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Tangent-point energies and ropelength as Gamma-limits of discrete tangent-point energies on biarc curves

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Abstract

Using interpolation with biarc curves we prove Γ -convergence of discretized tangent-point energies to the continuous tangent-point energies in the C^1 -topology, as well as to the ropelength functional. As a consequence, discrete almost minimizing biarc curves converge to minimizers of the continuous tangent-point energies, and to ropelength minimizers, respectively. In addition, taking point-tangent data from a given $C^{1,1}$ -curve γ , we establish convergence of the discrete energies evaluated on biarc curves interpolating these data, to the continuous tangent-point energy of γ , together with an explicit convergence rate.

Keywords: Ropelength; Tangent-point energy; Discretization; Biarcs; Gamma-convergence

1 Introduction

The ropelength¹ of a closed arclength parametrized curve $\gamma : \mathbb{R}/L\mathbb{Z} \rightarrow \mathbb{R}^3$ is defined as the quotient of its length and thickness,

$$\mathcal{R}(\gamma) := \frac{\mathcal{L}(\gamma)}{\Delta[\gamma]} = \frac{L}{\Delta[\gamma]}. \quad (1)$$

Here, for variational considerations, the thickness $\Delta[\gamma]$ is most conveniently expressed following Gonzalez and Maddocks [15]—without any regularity assumptions on the curve γ —as

$$\Delta[\gamma] := \inf_{s \neq t \neq \tau \neq s} R(\gamma(s), \gamma(t), \gamma(\tau)), \quad (2)$$

where $R(x, y, z)$ denotes the circumcircle radius of the three points $x, y, z \in \mathbb{R}^3$. Motivated by numerous applications in the natural sciences, ropelength is used in numerical computations (see [2, 10, 11, 20] and the references therein) to mathematically model long

¹This name is coined after the mathematical question, how long a thick rope has to be in order to tie it into a knot.

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and slender objects such as strings or macromolecules that do not self-intersect. In fact, it was proved rigorously in [9, 16] that a curve of finite ropelength is embedded and of class $C^{1,1}(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^3)$, which means that its curvature exists and is bounded a.e. on $\mathbb{R}/L\mathbb{Z}$. Moreover, a curve γ with positive thickness $\Delta[\gamma] > 0$ is surrounded by an embedded tube with radius equal to $\Delta[\gamma]$ as shown in [16, Lemma 3], which justifies the use of the non-smooth quantity $\Delta[\cdot]$ as a steric excluded volume constraint.

The minimization over all triples of curve points to evaluate thickness in (2) is costly, which leads to the idea to replace minimization by integration; see [15, p. 4773]. One such integral energy is the *tangent-point energy*

$$\text{TP}_q(\gamma) := \iint_{(\mathbb{R}/L\mathbb{Z})^2} \frac{1}{r_{\text{tp}}^q(\gamma(s), \gamma(t))} \, ds \, dt, \quad q \geq 2, \quad (3)$$

where the circumcircle radius is now replaced by the *tangent-point radius*

$$r_{\text{tp}}(\gamma(s), \gamma(t)) = \frac{|\gamma(s) - \gamma(t)|^2}{2 \, \text{dist}(\gamma(s) + \mathbb{R}\gamma'(s), \gamma(t))}, \quad (4)$$

i.e., the radius of the unique circle through the points $\gamma(s)$ and $\gamma(t)$ that is in addition tangent to the curve γ at $\gamma(s)$. Also, this energy implies self-avoidance and has regularizing properties. It was shown in [33] that if $\text{TP}_q(\gamma)$ is finite for some $q > 2$, then γ is embedded and of class $C^{1,1-\frac{2}{q}}(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^3)$. Later, Blatt [5] improved this regularity to the optimal fractional Sobolev² regularity $W^{2-\frac{1}{q},q}(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^3)$, which actually characterizes curves of finite TP_q -energy. The knowledge of the exact energy space was then used to establish continuous differentiability of the tangent-point energy [7, Remark 3.1], [34], and to find TP_q -critical knots by means of Palais's symmetric criticality principle [14]. Very recently, long-time existence for a suitably regularized gradient flow for TP_q was shown via a minimizing movement scheme [22].

But the tangent-point energy was also used in numerical simulations. Bartels et al. added a desingularized variant of the TP_q -energy in [3, 4] as a self-avoidance term to the bending energy to find elastic knots. The impressive simulations of Crane et al. in [36] use the TP_q -energy as well to avoid self-intersections, a higher-dimensional tangent-point energy allows for computations on self-avoiding surfaces; see [35].

In the present paper we address the mathematical question of variational convergence of suitably discretized tangent-point energies towards the continuous TP_q -energy, as well as towards ropelength. To account for the tangential information encoded in the tangent-point radius in (3) on the discrete level we use *biarcs*, i.e., pairs of circular arcs, which on the one hand, can interpolate *point-tangent data*

$$(\gamma(s_i), \gamma'(s_i)) \in \mathbb{R}^3 \times \mathbb{S}^2 \quad \text{for } i = 1, \dots, n \quad (5)$$

of a given arclength parametrized curve $\gamma \in C^1(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^3)$. Every *biarc curve* β consisting of n consecutive biarcs is therefore a $C^{1,1}$ -interpolant of the curve γ . On the other hand,

²For the definition see Appendix A; a condensed selection of pertinent results regarding periodic fractional Sobolev spaces can be found, e.g., in [21, Appendix A].

every biarc curve produces point-tangent data

$$(q_i, t_i) \in \mathbb{R}^3 \times \mathbb{S}^2 \quad \text{for } i = 1, \dots, n, \quad (6)$$

on its own, namely the points q_i and unit-tangents t_i at every junction of two consecutive biarcs. In principle, we believe that one could carry out the analysis with other splines, but biarcs are well adapted to discretizing geometric curvature energies such as the tangent-point energy or ropelength. The respective integrands are defined by circles, and relevant geometric quantities like arclength, curvature, torsion, or the global radius of curvature can be evaluated accurately pointwise *everywhere* on biarc curves, thus providing rigorous upper bounds for the ropelength of ideal knots, see [11, p. 10], [31, p. 81], [17]. In order to avoid degeneracies we restrict ourselves to those biarc curves β whose biarcs have lengths λ_i that are controlled in terms of the curve's length $\mathcal{L}(\gamma)$ by means of the inequality

$$\frac{\mathcal{L}(\gamma)}{2n} \leq \lambda_i \leq \frac{2\mathcal{L}(\gamma)}{n} \quad \text{for } i = 0, \dots, n-1. \quad (7)$$

Let \mathcal{B}_n be the class of biarc curves β satisfying (7). Accordingly, we define in a parameter-invariant fashion the *discrete tangent-point energy* \mathcal{E}_q^n for $n \in \mathbb{N}$ and $q \in [2, \infty)$ on closed C^1 -curves γ as

$$\mathcal{E}_q^n(\gamma) := \begin{cases} \sum_{i=0}^{n-1} \sum_{j=0, j \neq i}^{n-1} \left(\frac{2 \operatorname{dist}(l(q_j), q_i)}{|q_i - q_j|^2} \right)^q \lambda_i \lambda_j & \text{if } \gamma \in \mathcal{B}_n, \\ \infty & \text{otherwise,} \end{cases} \quad (8)$$

with the straight lines $l(q_i) := q_i + \mathbb{R}t_i$ for $i = 0, \dots, n-1$. Note that both TP_q and \mathcal{E}_q^n are invariant under reparametrization of the curves, and they have the same scaling behavior,

$$\operatorname{TP}_q(d\gamma) = d^{2-q} \operatorname{TP}_q(\gamma) \quad \text{and} \quad \mathcal{E}_q^n(d\gamma) = d^{2-q} \mathcal{E}_q^n(\gamma) \quad \text{for any } d > 0. \quad (9)$$

We restrict ourselves to injective C^1 -curves that are parametrized by arclength, denoted as the subset C_{ia}^1 to state our main results.

Theorem 1.1 (Γ -convergence to tangent-point energy) *For $q > 2$ and $L > 0$ the discrete tangent-point energies \mathcal{E}_q^n Γ -converge to the tangent-point energy TP_q on the space $C_{\text{ia}}^1(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^3)$ with respect to the $\|\cdot\|_{C^1}$ -norm as $n \rightarrow \infty$, i.e.,*

$$\mathcal{E}_q^n \xrightarrow[n \rightarrow \infty]{\Gamma} \operatorname{TP}_q \quad \text{on } (C_{\text{ia}}^1(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^3), \|\cdot\|_{C^1}). \quad (10)$$

As an immediate consequence we infer the convergence of almost minimizers in a given knot class \mathcal{K} of the discrete energies \mathcal{E}_q^n to a minimizer of the continuous tangent-point energy TP_q in the same knot class \mathcal{K} .

