}\sigma_3 Q \Psi^{(0)}. \tag{9}$$

For a simple calculation, we take $(x_0, t_0) = (0, 0)$. As a result, the asymptotic behavior for the eigenfunction Ψ given in Eq. (7) suggests introducing a new eigenfunction $\mu(x,t,\lambda)$,

$$\Psi(x,t,\lambda) = e^{i\int_{(0,0)}^{(x,t)}\Delta\hat{\sigma}_3}\mu(x,t,\lambda)\Psi^{(0)}(x,t). \tag{10}$$

We then have

$$\mu(x,t,\lambda) = I + O(1/\lambda) \quad (\lambda \to \infty).$$

Defining a closed differential one-form W given by

$$W(x,t,\lambda) = e^{i(f_1(\lambda)x+f_2(\lambda)t)\hat{\sigma}_3}V(x,t,\lambda)\mu(x,t,\lambda), \tag{11}$$

equations (6a)–(6b) can be written as

$$d\big(e^{i(f_1(\lambda)x+f_2(\lambda)t)\hat{\sigma}_3}\mu(x,t,\lambda)\big) = W(x,t,\lambda), \tag{12}$$

where

$$V(x,t,\lambda) = V_1(x,t,\lambda)\,dx + V_2(x,t,\lambda)\,dt = e^{-i\int_{(0,0)}^{(x,t)}\Delta\hat{\sigma}_3}[U_1\,dx + U_2\,dt - i\Delta\sigma_3].$$

More precisely,

$$V_1(x,t,\lambda) = e^{-i\int_{(0,0)}^{(x,t)}\Delta\hat{\sigma}_3}\left(U_1 - \frac{i}{2}\alpha|q|^2\sigma_3\right), \tag{13a}$$

$$V_2(x,t,\lambda) = e^{-i\int_{(0,0)}^{(x,t)}\Delta\hat{\sigma}_3}\left[U_2 - i\left(\frac{3}{4}\alpha^2|q|^4 - \frac{i}{2}(\bar{q}q_x - \bar{q}_x q)\right)\sigma_3\right]. \tag{13b}$$

We note that Eq. (12) is equivalent to the following modified Lax pair:

$$\mu_x + if_1(\lambda)[\sigma_3, \mu] = V_1\mu, \tag{14a}$$

$$\mu_t + if_2(\lambda)[\sigma_3, \mu] = V_2\mu. \tag{14b}$$

We define the Jost eigenfunction as the simultaneous solution for the both parts of the Lax pair (14a)–(14b) as

$$\mu_j(x,t,\lambda) = I + \int_{(x_j,t_j)}^{(x,t)} e^{-i(f_1(\lambda)x+f_2(\lambda)t)\hat{\sigma}_3}W_j(\xi,\tau,\lambda), \tag{15}$$

where W_j is the differential one-form defined by Eq. (11) with μ_j. Since the one-form W is exact, the function μ_j defined by Eq. (15) is independent of the path of integration. Hence, we choose three distinct normalization points (cf. Fig. 1)

$$(x_1, t_1) = (0, T), \qquad (x_2, t_2) = (0, 0), \qquad (x_3, t_3) = (\infty, t).$$

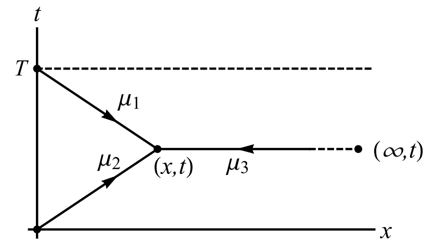

Figure 1 The eigenfunctions μ_1, μ_2, and μ_3 for the Lax pair (14a)–(14b)

More specifically, we define the Jost eigenfunctions that satisfy the integral equations

$$\mu_1(x,t,\lambda) = I + \int_0^x e^{if_1(\lambda)(\xi-x)\hat{\sigma}_3} V_1\mu_1(\xi,t,\lambda)\,d\xi \\ - e^{-if_1(\lambda)x\hat{\sigma}_3} \int_t^T e^{if_2(\lambda)(\tau-t)\hat{\sigma}_3} V_2\mu_1(0,\tau,\lambda)\,d\tau, \tag{16a}$$

$$\mu_2(x,t,\lambda) = I + \int_0^x e^{if_1(\lambda)(\xi-x)\hat{\sigma}_3} V_1\mu_2(\xi,t,\lambda)\,d\xi \\ + e^{-if_1(\lambda)x\hat{\sigma}_3} \int_0^t e^{if_2(\lambda)(\tau-t)\hat{\sigma}_3} V_2\mu_2(0,\tau,\lambda)\,d\tau, \tag{16b}$$

$$\mu_3(x,t,\lambda) = I - \int_x^\infty e^{if_1(\lambda)(\xi-x)\hat{\sigma}_3} V_1\mu_3(\xi,t,\lambda)\,d\xi. \tag{16c}$$

Note that the off-diagonal components of the matrix-valued eigenfunctions μ_j $(j = 1,2,3)$ involve the explicit exponential terms. Thus, we partition the complex λ-plane into the domains D_j $(j = 1,\dots,4)$ depending on the signs of the imaginary parts of the functions $f_1(\lambda)$ and $f_2(\lambda)$ (cf. Fig. 2)

$$D_1 = \{\lambda \in \mathbb{C} \mid \mathrm{Im} f_1(\lambda) > 0 \text{ and } \mathrm{Im} f_2(\lambda) > 0\},$$
$$D_2 = \{\lambda \in \mathbb{C} \mid \mathrm{Im} f_1(\lambda) > 0 \text{ and } \mathrm{Im} f_2(\lambda) < 0\},$$
$$D_3 = \{\lambda \in \mathbb{C} \mid \mathrm{Im} f_1(\lambda) < 0 \text{ and } \mathrm{Im} f_2(\lambda) > 0\},$$
$$D_4 = \{\lambda \in \mathbb{C} \mid \mathrm{Im} f_1(\lambda) < 0 \text{ and } \mathrm{Im} f_2(\lambda) < 0\}.$$

Letting $\lambda = \xi + i\eta$ $(\xi,\eta \in \mathbb{R})$, we find

$$\mathrm{Im} f_1(\lambda) = 2\eta(\alpha\xi - \beta), \qquad \mathrm{Im} f_2(\lambda) = 4\,\mathrm{Im} f_1(\lambda)\big[\alpha\big(\xi^2 - \eta^2\big) - 2\beta\xi\big],$$

and the boundaries of D_j are depicted in Fig. 2 for the case of $\alpha\beta > 0$. We denote by $\mu^{(1)}$ and $\mu^{(2)}$ the columns of 2×2 matrix $\mu(x,t,\lambda) = (\mu^{(1)},\mu^{(2)})$. We can determine regions where the eigenfunctions are analytic and bounded as follows:

- μ_1 is an entire function of λ if $T < \infty$; $\mu_1^{(1)}$ is bounded for $\lambda \in \bar{D}_2$, while $\mu_1^{(2)}$ is bounded for $\lambda \in \bar{D}_3$, where $\bar{D}$ is the closure of D. If $T = \infty$, $\mu_1^{(1)}$ is defined for $\lambda \in \bar{D}_2$ and $\mu_1^{(2)}$ is defined for $\lambda \in \bar{D}_3$.
- μ_2 is an entire function of λ; $\mu_2^{(1)}$ is bounded for $\lambda \in \bar{D}_1$, while $\mu_2^{(2)}$ is bounded for $\lambda \in \bar{D}_4$.
- $\mu_3^{(1)}$ is analytic for $\lambda \in D_3 \cup D_4$, and bounded for $\lambda \in \bar{D}_3 \cup \bar{D}_4$; $\mu_3^{(2)}$ is analytic for $\lambda \in D_1 \cup D_2$, and bounded for $\lambda \in \bar{D}_1 \cup \bar{D}_2$.

