Advances in Continuous and Discrete Models

REVIEW

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My collaboration with Jean Ginibre

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Abstract

The history of the long-standing scientific collaboration between Jean Ginibre and Giorgio Velo is presented. The most important results obtained for a number of nonlinear dispersive partial differential equations of interest in pure mathematics and applied physics are described.

Keywords: Partial differential equations; Cauchy problem; Scattering theory

The first time I met Jean was in September 1967 when we were both visiting the Physics Department of New York University (NYU) for one year. Jean came from the University of Paris-Sud in Orsay (France), and I was from the University of Bologna (Italy).

At that time, the theory part of the Department of Physics was housed on the fifth floor of the Courant Institute of Mathematical Sciences, which was the center for research in mathematics and computer science at NYU. Permanent senior members included some well-known physicists from Europe, K. Symanzik and W. Zimmermann from Germany, and B. Zumino from Italy. High-level analysts were members of the Department of Mathematics: K. Friedrichs, F. John, L. Nirenberg, and P. Lax. I was told that the creation of a strong group in theoretical physics was supported by Friedrichs, who, at the beginning of 1950, became interested in quantum field theory and wrote a remarkable book on that subject in a language unfortunately too personal to be widely circulated.

The environment was perfect for conducting research in mathematical physics on problems motivated by fundamental physics. By chance, we ended up sharing the same office. At that time, Jean was working in in the field of statistical mechanics, studying some aspects of the Bose–Einstein condensation. On my part, I was interested in solvable models of quantum field theory, and having just completed a paper on that subject, I started to pay my attention to the Bose–Einstein condensation. The result of several discussions on that topic was a small paper published in Physics Letters, marking the beginning of our collaboration [1]. In the summer of 1968, Jean returned to Orsay, and I decided to spend my second year at the Courant Institute.

It is now time for me to go briefly back to the beginning of Jean's scientific activity. In 1963, motivated by the use of the SU(3) group in the physics of elementary particles, he studied certain aspects of the Wigner–Eckart theorem for general simple groups. Shortly after, during his one year at Princeton (1963–1964), he became interested in random matrices and wrote a widely quoted paper on that subject. From 1964 to around 1971, Jean turned to statistical mechanics. We owe him the proof of the existence of the thermodynamic limit for a quantum system using the technique of functional integration as well as the proof of the existence of phase transition for sufficiently general quantum systems. Important tools for his last piece of work were some correlation inequalities, the proof of which he contributed in a very important way. During 1970–1971, inspired by the classical paper of L. Fadeev on the three-body problem in quantum mechanics, Jean started to study the Schrödinger operators in quantum mechanics with M. Combescure. They succeeded in improving Fadeev's results using Hilbert space methods, and, among other things, they provided a proof of asymptotic completeness for potentials with adequate decrease in space. This activity ended in 1976.

For my part, I returned to Italy by the end of 1969. Around 1973, motivated by the interest in theoretical physics for particular solutions of some nonlinear equations named solitons, I concentrated on studying certain aspects of the Klein–Gordon equation with nonlinearity. At the same time, Jean and I started visiting each other either in Bologna or in Orsay. Each of us had his main line of research, but we continued a soft form of collaboration on minor problems of statistical mechanics and constructive field theory. As a result of those visits, as well as our previous acquaintance in New York, we realized that our collaboration was working quite well and could be applied to a research project of wider scope. Coming from physics, we were well aware of the impressive progress of research in quantum field theory during 1970–1980: the discovery of the role of solitons, the development of non-abelian gauge theories, the various studies on perturbative renormalization, the results obtained in constructive field theory, etc. These circumstances suggested that it was time to start a serious study of nonlinear classical equations, which might eventually be submitted to the procedure of quantization.