Corollary 1.2 (Convergence of discrete almost minimizers) *Let $q > 2$, $L > 0$, and \mathcal{K} be a tame knot class and $b_n \in C^* := C_{\text{ia}}^1(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^3) \cap \mathcal{K}$ with*

$$\left| \inf_{C^*} \mathcal{E}_q^n - \mathcal{E}_q^n(b_n) \right| \rightarrow 0 \quad \text{and} \quad \|b_n - \gamma\|_{C^1} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Then, γ is a minimizer of TP_q in C^* and $\lim_{n \rightarrow \infty} \mathcal{E}_q^n(b_n) = \text{TP}_q(\gamma)$. Furthermore, it holds that $\gamma \in W^{2-\frac{1}{q},q}(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^3)$.

Moreover, the discrete tangent-point energies can also be used to approximate the non-smooth ropelength functional \mathcal{R} in the sense of Γ -convergence.

Theorem 1.3 (Γ -convergence to ropelength) *It holds that $L^{\frac{n-2}{n}}(\mathcal{E}_n^n)^{\frac{1}{n}} \xrightarrow[n \rightarrow \infty]{\Gamma} \mathcal{R}$ on $(C_{\text{ia}}^{1,1}(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^3), \|\cdot\|_{C^1})$.*

Also, here we can state the convergence of almost minimizers to ropelength-minimizing curves in a prescribed knot class, which could be of computational relevance for the minimization of ropelength.

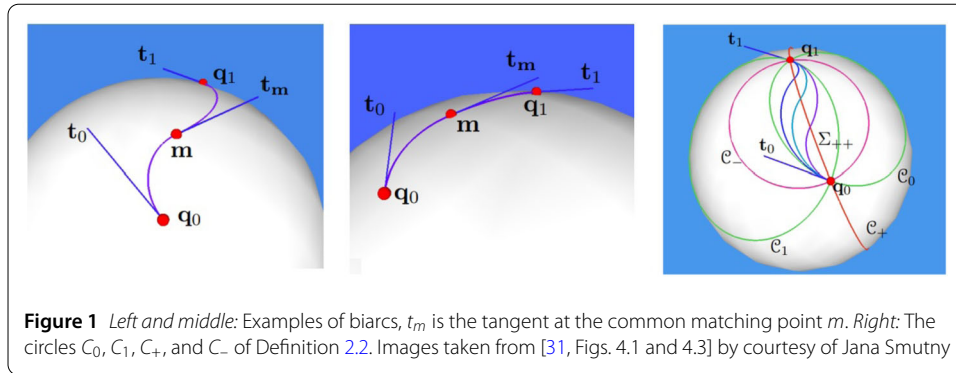
Corollary 1.4 (Discrete almost minimizers approximate ropelength minimizers) *Let \mathcal{K} be a tame knot class and $b_n, \gamma \in C^{**} := C_{\text{ia}}^{1,1}(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^3) \cap \mathcal{K}$ with*

$$\left| \inf_{C^{**}} \mathcal{E}_n^n - \mathcal{E}_n^n(b_n) \right| \rightarrow 0 \quad \text{and} \quad \|b_n - \gamma\|_{C^1} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

*Then, γ is a minimizer of \mathcal{R} in C^{**} and $\lim_{n \rightarrow \infty} L^{\frac{n-2}{n}}(\mathcal{E}_n^n)^{\frac{1}{n}}(b_n) = \mathcal{R}(\gamma)$.*

To the best of our knowledge, the only known contributions on variational convergence of discrete energies to continuous knot energies are the Γ -convergence results of Scholtes and Blatt. In [26] Scholtes proved the Γ -convergence of a discrete polygonal variant of the Möbius energy to the classic Möbius energy introduced by O'Hara [23]. This result was strengthened later by Blatt [6]. In [27, 28] Scholtes proved the Γ -convergence of polygonal versions of ropelength and of integral Menger curvature to ropelength and to continuous integral Menger curvature, respectively. It remains open at this point if stronger types of variational convergence such as Hausdorff convergence of sets of almost minimizers can be shown for the nonlocal knot energies treated here, as was, e.g., established in [29] for the classic bending energy under clamped boundary conditions. It would be also interesting to set up a numerical scheme for the discretized tangent-point energies \mathcal{E}_q^n to numerically approximate ropelength minimizers, in comparison to the simulated annealing computations in [11, 31], or to compute discrete (almost) minimizers of the tangent-point energy. The almost linear energy convergence rate established in Theorem 3.1 in Sect. 3 is identical with the one in [26, Proposition 3.1] for Scholtes' polygonal Möbius energy, which exceeds the $n^{-\frac{1}{4}}$ -convergence rate for the minimal distance approximation of the Möbius energy by Rawdon and Simon [24, Theorem 1].

The present paper is structured as follows. In Sect. 2 we provide the necessary background on biarcs—mainly following Smutny's work [31]. Section 3 is devoted to the convergence of the discretized energies \mathcal{E}_q^n including explicit convergence rates; see Theorem 3.1. In Sect. 4 we treat Γ -convergence towards the continuous tangent-point energies, as well as convergence of discrete almost minimizers, to prove Theorem 1.1 and Corollary 1.2. Finally, in Sect. 5 we prove Γ -convergence to the ropelength functional, Theorem 1.3 and convergence of discrete almost minimizers to ropelength minimizers, Corollary 1.4. In Appendix A we establish the convergence of rescaled and reparametrized convolutions in fractional Sobolev spaces. Appendix B contains some quantitative analysis of general C^1 -curves.



2 Biarcs and Biarc curves

The discrete tangent-point energy defined in (8) of the introduction is defined on biarc curves, which are space curves assembled from biarcs, i.e., from pairs of circular arcs. In this section we first present the basic definitions and a general existence result due to Smutny [31, Chap. 4], before specializing to the balanced proper biarc interpolations needed in our convergence proofs later.

Definition 2.1 (Point-tangent pairs and biarcs) Let $\mathcal{T} := \mathbb{R}^3 \times \mathbb{S}^2$ be the set of *point-tangent data* $[q, t]$, where \mathbb{S}^2 is the unit sphere in \mathbb{R}^3 .

(i) A *point-tangent pair* is a pair of tuples of the form $([q_0, t_0], [q_1, t_1]) \in \mathcal{T} \times \mathcal{T}$ with $q_0 \neq q_1$.

(ii) A *biarc* (a, \bar{a}) is a pair of circular arcs in \mathbb{R}^3 that are continuously joined with continuous tangents and that interpolate a point-tangent pair $([q_0, t_0], [q_1, t_1]) \in \mathcal{T} \times \mathcal{T}$. The common end point m of the two circular arcs a and \bar{a} is called the *matching point*. The interpolation is meant with orientation, such that t_0 points to the interior of the arc a and $-t_1$ points to the interior of the arc \bar{a} ; see Fig. 1.

For two points $q_0, q_1 \in \mathbb{R}^3$ we set $d := q_1 - q_0$ and $e := \frac{q_1 - q_0}{|q_1 - q_0|} = \frac{d}{|d|}$, and define

$$R(e) := 2e \otimes e - \text{Id} = 2ee^T - \text{Id}, \quad (11)$$

which is a symmetric, proper rotation matrix representing the reflection at the unit vector e . Moreover, for a point-tangent pair $([q_0, t_0], [q_1, t_1]) \in \mathcal{T} \times \mathcal{T}$ we set

$$t_0^* := R(e)t_0 \quad \text{and} \quad t_1^* := R(e)t_1. \quad (12)$$

Definition 2.2 Let $([q_0, t_0], [q_1, t_1]) \in \mathcal{T} \times \mathcal{T}$ be a point-tangent pair.

(i) Let C_0 be the circle through q_0 and q_1 with tangent t_0 at q_0 and let C_1 be the circle through both points with tangent t_1 at q_1 . If $t_0 + t_1^* \neq 0$, we denote the circle through q_0 and q_1 with tangent $t_0 + t_1^*$ at q_0 by C_+ , if $t_0 - t_1^* \neq 0$, we denote the circle through both points with tangent $t_0 - t_1^*$ at q_0 by C_- ; see Fig. 1 on the right.

(ii) A point-tangent pair $([q_0, t_0], [q_1, t_1]) \in \mathcal{T} \times \mathcal{T}$ is called *cocircular*, if $C_0 = C_1$ as point sets. A cocircular point-tangent pair is classified as *compatible*, if the orientations of the two circles induced by the tangents agree, and *incompatible* otherwise.