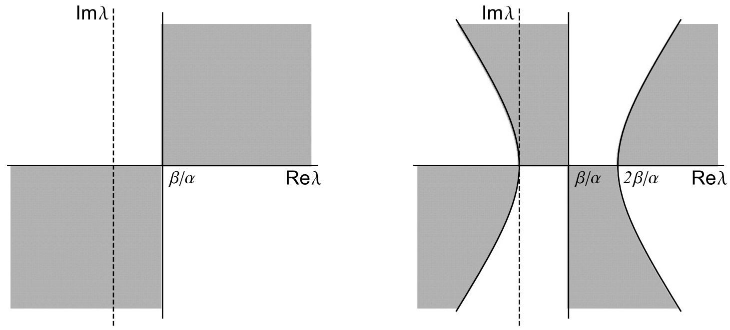

Figure 2 (Left) The regions (shaded) of the complex λ-plane, where $\mathrm{Im}\, f_1(\lambda) > 0$ with $\alpha\beta > 0$. (Right) The regions (shaded) of the λ-plane, where $\mathrm{Im}\, f_2(\lambda) > 0$ with $\alpha\beta > 0$ (see text for details)

Note that the potential functions V_1 and V_2 have the following symmetries:

$$\sigma_+ \overline{V_j(x,t,\bar{\lambda})}\sigma_+ = V_j(x,t,\lambda), \qquad -\sigma_- \overline{V_j(x,t,2\beta/\alpha - \bar{\lambda})}\sigma_- = V_j(x,t,\lambda), \quad j = 1,2,$$

where

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_- = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

As a consequence, the Jost solutions μ_j defined by Eqs. (16a)–(16c) ($j = 1,2,3$) satisfy the symmetry relations

$$\sigma_+ \overline{\mu_j(x,t,\bar{\lambda})}\sigma_+ = \mu_j(x,t,\lambda), \qquad -\sigma_- \overline{\mu_j(x,t,2\beta/\alpha - \bar{\lambda})}\sigma_- = \mu_j(x,t,\lambda). \tag{18}$$

Moreover, the asymptotic behavior of the eigenfunction as $\lambda \to \infty$ leads to the reconstruction formula for the solution of the mixed NLS equation on the half-line. It follows from Eq. (9) that the reconstruction formula for the solution is given by

$$q(x,t) = 2im(x,t)e^{2i\int_{(0,0)}^{(x,t)}\Delta}, \tag{19}$$

where

$$m(x,t) = \lim_{\lambda\to\infty} \lambda\mu_{12}(x,t,\lambda). \tag{20}$$

We note that $|q| = 2|m|$ and

$$q_x(x,t) = 2i\left(m_x + 4i\alpha m|m|^2\right)e^{2i\int_{(0,0)}^{(x,t)}\Delta},$$

which implies that

$$\bar{q}q_x - \bar{q}_x q = 4(\bar{m}m_x - m\bar{m}_x) + 32i\alpha|m|^4.$$

Thus the differential one-form $\Delta(x,t)$ is given in terms of m as

$$\Delta(x,t) = 2\alpha|m|^2\,dx + \left[12\alpha^2|m|^4 - 2i(\bar{m}m_x - m\bar{m}_x) + 16\alpha|m|^4\right]dt. \tag{21}$$

Thus the inverse problem can be solved in the following steps. (i) Use any of the eigenfunctions μ_j ($j = 1, 2, 3$) to find m given in Eq. (20). (ii) Determine $\Delta(x, t)$ given in Eq. (21). (iii) The solution for the mixed NLS equation on the half-line can be determined from Eq. (19).

Any two solutions of Eq. (15) are related by the so-called scattering matrices $s(\lambda)$ and $S(\lambda)$, also known as the spectral matrices, as follows:

$$\mu_3(x,t,\lambda) = \mu_2(x,t,\lambda)e^{-i(f_1(\lambda)x+f_2(\lambda)t)\hat{\sigma}_3}s(\lambda), \tag{22a}$$

$$\mu_1(x,t,\lambda) = \mu_2(x,t,\lambda)e^{-i(f_1(\lambda)x+f_2(\lambda)t)\hat{\sigma}_3}S(\lambda). \tag{22b}$$

Using $\mu_2(0,0,\lambda) = I$, Eq. (22a) implies that

$$s(\lambda) = \mu_3(0,0,\lambda). \tag{23}$$

Thus, the spectral matrix $s(\lambda)$ can be expressed in terms of the eigenfunction μ_3 as

$$s(\lambda) = I - \int_0^\infty e^{if_1(\lambda)x\hat{\sigma}_3} V_1\mu_3(x,0,\lambda)\,dx.$$

On the other hand, noting that $\mu_1(0,T,\lambda) = I$, Eq. (22b) yields

$$S(\lambda) = \left(e^{if_2(\lambda)T\hat{\sigma}_3}\mu_2(0,T,\lambda)\right)^{-1} \tag{24}$$

and hence the spectral matrix $S(\lambda)$ can be expressed in terms of the eigenfunction μ_2 as

$$S^{-1}(\lambda) = I + \int_0^T e^{if_2(\lambda)t\hat{\sigma}_3} V_2\mu_2(0,t,\lambda)\,dt.$$

We also remark that $\mu_2(0,0,\lambda) = I$ implies that $S(\lambda) = \mu_1(0,0,\lambda)$ and the spectral matrices inherit the symmetries given in Eq. (18), namely

$$\sigma_+\overline{s(\bar{\lambda})}\sigma_+ = s(\lambda), \qquad \sigma_+\overline{S(\bar{\lambda})}\sigma_+ = S(\lambda), \tag{25a}$$

$$-\sigma_-\overline{s(2\beta/\alpha - \bar{\lambda})}\sigma_- = s(\lambda), \qquad -\sigma_-\overline{S(2\beta/\alpha - \bar{\lambda})}\sigma_- = S(\lambda). \tag{25b}$$

Thus, we write the spectral matrices as

$$s(\lambda) = \begin{pmatrix} \overline{a(\bar{\lambda})} & b(\lambda) \\ \overline{b(\bar{\lambda})} & a(\lambda) \end{pmatrix}, \qquad S(\lambda) = \begin{pmatrix} \overline{A(\bar{\lambda})} & B(\lambda) \\ \overline{B(\bar{\lambda})} & A(\lambda) \end{pmatrix}.$$

We summarize the properties of the spectral functions $s(\lambda)$ and $S(\lambda)$ as

- $s^{(1)}(\lambda)$ is analytic for $\lambda \in D_3 \cup D_4$, and bounded for $\lambda \in \bar{D}_3 \cup \bar{D}_4$, while $s^{(2)}(\lambda)$ is analytic for $\lambda \in D_1 \cup D_2$, and bounded for $\lambda \in \bar{D}_1 \cup \bar{D}_2$. Moreover,

$$a(\lambda) = a(2\beta/\alpha - \lambda), \qquad b(\lambda) = -b(2\beta/\alpha - \lambda). \tag{26}$$

- $\det s(\lambda) = 1$ and hence $a(\lambda)\overline{a(\bar{\lambda})} - b(\lambda)\overline{b(\bar{\lambda})} = 1$. $s(\lambda) = I + O(1/\lambda)$ as $\lambda \to \infty$ in the respective domains of boundedness of the columns.

- $S(\lambda)$ is an entire function of λ if $T < \infty$; $S^{(1)}(\lambda)$ is bounded for $\lambda \in \bar{D}_2 \cup \bar{D}_4$, and $S^{(2)}(\lambda)$ is bounded for $\lambda \in \bar{D}_1 \cup \bar{D}_3$. If $T = \infty$, the spectral functions $S^{(1)}(\lambda)$ and $S^{(2)}(\lambda)$ are defined for $\lambda \in \bar{D}_2 \cup \bar{D}_4$ and $\lambda \in \bar{D}_1 \cup \bar{D}_3$, respectively. Moreover,

$$A(\lambda) = A(2\beta/\alpha - \lambda), \qquad B(\lambda) = -B(2\beta/\alpha - \lambda). \tag{27}$$

- $\det S(\lambda) = 1$ and hence $A(\lambda)\overline{A(\bar{\lambda})} - B(\lambda)\overline{B(\bar{\lambda})} = 1$. $S(\lambda) = I + O(1/\lambda)$ as $\lambda \to \infty$ in the respective domains of boundedness of the columns.

Furthermore, using Eq. (22a), we find

$$\mu_2(x,t,\lambda) = \mu_3(x,t,\lambda)e^{-i(f_1(\lambda)x+f_2(\lambda)t)\hat{\sigma}_3}s^{-1}(\lambda).$$

Substituting the above equation into Eq. (22b), we obtain the relation

$$\mu_1(x,t,\lambda) = \mu_3(x,t,\lambda)e^{-i(f_1(\lambda)x+f_2(\lambda)t)\hat{\sigma}_3}\big(s^{-1}(\lambda)S(\lambda)\big), \tag{28}$$

where the first column is defined for $\lambda \in D_3 \cup D_4$ and the second column holds for $\lambda \in D_1 \cup D_2$. Evaluating Eq. (28) at $(x,t) = (0,T)$, the spectral functions satisfy the following relation, called the global relation:

$$S^{-1}(\lambda)s(\lambda) = e^{if_2(\lambda)T\hat{\sigma}_3}\mu_3(0,T,\lambda), \quad \lambda \in (D_3 \cup D_4, D_1 \cup D_2). \tag{29}$$

The $(1,2)$-entry of Eq. (29) yields

$$A(\lambda)b(\lambda) - a(\lambda)B(\lambda) = e^{2if_2(\lambda)T}c(\lambda), \quad \lambda \in D_1 \cup D_2, \tag{30}$$

where $c(\lambda) = -\int_0^\infty e^{2if_1(\lambda)x}(V_1\mu_3)_{12}(x,T,\lambda)\,dx$. Note that $c(\lambda)$ is analytic for $\lambda \in D_1 \cup D_2$ and is bounded for $\lambda \in \bar{D}_1 \cup \bar{D}_2$ with $c(\lambda) = O(1/\lambda)$ as $\lambda \to \infty$. If $T = \infty$, Eq. (30) becomes

$$A(\lambda)b(\lambda) - a(\lambda)B(\lambda) = 0, \quad \lambda \in D_1 \cup D_2. \tag{31}$$