My sabbatical year in Orsay in 1976–1977 was the perfect occasion for a closer collaboration aimed at a deeper study of nonlinear *Partial Differential Equations* (PDE) of the evolution type. In this domain, our research focused mainly on the *Cauchy Problem* (CP) and the *Scattering Theory* (ST). For a given equation, the CP consists of proving the existence of a—possibly unique—solution with prescribed initial conditions at a fixed time. A solution, usually found to exist for a finite interval of time (referred to as a *local solution*), can, in suitable circumstances, be extended to all times (in this case, we refer to it as a *global solution*). When the CP is well controlled in the time interval (T,∞), one can try to determine the asymptotic behavior of the solution for $t \to \infty$ (similarly in the case of the interval $(-\infty, T)$). Most of the equations of our interest were nonlinear perturbations of linear equations whose solutions spread out in time, both in the past and in the future. A typical form of those equations can be written as

$$\frac{\partial}{\partial t} u(t) = Au(t) + f\big(u(t)\big), \tag{$\star$}$$

where A is a linear operator acting on the u's, and f is a nonlinear function of u. The linear equation part of ($\star$) is

$$\frac{\partial}{\partial t} v(t) = Av(t), \tag{$\star\star$}$$

where the u's and v's are supposed to belong to the same vector space. In favorable circumstances, we can compare the asymptotic behavior at $t \sim \pm\infty$ of the solutions of ($\star\star$)

with that of the solutions of ($\star$). (This is not always possible, and I will come back to that point at the end of this script). Now ST can be, in essence, summarized in the following two questions:

(i) Existence of the wave operators: given a $v_+(t)$ solution of ($\star\star$), find a $u(t)$ solution of ($\star$) such that

$$\big(u(t) - v_+(t)\big) \underset{t\to+\infty}{\longrightarrow} 0.$$

The wave operator Ω_+ is defined as the mapping $\Omega_+ : v_+(0) \longmapsto u(0)$. In the same way, we define the wave operator $\Omega_- : v_-(0) \longmapsto u(0)$ with analogous considerations for $t \to -\infty$.

(ii) Given a $u(t)$ solution of ($\star$), find $v_+(t)$ and $v_-(t)$ solutions of ($\star\star$) such that

$$\big(u(t) - v_\pm(t)\big) \underset{t\to\pm\infty}{\longrightarrow} 0.$$

The circumstance that this happens for "all" solutions with "any" initial data is usually referred to as *asymptotic completeness*.

The above rough ideas should be completed by neat statements regarding the functional framework where the equation is defined. This will be done whenever the situation is not too complicated.

In 1976, the only nonlinear dispersive equations for which interesting results were available for the CP and ST in $\mathbb{R}^n$ with $n > 1$ were the Klein–Gordon and wave equations with adequate nonlinearity. It seemed quite natural to us to explore the same problems for the Schrödinger equation perturbed by a nonlinear term:

$$i\partial_t u = -\frac{1}{2}\Delta u + f(u), \tag{NLS}$$

with $t \in \mathbb{R}$, $x \in \mathbb{R}^n$, $n = 1, 2, 3, \dots$, Δ the Laplace operator in $\mathbb{R}^n$, $u = u(t, x)$ a complex function of time and space, and f a complex-valued nonlinear function. The equation (NLS) arises in various physical settings, including—but not limited to—nonlinear optics, surface gravity waves, superconductivity, and Bose–Einstein condensation. The case $n = 1$ is simple for the CP, but it contains some subtleties for the ST. For brevity, we now restrict our attention to the case $n \geq 2$. The results obtained can be summed up in the following way [2, 3]: existence and uniqueness of local solutions by a contraction method; existence of global solutions with finite energy for a not too negative energy functional; existence of the wave operators by a contraction method; asymptotic completeness for a sufficiently repulsive case. For the scattering case, we used a new conservation law, which we called pseudoconformal. In the same way as the conservation of energy has enabled us to control locally in time the growth of the solution, the pseudoconformal law has proved to be a valuable tool in highlighting the time behavior at $t = \pm\infty$ of the solution.