Remark 2.3 For a point-tangent pair $([q_0, t_0], [q_1, t_1]) \in \mathcal{T} \times \mathcal{T}$, the compatible cocircular case is equivalent to $t_0 - t_1^* = 0$. In this case, the circle \mathcal{C}_- is not defined. The incompatible cocircular case is equivalent to $t_0 + t_1^* = 0$, thus the circle \mathcal{C}_+ is not defined.

The following central existence result of Smutny not only states that interpolating biarcs always exist, but it also characterizes geometrically the possible locations of the corresponding matching points depending on the type of the point-tangent pair. For the precise statement we denote for an arbitrary circle \mathcal{C} through q_0 and q_1 the punctured set $\mathcal{C}' := \mathcal{C} \setminus \{q_0, q_1\}$.

Proposition 2.4 ([31, Proposition 4.7]) *For a given point-tangent pair $([q_0, t_0], [q_1, t_1]) \in \mathcal{T} \times \mathcal{T}$, we denote by $\Sigma_+ \subset \mathbb{R}^3$ the set of matching points of all possible biarcs interpolating the point-tangent pair. Then:*

- (i) *If $([q_0, t_0], [q_1, t_1])$ is not cocircular, then $\Sigma_+ = \mathcal{C}'_+$.*
- (ii) *If $([q_0, t_0], [q_1, t_1])$ is cocircular, we distinguish between two cases:*
 - (a) *If the point-tangent pair is compatible, then $\Sigma_+ = \mathcal{C}'_+ = \mathcal{C}'_0 = \mathcal{C}'_1$.*
 - (b) *If the point-tangent pair is incompatible, then Σ_+ is the sphere passing through q_0 and q_1 perpendicular to the circle \mathcal{C}_- without the points q_0 and q_1 .*
- (iii) *Σ_+ is a straight line passing through q_0 and q_1 without the two points if and only if $t_0 = t_1$ and $\langle t_0, e \rangle \neq 0$.*
- (iv) *Σ_+ is a plane through q_0 and q_1 without the two points if and only if $t_0 = t_1$ and $\langle t_0, e \rangle = 0$.*

A particularly powerful interpolation is possible if the location of the matching point $m \in \Sigma_+$ of the biarc is roughly “in between” the points q_0 and q_1 . The following definition states this precisely for the relevant cases (i), (ii)(a), and (iii) of Proposition 2.4.

Definition 2.5 (Desired matching point location Σ_{++} and proper biarcs) (i) Let $([q_0, t_0], [q_1, t_1]) \in \mathcal{T} \times \mathcal{T}$ be a point-tangent pair that is not incompatible cocircular. Then, we denote by $\Sigma_{++} \subset \Sigma_+$ the subarc of Σ_+ from q_0 to q_1 with the orientation induced by the tangent $t_0 + t_1^*$ (see Fig. 2).

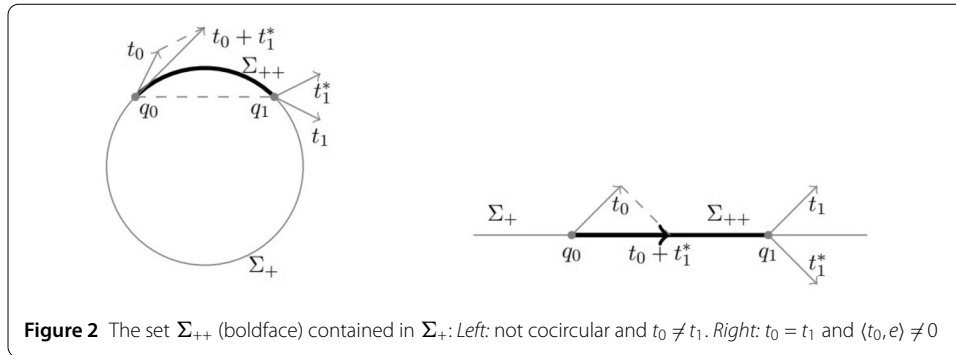
(ii) A point-tangent pair $([q_0, t_0], [q_1, t_1]) \in \mathcal{T} \times \mathcal{T}$ is called *proper* if $\langle q_1 - q_0, t_0 \rangle > 0$ and $\langle q_1 - q_0, t_1 \rangle > 0$.

(iii) A biarc is called *proper* if it interpolates a proper point-tangent pair with a matching point $m \in \Sigma_{++}$.

(iv) Let $\gamma \in C_{\text{ia}}^1(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^3)$. We call a biarc γ -*interpolating* and *balanced* if, for given $h > 0$ and $s \in \mathbb{R}$, it interpolates a point-tangent pair $([\gamma(s), \gamma'(s)], [\gamma(s+h), \gamma'(s+h)])$, such that the matching point $m_h \in \Sigma_{++}^h$ satisfies $|m_h - \gamma(s)| = |\gamma(s+h) - m_h|$, where we indicate the dependence of matching point and location by the index h .

Item (iv) of Definition 2.5 requires that the matching point m_h bisects the segment connecting $\gamma(s)$ and $\gamma(s+h)$. That this is indeed possible for sufficiently small h is the content of the following result. Note that here and throughout the paper we use the *periodic norm*

$$|s - t|_{\mathbb{R}/L\mathbb{Z}} := \min_{k \in \mathbb{Z}} |s + Lk - t| \quad (13)$$



to measure distances in the periodic domain $\mathbb{R}/L\mathbb{Z}$. Moreover, for a continuous function f on $[0, L]$ we denote by $\omega_f : [0, L] \rightarrow [0, \infty)$ its *modulus of continuity*, which satisfies $\omega_f(0) = 0$ and that can be chosen to be concave and non-decreasing.

Lemma 2.6 (Existence of γ -interpolating proper biarc) *Let $\gamma \in C_{\text{ia}}^1(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^3)$ and $h \in (0, \frac{L}{2}]$ such that $\omega_{\gamma'}(h) < \frac{1}{2}$. Then, there exists a proper γ -interpolating balanced biarc interpolating the point-tangent pair $([\gamma(s), \gamma'(s)], [\gamma(s+h), \gamma'(s+h)])$ for all $s \in \mathbb{R}$.*

Proof First, note that (79) and (80) of Lemma B.2 together with $\omega_{\gamma'}(h) < \frac{1}{2}$ and the injectivity of γ imply that

$$\langle \gamma'(s), \gamma(s+h) - \gamma(s) \rangle > 0 \quad \text{and} \quad \langle \gamma'(s+h), \gamma(s+h) - \gamma(s) \rangle > 0. \quad (14)$$

Thus, the point-tangent pair is proper according to Definition 2.5(ii).

If the point-tangent pair is not incompatible cocircular and $\gamma'(s) \neq \gamma'(s+h)$ holds, it follows from Proposition 2.4(i) and (ii) (a) that Σ_+^h is the circle C_+^h . Hence, Σ_{++}^h is a circular arc between $\gamma(s)$ and $\gamma(s+h)$. Thus, the matching point m_h can be chosen in Σ_{++}^h such that $|m_h - \gamma(s)| = |\gamma(s+h) - m_h|$ holds.

If $\gamma'(s) = \gamma'(s+h)$ holds, Σ_+^h is a straight line as a consequence of Proposition 2.4(iii), since we obtain $\langle \gamma'(s), e_h \rangle > 0$ by dividing (14) through $|\gamma(s+h) - \gamma(s)|$, thus excluding case (iv) of Proposition 2.4. Moreover, with $\gamma'(s) = \gamma'(s+h)$ we infer for the unit vector $e_h := \gamma(s+h) - \gamma(s) / |\gamma(s+h) - \gamma(s)|$ by means of (11) and (12)

$$\begin{aligned} \gamma'(s) + (\gamma'(s+h))^* &\stackrel{(11),(12)}{=} \gamma'(s) + 2(e_h \otimes e_h)\gamma'(s+h) - \gamma'(s+h) \\ &= 2\langle \gamma'(s), e_h \rangle e_h. \end{aligned}$$

Thus, the vector $\gamma'(s) + (\gamma'(s+h))^*$ is a positive multiple of the vector e_h . In particular, $\gamma'(s) + (\gamma'(s+h))^*$ has the same orientation as e_h . According to Definition 2.5, Σ_{++}^h is in this case the line segment between $\gamma(s)$ and $\gamma(s+h)$. Therefore, the matching point m_h in Σ_{++}^h can be also chosen such that $|m_h - \gamma(s)| = |\gamma(s+h) - m_h|$.