2.2 Riemann–Hilbert problem

We will formulate the matrix Riemann–Hilbert problem for the mixed NLS equation on the half-line. For later reference, we introduce the quantities

$$\begin{aligned}
&\theta(x,t,\lambda) = f_1(\lambda)x + f_2(\lambda)t, \qquad r_1(\lambda) = \frac{\overline{b(\bar{\lambda})}}{a(\lambda)},\\
&d(\lambda) = a(\lambda)\overline{A(\bar{\lambda})} - b(\lambda)\overline{B(\bar{\lambda})}, \qquad \Gamma(\lambda) = -\frac{\overline{B(\bar{\lambda})}}{a(\lambda)d(\lambda)}, \qquad r(\lambda) = r_1(\lambda) + \Gamma(\lambda).
\end{aligned}$$

Theorem 2.1 *Assume that $q(x,t,\lambda)$ is a sufficiently smooth function. Then $\mu_j(x,t,\lambda)$ ($j = 1,2,3$) given by Eqs. (16a)–(16c) define the following Riemann–Hilbert problem*

$$M_+(x,t,\lambda) = M_-(x,t,\lambda)J(x,t,\lambda), \quad \lambda \in L, \tag{33}$$

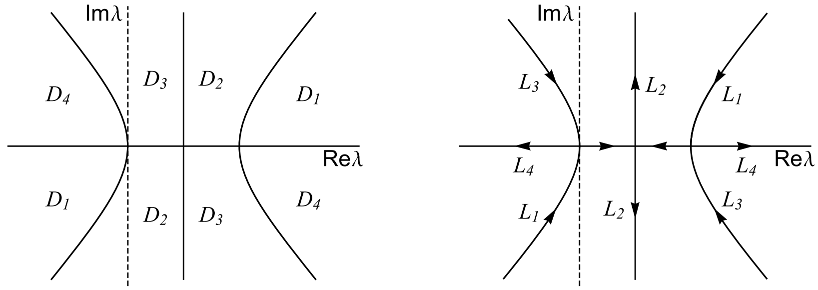

Figure 3 (Left) The regions $D_1,\ldots,D_4$ of the complex λ-plane for the case of $\alpha\beta > 0$. (Right) The contours $L_1,\ldots,L_4$ that define the Riemann–Hilbert problem for the case of $\alpha\beta > 0$

where the sectionally meromorphic functions $M_\pm$ are defined by

$$M_+(x,t,\lambda) = \begin{cases} (\frac{\mu_2^{(1)}(x,t,\lambda)}{a(\lambda)}, \mu_3^{(2)}(x,t,\lambda)), & \lambda \in D_1, \\ (\mu_3^{(1)}(x,t,\lambda), \frac{\mu_1^{(2)}(x,t,\lambda)}{\overline{d(\bar\lambda)}}), & \lambda \in D_3, \end{cases} \tag{34a}$$

$$M_-(x,t,\lambda) = \begin{cases} (\frac{\mu_1^{(1)}(x,t,\lambda)}{d(\lambda)}, \mu_3^{(2)}(x,t,\lambda)), & \lambda \in D_2, \\ (\mu_3^{(1)}(x,t,\lambda), \frac{\mu_2^{(2)}(x,t,\lambda)}{\overline{a(\bar\lambda)}}), & \lambda \in D_4, \end{cases} \tag{34b}$$

and the jump matrix is given by

$$J_1 = \begin{pmatrix} 1 & 0 \\ e^{2i\theta(x,t,\lambda)}\Gamma(\lambda) & 1 \end{pmatrix}, \quad \lambda \in L_1, \qquad J_3 = \begin{pmatrix} 1 & -e^{-2i\theta(x,t,\lambda)}\overline{\Gamma(\bar\lambda)} \\ 0 & 1 \end{pmatrix}, \quad \lambda \in L_3, \tag{35a}$$

$$J_4 = \begin{pmatrix} \frac{1}{a(\lambda)\overline{a(\bar\lambda)}} & e^{-2i\theta(x,t,\lambda)}\overline{r_1(\bar\lambda)} \\ -e^{2i\theta(x,t,\lambda)}r_1(\lambda) & 1 \end{pmatrix}, \quad \lambda \in L_4, \tag{35b}$$

$$J_2 = J_1J_4^{-1}J_3 = \begin{pmatrix} 1 & -\mathrm{e}^{-2i\theta(x,t,\lambda)}\overline{r(\bar\lambda)} \\ e^{2i\theta(x,t,\lambda)}r(\lambda) & 1 - r(\lambda)\overline{r(\bar\lambda)} \end{pmatrix}, \quad \lambda \in L_2, \tag{35c}$$

with the oriented contour L (see Fig. 3)

$$L_1 = \bar D_1 \cap \bar D_2, \qquad L_2 = \bar D_2 \cap \bar D_3, \qquad L_3 = \bar D_3 \cap \bar D_4, \qquad L_4 = \bar D_1 \cap \bar D_4. \tag{36}$$

Proof We can write Eqs. (22a) and (22b) as

$$\mu_3^{(1)}(x,t,\lambda) = \overline{a(\bar\lambda)}\mu_2^{(1)}(x,t,\lambda) + e^{2i\theta(\lambda)}\overline{b(\bar\lambda)}\mu_2^{(2)}(x,t,\lambda), \tag{37a}$$

$$\mu_3^{(2)}(x,t,\lambda) = e^{-2i\theta(\lambda)}b(\lambda)\mu_2^{(1)}(x,t,\lambda) + a(\lambda)\mu_2^{(2)}(x,t,\lambda), \tag{37b}$$

$$\mu_1^{(1)}(x,t,\lambda) = \overline{A(\bar\lambda)}\mu_2^{(1)}(x,t,\lambda) + e^{2i\theta(\lambda)}\overline{B(\bar\lambda)}\mu_2^{(2)}(x,t,\lambda), \tag{37c}$$

$$\mu_1^{(2)}(x,t,\lambda) = e^{-2i\theta(\lambda)}B(\lambda)\mu_2^{(1)}(x,t,\lambda) + A(\lambda)\mu_2^{(2)}(x,t,\lambda). \tag{37d}$$

From Eqs. (37a)–(37d), it is straightforward to define the Riemann–Hilbert problem (33) with the sectionally meromorphic functions $M_\pm$ and the the jump matrix given in Eqs. (34a)–(34b) and (35a)–(35c). □

We note that the function $M(x,t,\lambda)$ is sectionally meromorphic. The possible poles occur at the zeros of $a(\lambda)$ and $d(\lambda)$. If $\lambda_j \in D_1$ is a zero of $a(\lambda)$, then $\bar{\lambda}_j \in D_4$ is a zero of $\overline{a(\bar{\lambda})}$. Moreover, since $a(\lambda) = a(2\beta/\alpha - \lambda)$, $\lambda = 2\beta/\alpha - \lambda_j$ is also a zero of $a(\lambda)$. Similar facts also hold for zeros of $d(\lambda)$. Thus, we assume that

(i) $a(\lambda)$ has $2n$ simple zeros λ_j in D_1 such that λ_j lie in $D_1 \cap \{\operatorname{Im}\lambda > 0\}$ and $\lambda_{n+j} = 2\beta/\alpha - \lambda_j$ lie in $D_1 \cap \{\operatorname{Im}\lambda < 0\}$ $(j = 1,2,\ldots,n)$.

(ii) $d(\lambda)$ has $2N$ simple zeros z_j in D_2 such that z_j lie in $D_2 \cap \{\operatorname{Im}\lambda > 0\}$ and $z_{n+j} = 2\beta/\alpha - z_j$ lie in $D_2 \cap \{\operatorname{Im}\lambda < 0\}$ $(j = 1,2,\ldots,N)$.

(iii) None of the zeros of $a(\lambda)$ coincides with the zeros of $d(\lambda)$.

We then find the residue conditions

$$\operatorname*{Res}_{\lambda=\lambda_j} M_+^{(1)}(x,t,\lambda) = \frac{e^{2i\theta(x,t,\lambda_j)}}{\dot{a}(\lambda_j)b(\lambda_j)} M_+^{(2)}(x,t,\lambda_j), \tag{38a}$$

$$\operatorname*{Res}_{\lambda=\bar{\lambda}_j} M_-^{(2)}(x,t,\lambda) = \frac{e^{-2i\theta(x,t,\bar{\lambda}_j)}}{\overline{\dot{a}(\lambda_j)b(\lambda_j)}} M_-^{(1)}(x,t,\bar{\lambda}_j), \tag{38b}$$

$$\operatorname*{Res}_{\lambda=z_j} M_-^{(1)}(x,t,\lambda) = \frac{e^{2i\theta(x,t,z_j)}\overline{B(\bar{z}_j)}}{\dot{d}(z_j)a(z_j)} M_-^{(2)}(x,t,z_j), \tag{38c}$$

$$\operatorname*{Res}_{\lambda=\bar{z}_j} M_+^{(2)}(x,t,\lambda) = \frac{e^{-2i\theta(x,t,\bar{z}_j)}B(\bar{z}_j)}{\overline{\dot{d}(z_j)a(z_j)}} M_+^{(1)}(x,t,\bar{z}_j), \tag{38d}$$

where the overdot denotes differentiation with respect to λ. Indeed, from Eq. (37b), we can compute the residue

$$\operatorname*{Res}_{\lambda=\lambda_j} M_+^{(1)}(x,t,\lambda) = \frac{\mu_2^{(1)}(x,t,\lambda_j)}{\dot{a}(\lambda_j)} = e^{2i\theta(x,t,\lambda_j)}\frac{\mu_3^{(2)}(x,t,\lambda_j)}{\dot{a}(\lambda_j)b(\lambda_j)},$$

which yields Eq. (38a). For Eq. (38c), we use $M_+ = M_- J_2$, that is,

$$d(\lambda)\mu_3^{(1)}(x,t,\lambda) = \mu_1^{(1)}(x,t,\lambda) + e^{2i\theta(\lambda)}d(\lambda)r(\lambda)\mu_3^{(2)}(x,t,\lambda).$$

Thus, we find

$$\operatorname*{Res}_{\lambda=z_j} M_-^{(1)}(x,t,\lambda) = \frac{\mu_1^{(1)}(x,t,z_j)}{\dot{d}(z_j)} = e^{2i\theta(x,t,z_j)}\frac{\overline{B(\bar{z}_j)}\mu_3^{(2)}(x,t,z_j)}{\dot{d}(z_j)a(z_j)},$$

which yields Eq. (38c). Similarly, we can derive Eqs. (38b) and (38d).