For the specific case $f(u) = \lambda |u|^{p-1} u$ with $\lambda \in \mathbb{R}$, $p \geq 1$, the following precise conclusions are available:

(1) Existence and uniqueness of global solutions $u(t)$ continuous with respect to t in $H^1(\mathbb{R}^n)$ for

$$0 \leq p - 1 < \frac{4}{n-2} \quad \text{if } \lambda > 0,$$

$$0 \leq p-1 < \frac{4}{n} \quad \text{if } \lambda < 0.$$

(2) Existence of the wave operators and asymptotic completeness in $H^1(\mathbb{R}^n) \cap \mathcal{F}H^1(\mathbb{R}^n)$ for

$$\frac{4}{n} < p-1 < \frac{4}{n-2} \quad \text{if } \lambda > 0$$

($\mathcal{F}$ denotes the Fourier operator).

The upper bound on $p-1$ for $\lambda > 0$ in (1)—namely $4/(n-2)$—comes from the fact that the potential energy is controlled by the kinetic energy. The lower bound on $p-1$ in (2)—namely $4/n$—has been subsequently improved by shrinking the space where the equation is solved. However, it cannot go down to zero since the wave operators are proved not to exist in $L^2(\mathbb{R}^n)$ for $p-1 \leq 2/n$.

Less general results by J.-E. Lin and W. Strauss appeared at about the same time. For complete proofs and references, I suggest the book "Semilinear Schrödinger Equations" by T. Cazenave; *Courant Lecture Notes in Mathematics*, **10**. New York University, Courant Institute of Mathematical Sciences, New York, AMS, Providence, R.I., 2003.

A particularly interesting case of (NLS) equation is the one with $f(u) = |u|^2 u$, namely with $\lambda = 1$ and $p = 3$. The trilinear term in the right-hand side of that equation can be regularized by the introduction of a real even function V on $\mathbb{R}^n$ thereby leading to a new equation

$$i\partial_t u = -\frac{1}{2}\Delta u + \left(V * |u|^2\right)u, \tag{H}$$

the Hartree equation, named after D. Hartree who wrote it and used it in atomic physics. Here $*$ denotes the convolution in $\mathbb{R}^n$. For general results about the equation (H), we refer to [4] or, more extensively, to Cazenave's book. An important aspect of our interest in equation (H) is that it can describe a "classical limit" of a quantum theory for a many-body system. More precisely: consider an ensemble of nonrelativistic identical quantum particles obeying the Bose statistics, interacting through a two-body potential V, described by the formalism of second quantization and suitably parametrized in terms of Planck's constant h. Then that quantum system "converges" to the system described by equation (H) when $h \to 0$ [5, 6]. The results concern both the equations of motion and scattering theory and hold for a large class of Hartree potentials, thereby extending the previous important result by K. Hepp. I am aware that the presentation of this "classical limit" lacks definiteness even for a reader expert in quantum mechanics. I wanted to mention it because that line of research has attracted the interest of many mathematical physicists and has been pursued for decades under the name of *mean-field theory*.

In the following years, research activity on equation (NLS) developed considerably. The most important advance, thanks to T. Kato, systematically used some space-time estimates named after R.S. Strichartz for the solutions of the equation. Written by Strichartz in a specific case, both for the linear Schrödinger and linear wave equation, they were later extended to various situations by several people, including us [7, 8]. This new instrument made it possible to improve some of the previous results on equation (NLS). In particular, for $n \geq 3$, $f(u) = \lambda|u|^{p-1}u$ with $\lambda > 0$, the existence of the wave operators and asymptotic completeness have been proved to hold in $H^1(\mathbb{R}^n) \supset H^1(\mathbb{R}^n) \cap \mathcal{F}H^1(\mathbb{R}^n)$ with $4/n < p-1 <$

$4/(n-2)$ [9]. To achieve those conclusions, we have incorporated ideas from J.-E. Lin and W. Strauss.