Hence, we have completed the proof once we have shown that the smallness condition on h excludes case (ii) (b) of Proposition 2.4. Indeed, suppose that the point-tangent pair was incompatible cocircular. Then,

$$\gamma'(s) + (\gamma'(s+h))^* = 0, \quad (15)$$

and using (12) we can write

$$(\gamma'(s+h))^* = 2e_h \otimes e_h \gamma'(s+h) - \gamma'(s+h) = 2\langle e_h, \gamma'(s+h) \rangle e_h - \gamma'(s+h).$$

This representation inserted into (15) leads to $\gamma'(s+h) - \gamma'(s) = 2\langle e_h, \gamma'(s+h) \rangle e_h$ and hence,

$$|\gamma'(s+h) - \gamma'(s)| = 2|\langle e_h, \gamma'(s+h) \rangle| \underbrace{|e_h|}_{=1} = 2 \frac{\langle \gamma'(s+h) - \gamma'(s), \gamma'(s+h) \rangle}{|\gamma'(s+h) - \gamma'(s)|}. \quad (16)$$

By virtue of inequality (80) in Lemma B.2 we conclude that

$$2(1 - \omega_{\gamma'}(h)) \stackrel{(80)}{\leq} 2 \frac{\langle \gamma'(s+h) - \gamma'(s), \gamma'(s+h) \rangle}{|\gamma'(s+h) - \gamma'(s)|} \stackrel{(16)}{=} |\gamma'(s+h) - \gamma'(s)| \leq \omega_{\gamma'}(h),$$

which is equivalent to $\omega_{\gamma'}(h) \geq \frac{2}{3}$, contradicting our assumption on h . \square

Glueing together finitely many interpolating biarcs in a C^1 -fashion produces biarc curves precisely defined as follows.

Definition 2.7 ([31, cf. Definition 6.1]) (i) A closed *biarc curve* $\beta : J \rightarrow \mathbb{R}^3$ is a closed curve assembled from biarcs in a C^1 -fashion where the biarcs interpolate a sequence $([q_i, t_i])_{i \in I}$ of point-tangent tuples. J is a compact interval, $I \subset \mathbb{N}$ bounded, and the first and last point-tangent tuple coincide. The set of such biarc curves is denoted by $\tilde{\mathcal{B}}_n$ where n is the number of indices contained in I .

(ii) We call a closed biarc curve *proper* if every biarc of the curve is proper.

(iii) A biarc curve is γ -*interpolating* and *balanced* for a given curve $\gamma \in C_{\text{ia}}^1(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^3)$ if every biarc of the curve is γ -interpolating and balanced.

Note that the set \mathcal{B}_n of closed biarc curves satisfying (7) introduced in the introduction is a strict subset of $\tilde{\mathcal{B}}_n$.

Under suitable control of partitions of the periodic domain we can prove the existence of proper, γ -interpolating, and balanced biarc curves in Lemma 2.9 below.

Definition 2.8 Let $c_1, c_2 > 0$. A sequence $(\mathcal{M}_n)_{n \in \mathbb{N}}$ of partitions of $\mathbb{R}/L\mathbb{Z}$ with $\mathcal{M}_n := \{s_{n,0}, \dots, s_{n,n}\}$ and $0 = s_{n,0} < s_{n,1} < \dots < s_{n,n-1} < s_{n,n} = L$ is called $(c_1 - c_2)$ -*distributed* if for

$$h_n := \max_{k=0, \dots, n-1} |s_{n,k+1} - s_{n,k}| \quad \text{and} \quad \tilde{h}_n := \min_{k=0, \dots, n-1} |s_{n,k+1} - s_{n,k}|$$

one has

$$\frac{c_1}{n} \leq \tilde{h}_n \leq h_n \leq \frac{c_2}{n} \quad \text{and} \quad h_n \leq \frac{L}{2} \quad \text{for any } n \in \mathbb{N}. \quad (17)$$

Lemma 2.9 Let $\gamma \in C_{\text{ia}}^1(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^3)$, $c_1, c_2 > 0$ and $(\mathcal{M}_n)_{n \in \mathbb{N}}$ a sequence of $(c_1 - c_2)$ -distributed partitions. Then, there is some $N \in \mathbb{N}$ such that for all $n \geq N$ there exists a proper γ -interpolating and balanced biarc curve β_n interpolating the point-tangent pairs

$$([\gamma(s_{n,i}), \gamma'(s_{n,i})], [\gamma(s_{n,i+1}), \gamma'(s_{n,i+1})])_{i=0, \dots, n-1}.$$

Proof By means of the defining inequality (17) for the $(c_1 - c_2)$ -distributed sequence $(\mathcal{M}_n)_{n \in \mathbb{N}}$ we have $|s_{n,i+1} - s_{n,i}|_{\mathbb{R}/L\mathbb{Z}} = |s_{n,i+1} - s_{n,i}|$ for all $n \in \mathbb{N}$ and $i = 0, \dots, n-1$ (see (13)), and we can choose $N \in \mathbb{N}$ so large that the inequalities $\omega_{\gamma'}(\frac{c_2}{N}) < \frac{1}{2}$ and $\frac{c_2}{N} \leq \frac{L}{2}$ hold. Then, in particular,

$$\omega_{\gamma'}(|s_{n,i+1} - s_{n,i}|) \leq \omega_{\gamma'}(h_n) \stackrel{(17)}{\leq} \omega_{\gamma'}\left(\frac{c_2}{N}\right) < \frac{1}{2} \quad \text{for any } n \geq N.$$

As a consequence of Lemma 2.6, there exists for all $n \geq N$ and $i = 0, \dots, n-1$ a proper γ -interpolating and balanced biarc interpolating the point-tangent pair $([\gamma(s_{n,i}), \gamma'(s_{n,i})], [\gamma(s_{n,i+1}), \gamma'(s_{n,i+1})])$. Now, we assemble for $i = 0, \dots, n-1$ these n biarcs as in Definition 2.7 and obtain a biarc curve with the required properties. \square

From now on, whenever we write β_n for a given curve $\gamma \in C_{\text{ia}}^1(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^3)$, we mean a proper γ -interpolating and balanced biarc curve obtained in Lemma 2.9. By $\lambda_{n,i}$ we denote the length of the i th biarc of the curve β_n . In general, the elements β_n do not have the same length as the interpolated curve γ . However, Smutny showed in [31] that under certain assumptions the sequence of the lengths $(\mathcal{L}(\beta_n))_{n \in \mathbb{N}}$ of β_n converges towards the length $\mathcal{L}(\gamma)$ of γ . The following lemma is an essential ingredient for that proof.

Lemma 2.10 *Let $\gamma \in C_{\text{ia}}^{1,1}(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^3)$ and $(\beta_n)_{n \in \mathbb{N}}$ be a sequence of proper γ -interpolating and balanced biarc curves as in Lemma 2.9. Then,*

$$\lambda_{n,i} - |s_{n,i+1} - s_{n,i}| = O(|s_{n,i+1} - s_{n,i}|^3) \quad \text{for all } i = 1, \dots, n-1, \text{ as } n \rightarrow \infty,$$

where the constant on the right-hand side only depends on the Lipschitz constant of γ' .

Proof We identify the periodic domain $\mathbb{R}/L\mathbb{Z}$ with $[0, L]$ and check that $(c_1 - c_2)$ -distributed partitions of $\mathbb{R}/L\mathbb{Z}$ satisfy Smutny's requirements in [31, Notation 6.2, 6.3] apart from the nestedness of the mesh. The latter, however, is not necessary in her proof; whence we can apply [31, Lemma 6.8] to conclude the statement, where the dependence of the constant follows from the proof of [31, Lemma 6.8]. \square

Now, we show that the lengths $\mathcal{L}(\beta_n)$ of proper γ -interpolating and balanced biarc curves β_n converge towards the length $\mathcal{L}(\gamma)$ of γ .

Theorem 2.11 *Let $\gamma \in C_{\text{ia}}^{1,1}(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^3)$ and $(\beta_n)_{n \in \mathbb{N}}$ be a sequence of proper γ -interpolating and balanced biarc curves. Then, $|\frac{\mathcal{L}(\beta_n)}{\mathcal{L}(\gamma)} - 1| \rightarrow 0$ as $n \rightarrow \infty$.*

Proof This follows directly from [31, Corollary 6.9]; under the same preconditions as we verified in the proof of Lemma 2.10. \square

In order to address convergence of biarc curves β_n to the interpolated curve γ we need to reparametrize β_n for all $n \in \mathbb{N}$ such that those reparametrizations are defined on $\mathbb{R}/L\mathbb{Z}$ like γ is. An explicit reparametrization function that maps the arclength parameters of γ at the supporting points of the mesh to the arclength parameters of β_n is constructed in [31, Appendix A]. With that, we can show the C^1 -convergence of a reparametrized sequence of biarc curves to the interpolated curve γ .