We note that $\det M_\pm = 1$ and $M_\pm(x,t,\lambda) = I + O(1/\lambda)$ as $\lambda \to \infty$ in the respective domains of boundedness of their columns. The solution for the mixed NLS equation can be found from the solution of the Riemann–Hilbert problem. In this respect, we expand the solution $M(x,t,\lambda)$ of the Riemann–Hilbert problem as

$$M(x,t,\lambda) = I + \frac{M_1(x,t)}{\lambda} + \frac{M_2(x,t)}{\lambda^2} + O\bigl(1/\lambda^3\bigr) \quad (\lambda \to \infty).$$

Letting $M_+ - M_- = M_-\tilde{J}$, where $\tilde{J} = J - I$, the Riemann–Hilbert problem can be solved by the Cauchy-type integral equation

$$M(x,t,\lambda) = I + \frac{1}{2i\pi}\int_L \frac{M_-\tilde{J}(x,t,l)}{l-\lambda}\,dl,$$

which implies that

$$M_1(x,t) = -\frac{1}{2i\pi}\int_L M_-\tilde{J}(x,t,\lambda)\,d\lambda.$$

Thus, we can find the reconstruction formula for the solution of the mixed NLS equation on the half-line in terms of the solution of the Riemann–Hilbert problem as

$$q(x,t)e^{-2\int_{(0,0)}^{(x,t)}\Delta} = -\frac{1}{\pi}\int_L \bigl(M_-\tilde{J}(x,t,\lambda)\bigr)_{12}\,d\lambda. \tag{39}$$

3 Spectral functions

In this section, we define the spectral functions from the initial and boundary values.

Definition 3.1 Given $q_0(x) \in \mathcal{S}(\mathbb{R}^+)$, we define the map

$$\mathbb{S} : \bigl\{q_0(x)\bigr\} \to \bigl\{a(\lambda), b(\lambda)\bigr\} \tag{40}$$

by

$$\begin{pmatrix} b(\lambda) \\ a(\lambda) \end{pmatrix} = \mu_3^{(2)}(0,\lambda), \tag{41}$$

where

$$\mu_3(x,\lambda) = I - \int_x^\infty e^{if_1(\lambda)(\xi-x)\hat{\sigma}_3} V_1(\xi,\lambda)\mu_3(\xi,\lambda)\,d\xi \tag{42a}$$

with

$$V_1(x,\lambda) = e^{-\frac{i}{2}\int_0^x \alpha|q_0|^2\,d\xi\hat{\sigma}_3}\left[(\alpha\lambda-\beta)Q_0 - \frac{i}{2}\alpha|q_0|^2\sigma_3\right], \qquad Q_0(x) = \begin{pmatrix} 0 & q_0 \\ \bar{q}_0 & 0 \end{pmatrix}. \tag{42b}$$

Proposition 3.1 *The spectral functions $a(\lambda)$ and $b(\lambda)$ have the following properties*:

(i) *$a(\lambda)$ and $b(\lambda)$ are analytic for $D_1 \cup D_2$ and bounded on $\bar{D}_1 \cup \bar{D}_2$.*
(ii) *$\binom{b(\lambda)}{a(\lambda)} = \binom{0}{1} + O(1/\lambda)$ as $\lambda \to \infty$ for $\lambda \in \bar{D}_1 \cup \bar{D}_2$.*
(iii) *$a(\lambda)\overline{a(\bar{\lambda})} - b(\lambda)\overline{b(\bar{\lambda})} = 1$ for $\lambda \in L_2 \cup L_4$.*
(iv) *$a(\lambda) = a(2\beta/\alpha - \lambda)$ and $b(\lambda) = -b(2\beta/\alpha - \lambda)$ for $\lambda \in \bar{D}_1 \cup \bar{D}_2$.*
(v) *The inverse map $\mathbb{S}^{-1} : \{a(\lambda), b(\lambda)\} \to \{q_0(x)\}$ to the map $\mathbb{S}$ is defined by*

$$q_0(x) = 2im_1(x)e^{i\int_0^x \alpha|q_0|^2\,d\xi}, \tag{43}$$

where

$$m_1(x) = \lim_{\lambda\to\infty} \lambda M_{12}^{(x)}(x,\lambda)$$

and $M^{(x)}(x,\lambda)$ is the unique solution of the following Riemann–Hilbert problem:

- $$M^{(x)}(x,\lambda)=\begin{cases} M_+^{(x)}(x,\lambda), & \lambda\in D_1\cup D_2,\\ M_-^{(x)}(x,\lambda), & \lambda\in D_3\cup D_4\end{cases} \tag{44}$$

  is a meromorphic function for $\lambda\in\mathbb{C}\backslash(L_2\cup L_4)$.

- $$M_+^{(x)}(x,\lambda)=M_-^{(x)}(x,\lambda)J^{(x)}(x,\lambda),\quad \lambda\in L_2\cup L_4, \tag{45}$$

  where the jump matrix $J^{(x)}$ is given by

  $$J^{(x)}(x,\lambda)=\begin{pmatrix} \frac{1}{a(\lambda)\overline{a(\bar\lambda)}} & e^{-2if_1(\lambda)x}r_1(\bar\lambda)\\ -e^{2if_1(\lambda)x}r_1(\lambda) & 1\end{pmatrix} \tag{46}$$

  with

  $$r_1(\lambda)=\frac{\overline{b(\bar\lambda)}}{a(\lambda)}.$$

- *The first column of $M_+^{(x)}$ has $2n$ simple zeros $\lambda_j\in D_1\cup D_2$ such that $\lambda_j\in(D_1\cup D_2)\cap\{\operatorname{Im}\lambda>0\}$ and $\lambda_{n+j}=2\beta/\alpha-\lambda_j\in(D_1\cup D_2)\cap\{\operatorname{Im}\lambda<0\}$ $(j=1,2,\dots,n)$. The second column of $M_+^{(x)}$ has $2n$ simple zeros $\bar\lambda_j\in D_3\cup D_4$ such that $\bar\lambda_j\in(D_3\cup D_4)\cap\{\operatorname{Im}\lambda<0\}$ and $\bar\lambda_{n+j}=2\beta/\alpha-\bar\lambda_j\in(D_3\cup D_4)\cap\{\operatorname{Im}\lambda>0\}$ $(j=1,2,\dots,n)$. Then*

  $$\operatorname*{Res}_{\lambda=\lambda_j}M^{(x,1)}(x,\lambda)=\frac{e^{2if_1(\lambda_j)x}}{\dot a(\lambda_j)b(\lambda_j)}M^{(x,2)}(x,\lambda_j), \tag{47a}$$

  $$\operatorname*{Res}_{\lambda=\bar\lambda_j}M^{(x,2)}(x,\lambda)=\frac{e^{-2if_1(\bar\lambda_j)x}}{\overline{\dot a(\lambda_j)b(\lambda_j)}}M^{(x,1)}(x,\bar\lambda_j), \tag{47b}$$

  where $M^{(x,1)}$ and $M^{(x,2)}$ are the first column and the second column of the matrix $M^{(x)}$, respectively.