I will now move on to another important dispersive nonlinear PDE, the Klein–Gordon equation

$$\partial_t^2 u - \Delta u + f(u) = 0, \tag{NLKG}$$

with $t \in \mathbb{R}$, $x \in \mathbb{R}^n$, $n = 1, 2, 3, \ldots$, Δ the Laplace operator in $\mathbb{R}^n$, $u = u(t, x)$ a complex function of time and space, and f a complex valued and possibly nonlinear function. Here, unlike the case of (NLS) equation, the initial condition for the CP is represented by a pair of functions in $\mathbb{R}^n$, $(u(0), \partial_t u(0))$. For simplicity, I will quote just one result in the case of $f(u) = \lambda_0 u + \lambda |u|^{p-1} u$ with $n \geq 2$, $\lambda_0 \in \mathbb{R}$, $\lambda > 0$, $0 < p - 1 < 4/(n-2)$: existence and uniqueness of global solutions u of (NLKG) hold with $(u, \partial_t u)$ continuous functions in time in $(H^1(\mathbb{R}^n), L^2(\mathbb{R}^n))$ [10, 11]. The part concerning ST is rather complicated, depending both on the type of spaces we were allowed to use and on related to this the lower bounds on the values of p. That landscape has been less explored than the case of (NLS) equation [7, 12, 13].

But theoretical physics knocked at the door. The Yang–Mills equations (YM), a noncommutative generalization of the Maxwell equations, suitably coupled to a Klein–Gordon equation (or to a Dirac equation), form a system that, after quantization, describes the theory of fundamental interactions of elementary particles. Due to a gauge invariance similar to the one for the Maxwell equations, the system (YM+KG) is degenerate, so, to acquire a form appropriate to start a CP, a supplementary condition, usually called gauge condition, is needed. The two most important ones are the *temporal* and the *Lorenz* gauges. The local CP was solved in both gauges by various people, including Jean and me. The same people solved the global CP for $n = 1, 2$ in the temporal gauge [14, 15]. The real interesting problem, namely the case $n = 3$, was solved in 1982 by D. Eardley and V. Moncrief in the temporal gauge.

In 1987, Jean entered his Japanese period. One after the other, several talented Japanese mathematicians spent a year at the *Faculté des Sciences* of Orsay. I will mention Y. Tsutsumi, T. Ozawa, and N. Hayashi. In fact, our research on the equation (NLS) found fertile ground in Japan where there existed a tradition of studying dispersive equations, in particular, ST for the linear Schrödinger equation. Then a very fruitful partnership started between Europe and Japan, lasting for years.

In collaboration with Y. Tsutsumi, the existence and uniqueness of solutions for the generalized Kortweg-de Vries equation were established under quite general assumptions:

$$\partial_t u + \partial_x^3 u = \partial_x f(u), \tag{GKdV}$$

with $t \in \mathbb{R}$, $x \in \mathbb{R}$, $u = u(t, x)$ real function of time and space, and f a real-valued nonlinear function with $f(0) = f'(0) = 0$. Even though in a one-dimensional space, that equation poses some difficulties because of the derivative coupling [16]. The results obtained and the methods employed led naturally to the study of the generalized Benjamin–Ono equation, a variant of (GKdV) equation, where ∂_x^3 is replaced by the fractional derivative $-\partial_x(-\partial_x^2)^\mu$ with $\mu > 0$ [17]:

$$\partial_t u - \partial_x \left(-\partial_x^2\right)^\mu u = \partial_x f(u). \tag{GBO}$$

The two previous equations are used to describe the propagation of surface waves in water of relatively shallow depth in one-dimensional situations.

Again with Tsutsumi, we studied the local CP for the Zakharov system of equations [18]:

$$\begin{cases} i\partial_t u + \frac{1}{2}\Delta u = \nu u, \\ \partial_t^2 \nu - \Delta \nu = \Delta |u|^2, \end{cases} \tag{Z}$$

with $t \in \mathbb{R}$, $x \in \mathbb{R}^n$, $n = 1, 2, 3, \dots$, Δ the Laplace operator in $\mathbb{R}^n$, $u = u(t,x)$ and $\nu = \nu(t,x)$ complex and real functions of time and space, respectively, supposed to describe the Langmuir turbulence in a plasma. The function u represents the slowly varying envelope of a rapidly oscillating electric field, and the function ν denotes the deviation of a ion density from its mean value. The interest and the challenge of studying the system (Z) comes from the fact that the two equations have different transformation properties with respect to dilations of space-time. This makes it difficult to define a good critical dimension necessary in using contraction techniques to solve the local CP. The CP has been successfully studied with a method developed by J. Bourgain, of which Jean gave a very clear pedagogical account in a Bourbaki Seminar (1996).