Theorem 2.12 *Let $\gamma \in C_{\text{ia}}^{1,1}(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^3)$, and let $(\beta_n)_{n \in \mathbb{N}}$ be a sequence of proper γ -interpolating and balanced biarc curves parametrized by arclength. Then, for $B_n := \beta_n \circ \varphi_n$ with φ_n as constructed in [31, Appendix A] one has $\|\gamma - B_n\|_{C^1} \rightarrow 0$ as $n \rightarrow \infty$.*

Proof We want to apply [31, Theorem 6.13], where Smutny showed C^1 -convergence under certain assumptions. Additionally to the hypotheses checked before in the proof of Lemma 2.10, we need to show that the so-called biarc parameters $\Lambda_{n,i}$ of the i th biarc of the biarc curve β_n , representable as (cf. [31, Lemma 4.13])

$$\Lambda_{n,i} = \frac{\langle \gamma'(s_{n,i}), \gamma(s_{n,i+1}) - \gamma(s_{n,i}) \rangle |m_{n,i} - \gamma(s_{n,i})|^2}{\langle \gamma'(s_{n,i}), m_{n,i} - \gamma'(s_{n,i}) \rangle |\gamma(s_{n,i+1}) - \gamma(s_{n,i})|^2},$$

where the $m_{n,i}$ are the matching points of the i th biarc, are uniformly bounded from below and from above. In other words, we have to prove that there exist two constants $\Lambda_{\min}, \Lambda_{\max}$ such that

$$0 < \Lambda_{\min} \leq \Lambda_{n,i} \leq \Lambda_{\max} < 1 \quad \text{for any } n \in \mathbb{N}, i = 0, \dots, n-1.$$

Using the fact that the biarc curves are balanced, i.e., $|m_{n,i} - \gamma(s_{n,i})| = |m_{n,i} - \gamma(s_{n,i+1})|$, and that γ is parametrized by arclength, we can then estimate by means of (79) in Lemma B.2 in the appendix

$$\begin{aligned} \Lambda_{n,i} &\geq \frac{\langle \gamma'(s_{n,i}), \gamma(s_{n,i+1}) - \gamma(s_{n,i}) \rangle |m_{n,i} - \gamma(s_{n,i})|^2}{|m_{n,i} - \gamma(s_{n,i})| |\gamma(s_{n,i+1}) - \gamma(s_{n,i})|^2} \\ &= \frac{\langle \gamma'(s_{n,i}), \gamma(s_{n,i+1}) - \gamma(s_{n,i}) \rangle}{|\gamma(s_{n,i+1}) - \gamma(s_{n,i})|} \frac{|m_{n,i} - \gamma(s_{n,i})|}{|\gamma(s_{n,i+1}) - \gamma(s_{n,i})|} \\ &\geq \frac{\langle \gamma'(s_{n,i}), \gamma(s_{n,i+1}) - \gamma(s_{n,i}) \rangle}{|\gamma(s_{n,i+1}) - \gamma(s_{n,i})|} \underbrace{\frac{|m_{n,i} - \gamma(s_{n,i})|}{|m_{n,i} - \gamma(s_{n,i+1})| + |m_{n,i} - \gamma(s_{n,i})|}}_{=\frac{1}{2}} \\ &\stackrel{(79)}{\geq} \frac{1}{2} (1 - \omega_{\gamma'}(|s_{n,i+1} - s_{n,i}|)) \geq \frac{1}{2} (1 - \omega_{\gamma'}(h_n)). \end{aligned}$$

Hence, we can choose n sufficiently large such that $\frac{1}{2}(1 - \omega_{\gamma'}(h_n)) \geq \frac{1}{4} =: \Lambda_{\min}$. On the other hand, by [31, Lemma 5.6] we have

$$\Lambda_{n,i} = 1 - \bar{\Lambda}_{n,i} + O(h_{n,i}^2) \quad \text{for any } i = 0, \dots, n-1, \text{ as } n \rightarrow \infty,$$

where the constant hidden in the $O(h_{n,i}^2)$ -term only depends on the curve γ and where $\bar{\Lambda}_{n,i}$ is given by

$$\bar{\Lambda}_{n,i} = \frac{\langle \gamma'(s_{n,i+1}), \gamma(s_{n,i+1}) - \gamma(s_{n,i}) \rangle |m_{n,i} - \gamma(s_{n,i+1})|^2}{\langle \gamma'(s_{n,i+1}), \gamma'(s_{n,i+1}) - m_{n,i} \rangle |\gamma(s_{n,i+1}) - \gamma(s_{n,i})|^2}.$$

As for $\Lambda_{n,i}$, we can estimate $\bar{\Lambda}_{n,i} \geq \frac{1}{2}(1 - \omega_{\gamma'}(h_n))$, which yields

$$\Lambda_{n,i} \leq 1 - \frac{1}{2}(1 - \omega_{\gamma'}(h_n)) + O(h_n^2) \leq \frac{1}{2}(1 + \omega_{\gamma'}(h_n)) + O(h_n^2) \quad \text{as } n \rightarrow \infty.$$

Hence, again choosing n sufficiently large, we obtain $\frac{1}{2}(1 + \omega_{\gamma'}(h_n)) + O(h_n^2) \leq \frac{3}{4} =: \Lambda_{\max}$ as the necessary uniform bound on the biarc parameters. Therefore, [31, Theorem 6.13] is applicable and we obtain that $B_n \rightarrow \gamma$ in C^1 as $n \rightarrow \infty$. \square

3 Discrete energies on interpolating biarc curves that converge to the continuous TP_q energy

For the central convergence result of this section, Theorem 3.1, we work with discrete tangent-point energies $\tilde{\mathcal{E}}_q^n$ with the larger effective domain $\tilde{\mathcal{B}}_n$ (see Definition 2.7(i)), instead of with \mathcal{E}_q^n introduced in (8) of the introduction, whose effective domain \mathcal{B}_n is defined by the constraint (7). In other words,

$$\tilde{\mathcal{E}}_q^n(\gamma) := \begin{cases} \sum_{i=0}^{n-1} \sum_{j=0, j \neq i}^{n-1} \left(\frac{2 \text{dist}(l(q_j), q_i)}{|q_i - q_j|^2} \right)^q \lambda_i \lambda_j & \text{if } \gamma \in \tilde{\mathcal{B}}_n, \\ \infty & \text{otherwise.} \end{cases} \quad (18)$$

These discrete energies evaluated on a sequence $(\beta_n)_{n \in \mathbb{N}}$ of proper γ -interpolating and balanced biarc curves converge with a certain rate to the continuous TP_q -energy of γ if γ is sufficiently smooth. Some of the ideas in the proof of the theorem are based on [26, Proposition 3.1] by Scholtes.

Theorem 3.1 *Let $c_1, c_2 > 0$ and $\gamma \in C_{\text{ia}}^{1,1}(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^3)$, and $(\mathcal{M}_n)_{n \in \mathbb{N}}$ with $\mathcal{M}_n = \{s_{n,0}, \dots, s_{n,n}\}$ be a $(c_1 - c_2)$ -distributed sequence of partitions of $\mathbb{R}/L\mathbb{Z}$ (see Definition 2.8). Then, there is a constant $C > 0$ depending on q, c_1, c_2 , and γ , such that for a sequence $(\beta_n)_{n \in \mathbb{N}}$ of proper γ -interpolating and balanced biarc curves interpolating the point-tangent data*

$$(([\gamma(s_{n,i}), \gamma'(s_{n,i})], [\gamma(s_{n,i+1}), \gamma'(s_{n,i+1})]))_{i=0, \dots, n-1}$$

with $\beta_n \in \tilde{\mathcal{B}}_n$ for all $n \in \mathbb{N}$, there is an index $N \in \mathbb{N}$ with

$$|\text{TP}_q(\gamma) - \tilde{\mathcal{E}}_q^n(\beta_n)| \leq \frac{C \ln(n)}{n} \quad \text{for any } n \geq N.$$

Note that in particular, the convergence rate $\frac{\ln(n)}{n}$ implies the convergence rate $\frac{1}{n^{1-\varepsilon}}$ for any given $\varepsilon > 0$.