Proof The derivations of (i)–(iv) are given from the discussion in Sect. 2.2. Regarding the proof of (v), we consider the x-part of the Lax pair (14a) evaluated at $t=0$. We define two Jost solutions μ_2 and μ_3 as

$$\mu_2(x,\lambda)=I+\int_0^x e^{if_1(\lambda)(\xi-x)\hat\sigma_3}V_1(\xi,\lambda)\mu_2(\xi,\lambda)\,d\xi \tag{48}$$

and $\mu_3(x,\lambda)$ given in Eq. (42a). From Eq. (22a) evaluated at $t=0$, it follows that

$$\mu_3(x,\lambda)=\mu_2(x,\lambda)e^{-if_1(\lambda)x\hat\sigma_3}s(\lambda). \tag{49}$$

Since $\mu_2(0,\lambda) = I$, $s(\lambda) = \mu_3(0,\lambda)$. Moreover, letting

$$M_+^{(x)} = \left(\frac{\mu_2^{(1)}(x,\lambda)}{a(\lambda)}, \mu_3^{(2)}(x,\lambda)\right), \quad \lambda \in D_1 \cup D_2, \tag{50a}$$

$$M_-^{(x)} = \left(\mu_3^{(1)}(x,\lambda), \frac{\mu_2^{(2)}(x,\lambda)}{\overline{a(\bar{\lambda})}}\right), \quad \lambda \in D_3 \cup D_4, \tag{50b}$$

Eq. (49) can be written as the Riemann–Hilbert problem defined by Eq. (45) with the jump matrix given in Eq. (46). Equation (49) yields the residue conditions given by Eqs. (47a)–(47b). Moreover, expanding $M^{(x)}(x,\lambda)$ as

$$M^{(x)}(x,\lambda) = I + \frac{m_1(x)}{\lambda} + \frac{m_2(x)}{\lambda^2} + O\big(1/\lambda^3\big) \quad (\lambda \to \infty),$$

we can derive Eq. (43). □

Definition 3.2 Given smooth functions $g_j(t)$ $(j = 0,1)$, we define the map

$$\mathbb{Q} : \big\{g_0(t), g_1(t)\big\} \to \big\{A(\lambda), B(\lambda)\big\} \tag{51}$$

by

$$\begin{pmatrix} B(\lambda) \\ A(\lambda) \end{pmatrix} = \mu_1^{(2)}(0,\lambda), \tag{52}$$

where

$$\mu_1(t,\lambda) = I - \int_t^T e^{if_2(\lambda)(\tau-t)\hat{\sigma}_3} V_2(\tau,\lambda)\mu_1(\tau,\lambda)\,d\tau \tag{53}$$

with

$$V_2(t,\lambda) = e^{-i\int_0^t \Delta_2(\tau)\,d\tau\hat{\sigma}_3}\big[\tilde{U}_2 - i\Delta_2(t)\sigma_3\big], \tag{54a}$$

$$\begin{aligned} \tilde{U}_2(t,\lambda) = {}& i\big(\alpha^2\lambda^2 - 2\alpha\beta\lambda + \beta^2\big)\tilde{Q}_0^2\sigma_3 + \big(2\alpha^2\lambda^3 - 6\alpha\beta\lambda^2 + 4\beta^2\lambda\big)\tilde{Q}_0 \\ & + i(\alpha\lambda - \beta)\tilde{Q}_1\sigma_3 + \big(\alpha^2\lambda - \alpha\beta\big)\tilde{Q}_0^3, \end{aligned} \tag{54b}$$

and

$$\tilde{Q}_0 = \begin{pmatrix} 0 & g_0 \\ \bar{g}_0 & 0 \end{pmatrix}, \qquad \tilde{Q}_1 = \begin{pmatrix} 0 & g_1 \\ \bar{g}_1 & 0 \end{pmatrix}, \qquad \Delta_2(t) = \frac{3}{4}\alpha^2|g_0|^4 - \frac{i}{2}(\bar{g}_0 g_1 - g_0\bar{g}_1). \tag{54c}$$

Proposition 3.2 *The spectral functions $A(\lambda)$ and $B(\lambda)$ have the following properties*:

(i) *$A(\lambda)$ and $B(\lambda)$ are entire functions of λ if $T < \infty$ and are bounded for $\lambda \in \bar{D}_1 \cup \bar{D}_3$. If $T = \infty$, $A(\lambda)$ and $B(\lambda)$ are defined in $\bar{D}_1 \cup \bar{D}_3$.*

(ii) *$\binom{B(\lambda)}{A(\lambda)} = \binom{0}{1} + O(1/\lambda)$ as $\lambda \to \infty$ for $\lambda \in \bar{D}_1 \cup \bar{D}_3$.*

(iii) *$A(\lambda)\overline{A(\bar{\lambda})} - B(\lambda)\overline{B(\bar{\lambda})} = 1$ for $\lambda \in \mathbb{C}$ (if $T = \infty$, $\lambda \in L$).*

(iv) *$A(\lambda) = A(2\beta/\alpha - \lambda)$ and $B(\lambda) = -B(2\beta/\alpha - \lambda)$ for $\lambda \in \bar{D}_1 \cup \bar{D}_3$.*

(v) *The inverse map* $\mathbb{Q}^{-1} : \{A(\lambda), B(\lambda)\} \to \{g_0(t), g_1(t)\}$ *to the map* $\mathbb{Q}$ *is defined by*

$$g_0(t) = 2iM_{12}^{(1)}(t)e^{2i\int_0^t \Delta_2(\tau)\,d\tau}, \tag{55a}$$

$$\begin{aligned} g_1(t) &= \left(-4\alpha M_{12}^{(3)} + 8\beta M_{12}^{(2)}\right)e^{2i\int_0^t \Delta_2(\tau)\,d\tau} \\ &\quad - 2i\alpha g_0 M_{22}^{(2)} + 2i\beta g_0 M_{22}^{(1)} + \frac{\alpha}{2i} g_0 |g_0|^2, \end{aligned} \tag{55b}$$

where

$$\begin{aligned} \Delta_2(t) &= 12\alpha^2 \left|M_{12}^{(1)}\right|^4 - 2\,\mathrm{Re}\left[M_{12}^{(1)}\left(-4\alpha\overline{M_{12}^{(3)}} + 8\beta\overline{M_{12}^{(2)}}\right)\right] \\ &\quad - 8\left|M_{12}^{(1)}\right|^2\left(\alpha\,\mathrm{Re}\,M_{22}^{(2)} - \beta\,\mathrm{Re}\,M_{22}^{(1)} + \alpha\left|M_{12}^{(1)}\right|^2\right) \end{aligned} \tag{56}$$

and the matrix functions $M^{(1)}(t)$, $M^{(2)}(t)$, *and* $M^{(3)}(t)$ *are determined by the asymptotic expansion of* $M^{(t)}$

$$M^{(t)}(t,\lambda) = I + \frac{M^{(1)}(t)}{\lambda} + \frac{M^{(2)}(t)}{\lambda^2} + \frac{M^{(3)}(t)}{\lambda^3} + O\left(1/\lambda^4\right) \quad (\lambda \to \infty)$$

where $M^{(t)}(t,\lambda)$ *is the unique solution of the following Riemann–Hilbert problem*:

- $$M^{(t)}(t,\lambda) = \begin{cases} M_+^{(t)}(t,\lambda), & \lambda \in D_1 \cup D_3, \\ M_-^{(t)}(t,\lambda), & \lambda \in D_2 \cup D_4 \end{cases} \tag{57}$$

  is a meromorphic function for $\lambda \in \mathbb{C}\setminus L$.

- $$M_+^{(t)}(t,\lambda) = M_-^{(t)}(t,\lambda) J^{(t)}(t,\lambda), \quad \lambda \in L, \tag{58}$$

  where the jump matrix $J^{(t)}$ *is given by*

  $$J^{(t)}(t,\lambda) = \begin{pmatrix} \frac{1}{A(\lambda)\overline{A(\bar{\lambda})}} & e^{-2if_2(\lambda)t}\overline{R_1(\bar{\lambda})} \\ -e^{2if_2(\lambda)t}R_1(\lambda) & 1 \end{pmatrix} \tag{59}$$

  with

  $$R_1(\lambda) = \frac{\overline{B(\bar{\lambda})}}{A(\lambda)}.$$

- *The first column of* $M_+^{(t)}$ *has* $2N$ *simple zeros* $z_j \in D_1 \cup D_3$ *such that* $z_j \in (D_1 \cup D_3) \cap \{\mathrm{Im}\,\lambda > 0\}$ *and* $z_{n+j} = 2\beta/\alpha - z_j \in (D_1 \cup D_3) \cap \{\mathrm{Im}\,\lambda < 0\}$ $(j = 1, 2, \ldots, N)$. *The second column of* $M_+^{(t)}$ *has* $2N$ *simple zeros* $\bar{z}_j \in D_2 \cup D_4$ *such that* $\bar{z}_j \in (D_2 \cup D_4) \cap \{\mathrm{Im}\,\lambda < 0\}$ *and* $\bar{z}_{n+j} = 2\beta/\alpha - \bar{z}_j \in (D_2 \cup D_4) \cap \{\mathrm{Im}\,\lambda > 0\}$ $(j = 1, 2, \ldots, N)$. *Then*

  $$\operatorname*{Res}_{\lambda = z_j} M^{(t,1)}(t,\lambda) = \frac{e^{2if_2(z_j)t}}{\dot{A}(z_j)B(z_j)} M^{(t,2)}(t,z_j), \tag{60a}$$

$$\operatorname*{Res}_{\lambda=\bar{z}_j} M^{(t,2)}(t,\lambda) = \frac{e^{-2if_2(\bar{z}_j)t}}{\overline{\dot{A}(z_j)B(z_j)}} M^{(t,1)}(t,\bar{z}_j), \tag{60b}$$

where $M^{(t,1)}$ and $M^{(t,2)}$ are the first column and the second column of the matrix $M^{(t)}$, respectively.