Around the same time (1996–1997), we became interested in the CP for the complex Ginzburg–Landau equations

$$\partial_t u = \gamma u + (a + i\alpha)\Delta u - (b + i\beta)|u|^{2\sigma} u, \tag{cGL}$$

with $t \in \mathbb{R}^+$, $x \in \mathbb{R}^n$, $n = 1, 2, 3, \dots$, Δ the Laplace operator in $\mathbb{R}^n$, $u = u(t,x)$ a complex function of time and space, and $a, b, \alpha, \beta, \gamma, \sigma$ real parameters with $\gamma \geq 0$ and $a, b, \sigma > 0$. Clearly, the equation (cGL) interpolates between the nonlinear Schrödinger and the nonlinear heat equations. Equations of this type are used in hydrodynamics far from equilibrium to describe the start of space structures and the appearance of instabilities. In the applications, one is interested in describing extended systems that have some homogeneity in space and thus do not tend necessarily to zero at infinity in $\mathbb{R}^n$. Therefore, we have studied the CP in spaces defined by local conditions, and, possibly, with some bounds at infinity in $\mathbb{R}^n$. Inspired by a paper by P. Collet, we managed to obtain the existence and uniqueness of global CP in such spaces [19, 20].

The brief description of Scattering Theory summarized after the equation ($\star$) refers to the situation of short range, namely the case where the nonlinear (interaction) term in the equation ($\star$) decays sufficiently fast at infinity in space and/or in time, so that the set of solutions of the equation ($\star\star$) can be adequate as a set of asymptotic motions. The long-range situation is a complementary case where that set is inadequate and has to be replaced by a set of modified asymptotic motions. The last part of our research activity was mainly devoted to the so-called *long-range scattering*, for various equations and systems of equations. I refer to the equation (NLS) with $p = 1 + 2/n$, $n = 1, 2, 3$; the equation (H) with $V(x) = |x|^{-1}$, $n \geq 2$; the wave-Schrödinger system

$$\begin{cases} i\partial_t u = -\frac{1}{2}\Delta u + Au, \\ \partial_t^2 A - \Delta A = -|u|^2, \end{cases} \tag{WS}$$

where $t \in \mathbb{R}$, $x \in \mathbb{R}^3$, Δ the Laplace operator in $\mathbb{R}^3$, $u = u(t,x)$ and $A = A(t,x)$ complex and real functions of time and space, respectively; the Maxwell–Schrödinger system in $\mathbb{R}^3$; the system (Z) for $n = 2, 3$. In this kind of activity, important contributions were provided by the Japanese school and some Russian mathematicians. I refrain here from giving details that are too technical. For the reader's interest, I suggest references [21–23].

Finally, let me add a few words on some personal aspects of my long-standing collaboration with Jean. We know that research demands high levels of concentration and great perseverance to focus the mind on the same object for a long time. For that, considerable technical and psychological effort is required. When we started to study the Cauchy Problem and the Scattering Theory for the NLS equation in $\mathbb{R}^n$ for any n, as an essentially untouched issue, we were both convinced of the importance of digging around the NLS equation and, at the same time, confident of being able to cope with that problem using our expertise as mathematical physicists. In subsequent years, it became increasingly clear that we had a taste for the same type of problems and a kind of common affinity for how to deal with them. When Jean and I were working together, a typical situation was the following: we exchanged pieces of conversation, sometimes discussing in an animated way, getting up from our chairs to write some formulae on the blackboard. All this is interspersed with periods of silence or short remarks, often accompanied by a cup of coffee and peals of laughter due to funny comments to relieve the tension. Actually, I think that the process of doing research has to be a pleasant activity, and I admit that our collaboration was always a source of great amusement for both of us. Jean's insight and ability to concentrate on the basics of the questions were important elements of our long-lasting association. Mutual esteem and warm friendship allowed us to obtain great satisfaction. It has been a fantastic partnership!

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Abbreviations
PDE, partial differential equations; CP, Cauchy problem; ST, scattering theory.

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