Proof of Theorem 3.1 Set $\Upsilon = 4 \frac{c_2}{c_1}$ and define for $i, j \in \{0, \dots, n\}$ the periodic index distance

$$|i - j|_n := \min\{|i - j|, n - |i - j|\}.$$

We then decompose

$$\begin{aligned} \text{TP}_q(\gamma) - \tilde{\mathcal{E}}_q^n(\beta_n) &= \sum_{i=0}^{n-1} \sum_{j, |i-j|_n \leq \Upsilon} \int_{s_{n,j}}^{s_{n,j+1}} \int_{s_{n,i}}^{s_{n,i+1}} \left(2 \frac{\text{dist}(l(\gamma(t)), \gamma(s))}{|\gamma(s) - \gamma(t)|^2} \right)^q ds dt \\ &\quad - \sum_{i=0}^{n-1} \sum_{j, 0 < |i-j|_n \leq \Upsilon} \left(2 \frac{\text{dist}(l(\gamma(s_{n,j})), \gamma(s_{n,i}))}{|\gamma(s_{n,i}) - \gamma(s_{n,j})|^2} \right)^q \lambda_{n,i} \lambda_{n,j} \\ &\quad + 2^q \sum_{i=0}^{n-1} \sum_{j, |i-j|_n > \Upsilon} (A_{i,j} + B_{i,j} + C_{i,j}), \end{aligned} \quad (19)$$

with

$$\begin{aligned} A_{i,j} &:= \int_{s_{n,j}}^{s_{n,j+1}} \int_{s_{n,i}}^{s_{n,i+1}} \frac{\text{dist}(l(\gamma(t)), \gamma(s))^q - \text{dist}(l(\gamma(s_{n,j})), \gamma(s_{n,i}))^q}{|\gamma(s) - \gamma(t)|^{2q}} \, ds \, dt, \\ B_{i,j} &:= \int_{s_{n,j}}^{s_{n,j+1}} \int_{s_{n,i}}^{s_{n,i+1}} \left(\frac{\text{dist}(l(\gamma(s_{n,j})), \gamma(s_{n,i}))^q}{|\gamma(s) - \gamma(t)|^{2q}} - \frac{\text{dist}(l(\gamma(s_{n,j})), \gamma(s_{n,i}))^q}{|\gamma(s_{n,i}) - \gamma(s_{n,j})|^{2q}} \right) \, ds \, dt, \\ C_{i,j} &:= \frac{\text{dist}(l(\gamma(s_{n,j})), \gamma(s_{n,i}))^q}{|\gamma(s_{n,i}) - \gamma(s_{n,j})|^{2q}} [s_{n,i+1} - s_{n,i} |s_{n,j+1} - s_{n,j}| - \lambda_{n,i} \lambda_{n,j}]. \end{aligned}$$

Step 1: Since γ is an injective C^1 -curve it is bi-Lipschitz (see Lemma B.1), i.e., there exists a constant $c_\gamma \in (0, \infty)$ such that

$$|t - s|_{\mathbb{R}/L\mathbb{Z}} \leq c_\gamma |\gamma(t) - \gamma(s)| \quad \text{for any } t, s \in \mathbb{R}. \quad (20)$$

Step 2: Now, we give an upper bound for $2 \frac{\text{dist}(l(\gamma(t)), \gamma(s))}{|\gamma(s) - \gamma(t)|^2}$ for all $s, t \in \mathbb{R}$ with $s \neq t$. Without loss of generality we assume $t < s$. Then, there exists a number $k = k(s, t) \in \mathbb{Z}$ satisfying $|t - s|_{\mathbb{R}/L\mathbb{Z}} = |kL + t - s|$. We use the periodicity of γ and $K := \|\gamma''\|_{L^\infty} < \infty$ (since $\gamma \in C^{1,1}(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^3) \simeq W^{1,\infty}(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^3)$) to estimate

$$\begin{aligned} &\text{dist}(l(\gamma(t)), \gamma(s)) \\ &= \inf_{\mu \in \mathbb{R}} |\gamma(s) - \gamma(t) - \mu \gamma'(t)| \\ &\leq |\gamma(s) - \gamma(kL + t) - (s - (kL + t)) \gamma'(kL + t)| \\ &\leq \int_{kL+t}^s \int_{kL+t}^u |\gamma''(v)| \, dv \, du \leq K(kL + t - s)^2 = K|t - s|_{\mathbb{R}/L\mathbb{Z}}^2, \end{aligned} \quad (21)$$

where we assumed, without loss of generality, that $kL + t < s$ for the integrals. Therefore, by means of (20)

$$\left(2 \frac{\text{dist}(l(\gamma(t)), \gamma(s))}{|\gamma(s) - \gamma(t)|^2} \right)^q \leq (2c_\gamma^2 K)^q \quad \text{for any } s, t \in \mathbb{R}, s \neq t. \quad (22)$$

Define $C_1 := (2c_\gamma^2 K)^q c_2 (2\Upsilon + 1)L$. Applying the calculations above we can estimate the first term on the right-hand side of (19) from above by

$$\begin{aligned} &\stackrel{(22)}{\leq} (2c_\gamma^2 K)^q \sum_{i=0}^{n-1} \sum_{j, |i-j|_n \leq \Upsilon} |s_{n,i+1} - s_{n,i}| |s_{n,j+1} - s_{n,j}| \\ &\stackrel{(17)}{\leq} (2c_\gamma^2 K)^q \frac{c_2}{n} \sum_{i=0}^{n-1} |s_{n,i+1} - s_{n,i}| \underbrace{\sum_{\substack{j, |i-j|_n \leq \Upsilon \\ \leq 2\Upsilon+1}} 1}_{\leq 2\Upsilon+1} \\ &\leq (2c_\gamma^2 K)^q \frac{c_2}{n} (2\Upsilon + 1) \underbrace{\sum_{i=0}^{n-1} |s_{n,i+1} - s_{n,i}|}_{=L} = \frac{C_1}{n}. \end{aligned} \quad (23)$$

Step 3: By Lemma 2.10 there exists a constant c_K only depending on K such that

$$\left| \frac{\lambda_{n,i}}{|s_{n,i+1} - s_{n,i}|} - 1 \right| \leq c_K |s_{n,i+1} - s_{n,i}|^2$$

for all $n \geq N$ and $i = 0, \dots, n-1$, where N depends on the given sequence of biarc curves. Using the fact that $|s_{n,i+1} - s_{n,i}| \leq \frac{L}{2}$ yields

$$\lambda_{n,i} \leq \underbrace{c_K \left(\frac{L}{2} \right)^2}_{=: d_K} |s_{n,i+1} - s_{n,i}| \quad \text{for } n \geq N \text{ and } i = 0, \dots, n-1. \quad (24)$$

Without loss of generalization we can assume that $d_K \geq 1$. Define $C_2 := d_K^2 C_1$. Thus,

$$\begin{aligned} & \sum_{i=0}^{n-1} \sum_{j, 0 < |i-j|_n \leq \Upsilon} \left(2 \frac{\text{dist}(l(\gamma(s_{n,j})), \gamma(s_{n,i}))}{|\gamma(s_{n,i}) - \gamma(s_{n,j})|^2} \right)^q \lambda_{n,i} \lambda_{n,j} \\ & \stackrel{(24)}{\leq} d_K^2 \sum_{i=0}^{n-1} \sum_{j, 0 < |i-j|_n \leq \Upsilon} \left(2 \frac{\text{dist}(l(\gamma(s_{n,j})), \gamma(s_{n,i}))}{|\gamma(s_{n,i}) - \gamma(s_{n,j})|^2} \right)^q |s_{n,i+1} - s_{n,i}| |s_{n,j+1} - s_{n,j}| \\ & \stackrel{(22)}{\leq} d_K^2 (2c_\gamma^2 K)^q \sum_{i=0}^{n-1} \sum_{j, 0 < |i-j|_n \leq \Upsilon} |s_{n,i+1} - s_{n,i}| |s_{n,j+1} - s_{n,j}| \stackrel{(23)}{\leq} \frac{C_2}{n}, \end{aligned} \quad (25)$$

which deals with the second term on the right-hand side of (19).