Proof It is enough to prove (v). We define two eigenfunctions μ_1 and μ_2 from the t-part of the Lax pair (6b) evaluated at $x = 0$ as

$$\mu_2(t,\lambda) = I + \int_0^t e^{if_2(\lambda)(\tau-t)\hat{\sigma}_3} V_2(\tau,\lambda)\mu_2(\tau,\lambda)\,d\tau \tag{61}$$

and $\mu_1(t,\lambda)$ given in Eq. (53). From Eq. (22b) evaluated at $x = 0$, it follows that

$$\mu_1(t,\lambda) = \mu_2(t,\lambda)e^{-if_2(\lambda)t\hat{\sigma}_3} S(\lambda). \tag{62}$$

Since $\mu_2(0,\lambda) = I$, we find $S(\lambda) = \mu_1(0,\lambda)$. Letting

$$M_+^{(t)} = \left(\frac{\mu_2^{(1)}(t,\lambda)}{A(\lambda)}, \mu_1^{(2)}(t,\lambda)\right), \quad \lambda \in D_1 \cup D_3, \tag{63a}$$

$$M_-^{(t)} = \left(\mu_1^{(1)}(t,\lambda), \frac{\mu_2^{(2)}(t,\lambda)}{\overline{A(\bar{\lambda})}}\right), \quad \lambda \in D_2 \cup D_4, \tag{63b}$$

Eq. (62) can be written as the Riemann–Hilbert problem defined by Eq. (58) with the jump matrix given in Eq. (59). Moreover, Eq. (62) yields the residue conditions given by Eqs. (60a)–(60b).

Regarding the inverse map for $g_1(t)$, we consider Eq. (6b),

$$\Psi_t + if_2(\lambda)[\sigma_3, \Psi] = U_2\Psi. \tag{64}$$

Substituting the expansion of Ψ given in Eq. (7) into Eq. (64), we find

$$O(\lambda^3): \quad i[\sigma_3, \Psi^{(1)}] = Q\Psi^{(0)}, \tag{65a}$$

$$O(\lambda^2): \quad 2i\alpha[\sigma_3, \Psi^{(2)}] - 8i\beta[\sigma_3, \Psi^{(1)}] = -i\alpha Q^2\sigma_3\Psi^{(0)} + 2\alpha Q\Psi^{(1)} - 6\beta Q\Psi^{(0)}. \tag{65b}$$

From the off-diagonal entries of Eqs. (65a)–(65b), it follows that

$$\Psi^{(1,\mathrm{O})} = \frac{1}{2i}\sigma_3 Q\Psi^{(0)}, \tag{66a}$$

$$2i\alpha\sigma_3\Psi^{(2,\mathrm{O})} - \alpha Q\Psi^{(1,\mathrm{D})} = \beta Q\Psi^{(0)}, \tag{66b}$$

where $\Psi^{(j,\mathrm{O})}$ and $\Psi^{(j,\mathrm{D})}$ denote the off-diagonal part and the diagonal part of $\Psi^{(j)}$, respectively. At $O(\lambda)$, we have

$$\begin{aligned} &2i\alpha^2[\sigma_3, \Psi^{(3)}] - 8i\alpha\beta[\sigma_3, \Psi^{(2)}] + 8i\beta^2[\sigma_3, \Psi^{(1)}] \\ &\quad = 2\alpha^2 Q\Psi^{(2)} - (i\alpha^2 Q^2\sigma_3 + 6\alpha\beta Q)\Psi^{(1)} \\ &\qquad + (2i\alpha\beta Q^2\sigma_3 + 4\beta^2 Q\Psi^{(0)} + i\alpha Q_x\sigma_3 + \alpha^2 Q^3)\Psi^{(0)}. \end{aligned}$$

Collecting the off-diagonal part of the above equation, we find

$$\begin{aligned}&4i\alpha^2\sigma_3\Psi^{(3,\mathrm{O})}-16i\alpha\beta\sigma_3\Psi^{(2,\mathrm{O})}+16i\beta^2\sigma_3\Psi^{(1,\mathrm{O})}\\&\quad=-i\alpha^2Q^2\sigma_3\Psi^{(1,\mathrm{O})}+2\alpha^2Q\Psi^{(2,\mathrm{D})}-6\alpha\beta Q\Psi^{(1,\mathrm{D})}+\left(4\beta^2Q+i\alpha Q_x\sigma_3+\alpha^2Q^3\right)\Psi^{(0)}.\end{aligned}\tag{67}$$

Using Eqs. (66a)–(66b) and simplifying the resulting equation, Eq. (67) can be written as

$$Q_x\sigma_3\Psi^{(0)}=4\alpha\sigma_3\Psi^{(3,\mathrm{O})}-8\beta\sigma_3\Psi^{(2,\mathrm{O})}+2i\alpha Q\Psi^{(2,\mathrm{D})}-2i\beta Q\Psi^{(1,\mathrm{D})}-\frac{\alpha}{2i}Q^3\Psi^{(0)}.\tag{68}$$

Recalling $\Psi=e^{i\int_{(0,0)}^{(x,t)}\Delta\hat{\sigma}_3}\mu\Psi^{(0)}$ with $\Psi^{(0)}=e^{i\int_{(0,0)}^{(x,t)}\Delta\sigma_3}$, the $(1,2)$-entry yields

$$q_x(x,t)=\left(-4\alpha m_{12}^{(3)}+8\beta m_{12}^{(2)}\right)e^{2i\int_{(0,0)}^{(x,t)}\Delta}-2i\alpha qm_{22}^{(2)}+2i\beta qm_{22}^{(2)}+\frac{\alpha}{2i}q^2\bar{q},\tag{69}$$

where the functions $m^{(j)}$ $(j=1,2,3)$ are given from the asymptotic expansion of μ

$$\mu(x,t,\lambda)=I+\frac{m^{(1)}(x,t)}{\lambda}+\frac{m^{(2)}(x,t)}{\lambda^2}+\frac{m^{(3)}(x,t)}{\lambda^3}+O\left(1/\lambda^4\right),\quad \lambda\to\infty.$$

We note that $q(x,t)=2im_{12}^{(1)}e^{2i\int_{(0,0)}^{(x,t)}\Delta}$ and $|q|=2|m_{12}^{(1)}|$. Then, we obtain

$$\begin{aligned}\bar{q}q_x-\bar{q}_xq&=-4i\operatorname{Re}\left[m_{12}^{(1)}\left(-4\alpha\overline{m_{12}^{(3)}}+8\beta\overline{m_{12}^{(2)}}\right)\right]\\&\quad-16i\left|m_{12}^{(1)}\right|^2\left(\alpha\operatorname{Re}m_{22}^{(2)}-\beta\operatorname{Re}m_{22}^{(1)}+\alpha\left|m_{12}^{(1)}\right|^2\right).\end{aligned}$$

Thus, we find

$$\begin{aligned}\Delta(x,t)&=2\alpha\left|m_{12}^{(1)}\right|^2dx+\left[12\alpha^2\left|m_{12}^{(1)}\right|^4-2\operatorname{Re}\left[m_{12}^{(1)}\left(-4\alpha\overline{m_{12}^{(3)}}+8\beta\overline{m_{12}^{(2)}}\right)\right]\right.\\&\quad\left.-8\left|m_{12}^{(1)}\right|^2\left(\alpha\operatorname{Re}m_{22}^{(2)}-\beta\operatorname{Re}m_{22}^{(1)}+\alpha\left|m_{12}^{(1)}\right|^2\right)\right]dt.\end{aligned}\tag{70}$$

Finally, evaluating Eqs. (69) and (70) at $x=0$, Eqs. (55b) and (56) can be derived. □

4 The Riemann–Hilbert problem

In this section, we establish the existence theorem for the solution of the mixed NLS equation posed on the half-line.

Theorem 4.1 *Let $q_0(x)\in\mathcal{S}(\mathbb{R}^+)$. Assume that functions $g_0(t)$ and $g_1(t)$ are compatible with $q_0(x)$ at $x=t=0$. Let functions $\{a(\lambda),b(\lambda),A(\lambda),B(\lambda)\}$ be given by Eqs.* (40) *and* (52) *in Definitions* 3.1 *and* 3.2*, respectively. Suppose that the functions $\{a(\lambda),b(\lambda),A(\lambda),B(\lambda)\}$ satisfy the global relation given in Eq.* (30) *(if $T=\infty$, the global relation is replaced by Eq.* (31)*), where $c(\lambda)$ is analytic function for $\lambda\in D_1\cup D_2$ and is bounded for $\lambda\in\bar{D}_1\cup\bar{D}_2$ with $O(1/\lambda)$ as $\lambda\to\infty$.*

Let $M(x,t,\lambda)$ be the solution of the following 2×2 matrix Riemann–Hilbert problem:

- *M is sectionally meromorphic for $\lambda\in\mathbb{C}\backslash L$, where the oriented contour L is defined by Eq.* (36).