Step 4: We assume from now on that $|i-j|_n > \Upsilon$. The sequence $(\mathcal{M}_n)_n$ is assumed to be $(c_1 - c_2)$ -distributed, so that in view of (17)

$$|s_{n,k+1} - s_{n,k}| = |s_{n,k+1} - s_{n,k}|_{\mathbb{R}/L\mathbb{Z}} \quad \text{for any } n \in \mathbb{N} \text{ and } k = 0, \dots, n-1.$$

For $s \in [s_{n,i}, s_{n,i+1})$ and $t \in [s_{n,j}, s_{n,j+1})$ with $i \neq j$ we use $|s - t|_{\mathbb{R}/L\mathbb{Z}} \leq |s - s_{n,i}|_{\mathbb{R}/L\mathbb{Z}} + |s_{n,i} - s_{n,j}|_{\mathbb{R}/L\mathbb{Z}} + |s_{n,j} - t|_{\mathbb{R}/L\mathbb{Z}}$ to infer the inequality

$$\begin{aligned} |t - s|_{\mathbb{R}/L\mathbb{Z}} & \leq |s_{n,i} - s_{n,j}|_{\mathbb{R}/L\mathbb{Z}} + 2 \max_{k=0, \dots, n-1} |s_{n,k+1} - s_{n,k}|_{\mathbb{R}/L\mathbb{Z}} \\ & \stackrel{(17)}{\leq} |s_{n,i} - s_{n,j}|_{\mathbb{R}/L\mathbb{Z}} + 2 \frac{c_2}{c_1} \underbrace{\min_{k=0, \dots, n-1} |s_{n,k+1} - s_{n,k}|_{\mathbb{R}/L\mathbb{Z}}}_{\leq |s_{n,i} - s_{n,j}|_{\mathbb{R}/L\mathbb{Z}}} \\ & \leq \left(1 + 2 \frac{c_2}{c_1} \right) |s_{n,i} - s_{n,j}|_{\mathbb{R}/L\mathbb{Z}}. \end{aligned}$$

From $|i-j|_n > \Upsilon = 4 \frac{c_2}{c_1}$ we have in particular

$$2 \frac{c_2}{c_1} < \frac{1}{2} |i-j|_n. \quad (26)$$

Then, similarly as before,

$$|t - s|_{\mathbb{R}/L\mathbb{Z}} \geq |s_{n,i} - s_{n,j}|_{\mathbb{R}/L\mathbb{Z}} - 2 \max_{k=0, \dots, n-1} |s_{n,k+1} - s_{n,k}|_{\mathbb{R}/L\mathbb{Z}}$$

$$\begin{aligned}
&\stackrel{(17)}{\geq} |i-j|_n \min_{k=0,\dots,n-1} |s_{n,k+1} - s_{n,k}|_{\mathbb{R}/L\mathbb{Z}} - \frac{2c_2}{c_1} \min_{k=0,\dots,n-1} |s_{n,k+1} - s_{n,k}|_{\mathbb{R}/L\mathbb{Z}} \\
&= \left(|i-j|_n - \frac{2c_2}{c_1} \right) \min_{k=0,\dots,n-1} |s_{n,k+1} - s_{n,k}|_{\mathbb{R}/L\mathbb{Z}} \\
&\stackrel{(26)}{>} \frac{1}{2} |i-j|_n \min_{k=0,\dots,n-1} |s_{n,k+1} - s_{n,k}|_{\mathbb{R}/L\mathbb{Z}} \geq \frac{c_1}{2c_2} |s_{n,i} - s_{n,j}|_{\mathbb{R}/L\mathbb{Z}}.
\end{aligned}$$

In total, we conclude for $|i-j|_n > \Upsilon$

$$\frac{c_1}{2c_2} |s_{n,i} - s_{n,j}|_{\mathbb{R}/L\mathbb{Z}} \leq |t-s|_{\mathbb{R}/L\mathbb{Z}} \leq \left(1 + 2\frac{c_2}{c_1} \right) |s_{n,i} - s_{n,j}|_{\mathbb{R}/L\mathbb{Z}} \quad (27)$$

for $s \in [s_{n,i}, s_{n,i+1})$ and $t \in [s_{n,j}, s_{n,j+1})$, which we consider also in Steps 5 and 6.

Step 5: In order to estimate $A_{i,j}$, we initially estimate for arbitrary $a, b \geq 0$

$$|b^q - a^q| = \left| \int_a^b \frac{d}{dx} x^q dx \right| = \left| \int_a^b q x^{q-1} dx \right| \leq q|b-a| \max\{a, b\}^{q-1}, \quad (28)$$

since the function $f: [0, \infty) \rightarrow [0, \infty), x \rightarrow x^{q-1}$ is nondecreasing for $q \geq 2$. We abbreviate $d(\cdot, \cdot) := \text{dist}(l(\gamma(\cdot)), \gamma(\cdot))$ and use estimate (28) to find for $s \in [s_{n,i}, s_{n,i+1})$ and $t \in [s_{n,j}, s_{n,j+1})$

$$|d^q(t, s) - d^q(s_{n,j}, s_{n,i})| \leq q|d(t, s) - d(s_{n,j}, s_{n,i})| (\max\{d(t, s), d(s_{n,j}, s_{n,i})\})^{q-1}. \quad (29)$$

Furthermore, combining (21) with (27) yields

$$\begin{aligned}
d(t, s) &\stackrel{(21)}{\leq} K|t-s|_{\mathbb{R}/L\mathbb{Z}}^2 \stackrel{(27)}{\leq} K \left(1 + 2\frac{c_2}{c_1} \right)^2 |s_{n,i} - s_{n,j}|_{\mathbb{R}/L\mathbb{Z}}^2, \\
d(s_{n,j}, s_{n,i}) &\leq K|s_{n,i} - s_{n,j}|_{\mathbb{R}/L\mathbb{Z}}^2 \leq K \left(1 + 2\frac{c_2}{c_1} \right)^2 |s_{n,i} - s_{n,j}|_{\mathbb{R}/L\mathbb{Z}}^2.
\end{aligned} \quad (30)$$

Hence,

$$(\max\{d(t, s), d(s_{n,j}, s_{n,i})\})^{q-1} \leq K^{q-1} \left(1 + 2\frac{c_2}{c_1} \right)^{2q-2} |s_{n,i} - s_{n,j}|_{\mathbb{R}/L\mathbb{Z}}^{2q-2}. \quad (31)$$

Moreover, we estimate again by virtue of (27) now for $s := s_{n,i}$

$$|t - s_{n,i}|_{\mathbb{R}/L\mathbb{Z}} \stackrel{(27)}{\leq} \left(1 + 2\frac{c_2}{c_1} \right) |s_{n,i} - s_{n,j}|_{\mathbb{R}/L\mathbb{Z}},$$

and we use (17) to find for $t \in [s_{n,j}, s_{n,j+1})$

$$\begin{aligned}
|t - s_{n,j}|_{\mathbb{R}/L\mathbb{Z}} &\leq \max_{k=0,\dots,n-1} |s_{n,k+1} - s_{n,k}|_{\mathbb{R}/L\mathbb{Z}} \\
&\stackrel{(17)}{\leq} \frac{c_2}{c_1} \min_{k=0,\dots,n-1} |s_{n,k+1} - s_{n,k}|_{\mathbb{R}/L\mathbb{Z}} \leq \left(1 + 2\frac{c_2}{c_1} \right) |s_{n,i} - s_{n,j}|_{\mathbb{R}/L\mathbb{Z}}.
\end{aligned}$$

Combining these last two estimates with (27) leads to

$$\begin{aligned}
&\max\{|t-s|_{\mathbb{R}/L\mathbb{Z}}, |t-s_{n,i}|_{\mathbb{R}/L\mathbb{Z}}\} + |t-s_{n,j}|_{\mathbb{R}/L\mathbb{Z}} + |t-s_{n,i}|_{\mathbb{R}/L\mathbb{Z}} \\
&\leq 3 \left(1 + 2\frac{c_2}{c_1} \right) |s_{n,i} - s_{n,j}|_{\mathbb{R}/L\mathbb{Z}} \quad \text{for } s \in [s_{n,i}, s_{n,i+1}), t \in [s_{n,j}, s_{n,j+1}).
\end{aligned} \quad (32)$$

For arbitrary $\tau \in \mathbb{R}$ the mapping $P_{\gamma'(\tau)} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined as

$$P_{\gamma'(\tau)}(v) := \langle v, \gamma'(\tau) \rangle \gamma'(\tau), \quad \text{for } v \in \mathbb{R}^3 \quad (33)$$

is the orthogonal projection onto the subspace $\mathbb{R}\gamma'(\tau)$ since $|\gamma'| = 1$, and we have

$$|P_{\gamma'(\tau)}(v) - v| \leq |w - v| \quad \text{for all } w \in \mathbb{R}\gamma'(\tau), v \in \mathbb{R}^3. \quad (34)$$