- *The first column of M has simple zeros at $\lambda = \lambda_j$ $(j = 1, 2, \dots, 2n)$ and at $\lambda = z_j$ $(j = 1, 2, \dots, 2N)$. The second column of M has simple zeros at $\lambda = \bar{\lambda}_j$ $(j = 1, 2, \dots, 2n)$ and at $\lambda = \bar{z}_j$ $(j = 1, 2, \dots, 2N)$.*
- *M satisfies the jump condition*

$$M_+(x,t,\lambda) = M_-(x,t,\lambda)J(x,t,\lambda), \quad \lambda \in L, \tag{71}$$

where M_+ and M_- are defined for $\lambda \in D_1 \cup D_3$, and for $\lambda \in D_2 \cup D_4$, respectively, and the jump matrix is defined by Eqs. (35a)–(35c).
- *$M(x,t,\lambda) = I + O(1/\lambda)$ as $\lambda \to \infty$.*

Then the Riemann–Hilbert problem is uniquely solvable and the function $q(x,t)$ defined by

$$q(x,t) = 2im(x,t)e^{2i\int_{(0,0)}^{(x,t)} \Delta}, \qquad m(x,t) = \lim_{\lambda\to\infty} \lambda M(x,t,\lambda)_{12}, \tag{72}$$

where

$$\Delta(x,t) = 2\alpha|m|^2\,dx + \left[12\alpha^2|m|^4 - 2i(\bar{m}m_x - m\bar{m}_x) + 16\alpha|m|^4\right]dt, \tag{73}$$

solves the mixed NLS equation satisfying

$$q(x,0) = q_0(x), \qquad q(0,t) = g_0(t), \qquad q_x(0,t) = g_1(t). \tag{74}$$

Proof In the absence of poles, the unique solvability of the Riemann–Hilbert problem follows from the vanishing lemma. □

Lemma 4.2 *The Riemann–Hilbert problem in Theorem 4.1 with $M = O(1/\lambda)$ as $\lambda \to \infty$ has only the zero solution.*

Proof Let $A^\dagger$ denote the complex conjugate transpose of a matrix A and $\hat{\lambda} = 2\beta/\alpha - \bar{\lambda}$. Define

$$H_+(x,t,\lambda) = M_+(x,t,\lambda)M_-^\dagger(x,t,\hat{\lambda}), \quad \lambda \in D_1 \cup D_3, \tag{75a}$$

$$H_-(x,t,\lambda) = M_-(x,t,\lambda)M_+^\dagger(x,t,\hat{\lambda}), \quad \lambda \in D_2 \cup D_4. \tag{75b}$$

Note that H_+ and H_- are analytic in $D_1 \cup D_3$ and $D_2 \cup D_4$, respectively. Hereafter, we suppress the x and t dependence for simplicity. From Eqs. (26) and (27), it follows that

$$J_1(\lambda) = J_3^\dagger(\hat{\lambda}), \qquad J_2(\lambda) = J_2^\dagger(\hat{\lambda}), \qquad J_4(\lambda) = J_4^\dagger(\hat{\lambda}).$$

By the definitions of $H_\pm$, we find

$$H_+(\lambda) = M_-(\lambda)J(\lambda)M_-^\dagger(\hat{\lambda}), \qquad H_-(\lambda) = M_-(\lambda)J^\dagger(\hat{\lambda})M_-^\dagger(\hat{\lambda}), \quad \lambda \in L,$$

which implies that $H_+(\lambda) = H_-(\lambda)$ for $\lambda \in L$. Thus $H_+(\lambda)$ and $H_-(\lambda)$ define an entire function vanishing at infinity and hence $H_\pm(\lambda) = 0$. Since the line $\operatorname{Re}\lambda = \beta/\alpha$ is invariant under the transformation $\lambda \to \hat{\lambda}$, the matrix $J_2(\beta/\alpha + i\kappa)$ is Hermitian for $\kappa \in \mathbb{R}$. As a result, $J_2(\beta/\alpha + i\kappa)$ is positive definite. Furthermore, note that $H_+(\beta/\alpha + i\kappa) = 0$ and $M_-(\beta/\alpha + i\kappa)J_2(\beta/\alpha + i\kappa)M_-^\dagger(\beta/\alpha + i\kappa) = 0$ for $\kappa \in \mathbb{R}$. Thus, $M_-(\beta/\alpha + i\kappa) = 0$ for $\kappa \in \mathbb{R}$, which concludes that $M_\pm(x,t,\lambda) = 0$. □

Proof of Theorem 4.1 In the presence of poles, $M(x,t,\lambda)$ is a meromorphic function of λ. In this case, this singular matrix Riemann–Hilbert problem can be mapped to a regular Riemann–Hilbert problem [18]. By using the dressing method presented in [18, 28], one can prove that if M solves the Riemann–Hilbert problem, then $q(x,t)$ defined by Eq. (72) solves the mixed NLS equation and satisfies that $q(x,0) = q_0(x)$, $q(0,t) = g_0(t)$ and $q_x(0,t) = g_1(t)$ (cf. Propositions 3.1 and 3.2). In what follows, we will show that the Riemann–Hilbert problem given in Eq. (71) is related with the Riemann–Hilbert problems defined by Eqs. (45) and (58), respectively.

We define $M^{(x)}(x,\lambda)$ in terms of $M(x,0,\lambda)$ by

$$M^{(x)}(x,\lambda) = \begin{cases} M(x,0,\lambda), & \lambda \in D_1 \cup D_4, \\ M(x,0,\lambda)J_1(x,0,\lambda), & \lambda \in D_2, \\ M(x,0,\lambda)J_3^{-1}(x,0,\lambda), & \lambda \in D_3. \end{cases} \tag{76}$$

Let $M_j(x,0,\lambda)$ and $M_j^{(x)}(x,\lambda)$ denote $M(x,0,\lambda)$ and $M^{(x)}(x,\lambda)$ for $\lambda \in D_j$ ($j = 1,\dots,4$). Then

$$M_1(x,0,\lambda) = M_2(x,0,\lambda)J_1(x,0,\lambda), \qquad M_3(x,0,\lambda) = M_2(x,0,\lambda)J_2(x,0,\lambda), \tag{77a}$$

$$M_1(x,0,\lambda) = M_4(x,0,\lambda)J_4(x,0,\lambda), \qquad M_3(x,0,\lambda) = M_4(x,0,\lambda)J_3(x,0,\lambda), \tag{77b}$$

and

$$M_1^{(x)}(x,\lambda) = M_1(x,0,\lambda), \qquad M_2^{(x)}(x,\lambda) = M_2(x,0,\lambda)J_1(x,0,\lambda), \tag{78a}$$

$$M_3^{(x)}(x,\lambda) = M_3(x,0,\lambda)J_3^{-1}(x,0,\lambda), \qquad M_4^{(x)}(x,\lambda) = M_4(x,0,\lambda). \tag{78b}$$

Using the above equations, we find

$$\begin{aligned} &M_1^{(x)}(x,\lambda) = M_2^{(x)}(x,\lambda), \qquad M_3^{(x)}(x,\lambda) = M_4^{(x)}(x,\lambda), \\ &M_1^{(x)}(x,\lambda) = M_4^{(x)}(x,\lambda)J_4(x,0,\lambda), \qquad M_2^{(x)}(x,\lambda) = M_3^{(x)}(x,\lambda)\big(J_3J_2^{-1}J_1\big)(x,0,\lambda). \end{aligned}$$

Note that no jumps occurs along the contours L_1 and L_3 and that $J_3J_2^{-1}J_1 = J_4$ and $J_4(x,0,\lambda) = J^{(x)}(x,\lambda)$, where $J^{(x)}$ is given in Eq. (46). Thus, if we define $M^{(x)} = M_+^{(x)}$ for $\lambda \in D_1 \cup D_2$ and $M^{(x)} = M_-^{(x)}$ for $\lambda \in D_3 \cup D_4$, Eqs. (78a)–(78b) yields Eq. (46).

On the other hand, let $M_j(0,t,\lambda)$ denote $M(0,t,\lambda)$ for $\lambda \in D_j$ ($j = 1,\dots,4$). Then

$$M_1(0,t,\lambda) = M_2(0,t,\lambda)J_1(0,t,\lambda), \quad M_3(0,t,\lambda) = M_2(0,t,\lambda)J_2(0,t,\lambda), \tag{80a}$$

$$M_1(0,t,\lambda) = M_4(0,t,\lambda)J_4(0,t,\lambda), \quad M_3(0,t,\lambda) = M_4(0,t,\lambda)J_3(0,t,\lambda). \tag{80b}$$

Let $M^{(t)}(t,\lambda)$ be defined by

$$M_j^{(t)}(t,\lambda) = M_j(0,t,\lambda)F_j(t,\lambda), \quad \lambda \in D_j, \tag{81}$$

where F_j ($j = 1, \ldots, 4$) are analytic and bounded in the domains of their definition and $F_j(t,\lambda) = I + O(1/\lambda)$ as $\lambda \to \infty$. Moreover, they must satisfy

$$J_1(0,t,\lambda)F_1(t,\lambda) = F_2(t,\lambda)J^{(t)}(t,\lambda), \qquad J_2(0,t,\lambda)F_3(t,\lambda) = F_2(t,\lambda)J^{(t)}(t,\lambda), \tag{82a}$$

$$J_3(0,t,\lambda)F_3(t,\lambda) = F_4(t,\lambda)J^{(t)}(t,\lambda), \qquad J_4(0,t,\lambda)F_1(t,\lambda) = F_4(t,\lambda)J^{(t)}(t,\lambda), \tag{82b}$$

where $J^{(t)}(t,\lambda)$ is given in Eq. (59). The functions F_j can be determined by

$$F_1(t,\lambda) = \begin{pmatrix} \frac{a(\lambda)}{A(\lambda)} & c(\lambda)e^{2if_2(\lambda)(T-t)} \\ 0 & \frac{A(\lambda)}{a(\lambda)} \end{pmatrix}, \qquad F_2(t,\lambda) = \begin{pmatrix} d(\lambda) & -\frac{b(\lambda)}{\overline{A(\bar{\lambda})}}e^{-2if_2(\lambda)t} \\ 0 & \frac{1}{d(\lambda)} \end{pmatrix}, \tag{83a}$$

$$F_3(t,\lambda) = \begin{pmatrix} \frac{1}{\overline{d(\bar{\lambda})}} & 0 \\ -\frac{\overline{b(\bar{\lambda})}}{A(\lambda)}e^{2if_2(\lambda)t} & \overline{d(\bar{\lambda})} \end{pmatrix}, \qquad F_4(t,\lambda) = \begin{pmatrix} \frac{\overline{A(\bar{\lambda})}}{\overline{a(\bar{\lambda})}} & 0 \\ \overline{c(\bar{\lambda})}e^{-2if_2(\lambda)(T-t)} & \frac{\overline{a(\bar{\lambda})}}{\overline{A(\bar{\lambda})}} \end{pmatrix}. \tag{83b}$$

We can show that the F_j given in Eqs. (83a)–(83b) satisfy Eqs. (82a)–(82b) if and only if the global relation given in Eq. (30) (Eq. (31) if $T = \infty$) is valid for the spectral functions $\{a(\lambda), b(\lambda), A(\lambda), B(\lambda)\}$, where we have used the fact that the global relation yields

$$d(\lambda) = \frac{a(\lambda)}{A(\lambda)} + \frac{\overline{B(\bar{\lambda})}}{A(\lambda)}e^{2if_2(\lambda)T}c(\lambda) = \frac{\overline{A(\bar{\lambda})}}{\overline{a(\bar{\lambda})}} - \frac{b(\lambda)}{\overline{a(\bar{\lambda})}}e^{-2if_2(\lambda)T}\overline{c(\bar{\lambda})}. \qquad \square$$

5 Concluding remarks

In this work, we have studied the initial-boundary value problem for the mixed NLS equation posed on the half-line by using the Fokas method. Specifically, we have investigated the spectral functions defined from the initial and boundary values and we have derived the global relation in terms of the spectral functions. It has been shown that if the spectral functions satisfy the global relation, there exists the solution for the mixed NLS equation on the half-line in terms of the unique solution of the matrix Riemann–Hilbert problem with the jump matrix defined by the spectral functions. It should be remarked that the mixed NLS equation can be viewed as a modification of the classical NLS equation. Moreover, the mixed NLS equation can be analogous to the derivative NLS equation under a certain change of variables [28]. Thus, the present analysis is similar to what has been addressed in the NLS equation. Nevertheless, we have demonstrated directly the Fokas method in solving the mixed NLS equation on the half-line so that the result of the present work can cover the NLS-type equations including the derivative NLS and modified NLS equations. Furthermore, the present result in this paper will help lead us toward further analysis. For example, it can be applied to determine the long-time asymptotics for the solution of the mixed NLS equation on the half-line by employing the nonlinear steepest descent method in the associated Riemann–Hilbert problem [23, 24]. It can be also used to characterize the long-time asymptotics for the unknown boundary values as discussed in [29, 30].

Acknowledgements
The author would like to thank anonymous reviewers, whose comments and suggestions helped improve the manuscript.

Funding
The author is supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2020R1F1A1A01067944).

Availability of data and materials
No datasets were generated or analyzed during the current study.

Declarations

Ethics approval and consent to participate
Not applicable.

Consent for publication
Not applicable.

Competing interests
The author declares no competing interests.

Author contribution
The author confirms sole responsibility for study conception and design, analysis, and interpretation of results, as well as manuscript preparation and writing. The author read and approved the final manuscript.

Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 25 March 2022 Accepted: 21 November 2022 Published online: 05 December 2022

References
1. Ablowitz, M.J., Segur, H.: Solitons and the Inverse Scattering Transform. SIAM, Philadelphia (1981)
2. Newell, A.C., Moloney, J.V.: Nonlinear Optics. Addison-Wesley, Reading (1992)
3. Dalfovo, F., Giorgini, S., Pitaevskii, L.P., Stringari, S.: Theory of Bose–Einstein condensation in trapped gases. Rev. Mod. Phys. **71**, 463–512 (1999)
4. Kivshar, Y.S., Agrawal, G.P.: Optical Solitons: From Fibers to Photonic Crystals. Academic Press, San Diego (2003)
5. Yang, J.: Nonlinear Waves in Integrable and Nonintegrable Systems. SIAM, Philadelphia (2010)
6. Kevrekidis, P.G., Frantzeskakis, D.J., Carretero-González, R.: The Defocusing Nonlinear Schrödinger Equation: From Dark-Solitons to Vortices and Vortex Rings. SIAM, Philadelphia (2015)
7. Mjolhus, E.: On the modulational instability of hydromagnetic waves parallel to the magnetic field. J. Plasma Phys. **16**, 321–334 (1976)
8. Nakatsuka, H., Grischkowsky, D., Balant, A.C.: Nonlinear picosecond-pulse propagation through optical fibers with positive group velocity dispersion. Phys. Rev. Lett. **47**, 910–913 (1981)
9. Tzoar, N., Jain, M.: Self-phase modulation in long-geometry optical waveguides. Phys. Rev. A **23**, 1266–1270 (1981)
10. Kaup, D.J., Newell, A.C.: An exact solution for a derivative nonlinear Schrödinger equation. J. Math. Phys. **19**, 789–801 (1978)
11. Kawata, T., Inoue, H.: Exact solutions for the derivative nonlinear Schrödinger equation under the nonvanishing conditions. J. Phys. Soc. Jpn. **44**, 1968–1976 (1978)
12. Kawata, T., Sakai, J., Kobayashi, N.: Inverse method for the mixed nonlinear Schrödinger equation and soliton solutions. J. Phys. Soc. Jpn. **48**, 1371–1379 (1980)
13. Wadati, M., Konno, K., Ichikawa, Y.: A generalization of inverse scattering method. J. Phys. Soc. Jpn. **46**, 1965–1966 (1979)
14. Lü, X.: Soliton behavior for a generalized mixed nonlinear Schrödinger model with *N*-fold Darboux transformation. Chaos **23**, 033137 (2013)
15. Lü, X.: Madelung fluid description on a generalized mixed nonlinear Schrödinger equation. Nonlinear Dyn. **81**, 239–247 (2015)
16. Cheng, Q., Fan, E.: Long-time asymptotics for a mixed nonlinear Schrödinger equation with the Schwartz initial data. J. Math. Anal. Appl. **489**, 124188 (2020)
17. Fokas, A.S.: A unified transform method for solving linear and certain nonlinear PDEs. Proc. R. Soc. Lond. A **453**, 1411–1443 (1997)
18. Fokas, A.S.: Integrable nonlinear evolution equations on the half-line. Commun. Math. Phys. **230**, 1–39 (2002)
19. Fokas, A.S.: A Unified Approach to Boundary Value Problems. CBMS-NSF Regional Conference Series in Applied Mathematics. SIAM, Philadelphia (2008)
20. Deift, P., Zhou, X.: A steepest descent method for oscillatory Riemann–Hilbert problems. Bull. Am. Math. Soc. **26**, 119–123 (1992)
21. Lenells, J.: The nonlinear steepest descent method: asymptotics for initial-boundary value problems. SIAM J. Math. Anal. **48**, 2076–2118 (2016)
22. Fokas, A.S., Sung, L.: The nonlinear Schrödinger equation on the half-line. Nonlinearity **18**, 1771–1822 (2005)
23. Arruda, L.K., Lenells, J.: Long-time asymptotics for the derivative nonlinear Schrödinger equation on the half-line. Nonlinearity **30**, 4141–4172 (2017)
24. Guo, B., Liu, N.: Long-time asymptotics for the Kundu–Eckhaus equation on the half-line. J. Math. Phys. **59**, 061505 (2018)

25. Fokas, A.S.: The generalized Dirichlet-to-Neumann map for certain nonlinear evolution PDEs. Commun. Pure Appl. Math. **LVIII**, 639–670 (2005)
26. Fokas, A.S., Pelloni, B.: The Dirichlet-to-Neumann map for the elliptic sine-Gordon equation. Nonlinearity **25**, 1011–1031 (2012)
27. Hwang, G.: The Fokas method: the Dirichlet to Neumann map for the sine-Gordon equation. Stud. Appl. Math. **132**, 381–406 (2014)
28. Lenells, J.: The derivative nonlinear Schrödinger equation on the half-line. Physica D **237**, 3008–3019 (2008)
29. Lenells, J., Fokas, A.S.: The nonlinear Schrödinger equation with *t*-periodic data: II. Perturbative results. Proc. R. Soc. A **471**, 20140926 (2015)
30. Hwang, G.: A perturbative approach for the asymptotic evaluation of the Neumann value corresponding to the Dirichlet datum of a single periodic exponential for the NLS. J. Nonlinear Math. Phys. **21**, 225–247 (2014)