Moreover, we have for any $\tau, \sigma \in \mathbb{R}$

$$d(\tau, \sigma) = \text{dist}(l(\gamma(\tau)), \gamma(\sigma)) = |P_{\gamma'(\tau)}(\gamma(\sigma) - \gamma(\tau)) - (\gamma(\sigma) - \gamma(\tau))|. \quad (35)$$

Furthermore, we calculate for $s \in [s_{n,i}, s_{n,i+1})$ and $t \in [s_{n,j}, s_{n,j+1})$ using the linearity of the projection

$$\begin{aligned} & P_{\gamma'(t)}(\gamma(s) - \gamma(t)) - P_{\gamma'(s_{n,j})}(\gamma(s_{n,i}) - \gamma(s_{n,j})) \\ &= P_{\gamma'(t)}(\gamma(s) - \gamma(s_{n,i})) + P_{\gamma'(t)}(\gamma(s_{n,i}) - \gamma(t)) \\ &\quad - P_{\gamma'(s_{n,j})}(\gamma(t) - \gamma(s_{n,j})) - P_{\gamma'(s_{n,j})}(\gamma(s_{n,i}) - \gamma(t)) \\ &\stackrel{(33)}{=} P_{\gamma'(t)}(\gamma(s) - \gamma(s_{n,i})) - P_{\gamma'(s_{n,j})}(\gamma(t) - \gamma(s_{n,j})) \\ &\quad + \langle \gamma(s_{n,i}) - \gamma(t), \gamma'(t) - \gamma'(s_{n,j}) \rangle \gamma'(t) \\ &\quad + \langle \gamma(s_{n,i}) - \gamma(t), \gamma'(s_{n,j}) \rangle (\gamma'(t) - \gamma'(s_{n,j})). \end{aligned} \quad (36)$$

In conclusion, by (35) and the elementary inequality $\|a\| - \|b\| \leq \|a - b\|$, this yields for the expression $|d(t, s) - d(s_{n,j}, s_{n,i})|$ (for $s \in [s_{n,i}, s_{n,i+1})$ and $t \in [s_{n,j}, s_{n,j+1})$) the upper bound

$$|P_{\gamma'(t)}(\gamma(s) - \gamma(t)) - P_{\gamma'(s_{n,j})}(\gamma(s_{n,i}) - \gamma(s_{n,j})) - (\gamma(s) - \gamma(s_{n,i})) + (\gamma(t) - \gamma(s_{n,j}))|,$$

which in turn by means of (36) and (34) can be bounded from above by

$$\begin{aligned} & |P_{\gamma'(t)}(\gamma(s) - \gamma(s_{n,i})) - (\gamma(s) - \gamma(s_{n,i}))| \\ &+ |P_{\gamma'(s_{n,j})}(\gamma(t) - \gamma(s_{n,j})) - (\gamma(t) - \gamma(s_{n,j}))| \\ &+ |\langle \gamma(s_{n,i}) - \gamma(t), \gamma'(t) - \gamma'(s_{n,j}) \rangle \gamma'(t)| \\ &+ |\langle \gamma(s_{n,i}) - \gamma(t), \gamma'(s_{n,j}) \rangle (\gamma'(t) - \gamma'(s_{n,j}))| \\ &\stackrel{(34)}{\leq} |(\gamma(s) - \gamma(s_{n,i})) - (s - s_{n,i})\gamma'(t)| + |(\gamma(t) - \gamma(s_{n,j})) - (t - s_{n,j})\gamma'(s_{n,j})| \\ &\quad + 2|\gamma(s_{n,i}) - \gamma(t)| |\gamma'(s_{n,j}) - \gamma'(t)|. \end{aligned}$$

The last summand is bounded by $2K|t - s_{n,i}|_{\mathbb{R}/\mathbb{L}\mathbb{Z}}|t - s_{n,j}|_{\mathbb{R}/\mathbb{L}\mathbb{Z}}$ since $K = \|\gamma''\|_{L^\infty}$ and 1 are the Lipschitz constants of γ' and γ , respectively. The first summand on the right-hand side of the above equals $|\int_{s_{n,i}}^s \int_t^u \gamma''(v) dv du|$, whereas the second is bounded by $\int_{s_{n,j}}^t \int_{s_{n,j}}^u |\gamma''(v)| dv du$, so that we can summarize the estimate

$$|d(t, s) - d(s_{n,j}, s_{n,i})|$$

$$\begin{aligned}
&\leq K|s_{n,i} - s|_{\mathbb{R}/L\mathbb{Z}} \max\{|s - t|_{\mathbb{R}/L\mathbb{Z}}, |t - s_{n,i}|_{\mathbb{R}/L\mathbb{Z}}\} \\
&\quad + K|t - s_{n,j}|_{\mathbb{R}/L\mathbb{Z}}^2 + 2K|t - s_{n,i}|_{\mathbb{R}/L\mathbb{Z}}|t - s_{n,j}|_{\mathbb{R}/L\mathbb{Z}} \\
&\leq 2K \max_{k=0,\dots,n-1} |s_{n,k+1} - s_{n,k}| \\
&\quad \times [\max\{|s - t|_{\mathbb{R}/L\mathbb{Z}}, |t - s_{n,i}|_{\mathbb{R}/L\mathbb{Z}}\} + |t - s_{n,j}|_{\mathbb{R}/L\mathbb{Z}} + |t - s_{n,i}|_{\mathbb{R}/L\mathbb{Z}}] \\
&\stackrel{(32)}{\leq} 6K \left(1 + 2\frac{c_2}{c_1}\right) |s_{n,i} - s_{n,j}|_{\mathbb{R}/L\mathbb{Z}} \max_{k=0,\dots,n-1} |s_{n,k+1} - s_{n,k}|. \tag{37}
\end{aligned}$$

Inserting (31) and (37) into (29) yields

$$\begin{aligned}
&|d^q(t, s) - d^q(s_{n,j}, s_{n,i})| \\
&\leq 6qK^q \left(1 + 2\frac{c_2}{c_1}\right)^{2q-1} |s_{n,i} - s_{n,j}|_{\mathbb{R}/L\mathbb{Z}}^{2q-1} \max_{k=0,\dots,n-1} |s_{n,k+1} - s_{n,k}|. \tag{38}
\end{aligned}$$

In order to obtain an estimate for the denominator of $A_{i,j}$ we consider

$$|\gamma(s) - \gamma(t)|^{2q} \stackrel{(20)}{\geq} \frac{1}{c_\gamma^{2q}} |t - s|_{\mathbb{R}/L\mathbb{Z}}^{2q} \stackrel{(27)}{\geq} \left(\frac{c_1}{2c_2c_\gamma}\right)^{2q} |s_{n,i} - s_{n,j}|_{\mathbb{R}/L\mathbb{Z}}^{2q}. \tag{39}$$

Setting $C_A := \frac{c_2^3}{c_1^3} 6qK^q (1 + 2\frac{c_2}{c_1})^{2q-1} (\frac{2c_2c_\gamma}{c_1})^{2q}$ we obtain from (38) and (39)

$$\begin{aligned}
|A_{i,j}| &\stackrel{(38),(39)}{\leq} \frac{c_1}{(c_2)^3} C_A \int_{s_{n,j}}^{s_{n,i+1}} \int_{s_{n,i}}^{s_{n,i+1}} \frac{|s_{n,i} - s_{n,j}|_{\mathbb{R}/L\mathbb{Z}}^{2q-1} \max_{k=0,\dots,n-1} |s_{n,k+1} - s_{n,k}|}{|s_{n,i} - s_{n,j}|_{\mathbb{R}/L\mathbb{Z}}^{2q}} \, ds \, dt \\
&\leq \frac{c_1}{(c_2)^3} C_A \left(\max_{k=0,\dots,n-1} |s_{n,k+1} - s_{n,k}|\right)^3 \frac{1}{|s_{n,i} - s_{n,j}|_{\mathbb{R}/L\mathbb{Z}}} \\
&\stackrel{(17)}{\leq} c_1 C_A \frac{1}{n^3} \frac{1}{|i - j|_n \min_{k=0,\dots,n-1} |s_{n,k+1} - s_{n,k}|} \stackrel{(17)}{\leq} C_A \frac{1}{n^2} \frac{1}{|i - j|_n}. \tag{40}
\end{aligned}$$

Step 6: To estimate $B_{i,j}$, we use (28) and twice (27) leading to

$$\begin{aligned}
&||\gamma(s_{n,i}) - \gamma(s_{n,j})|^{2q} - |\gamma(s) - \gamma(t)|^{2q}| \\
&= ||\gamma(s_{n,i}) - \gamma(s_{n,j})|^q + |\gamma(s) - \gamma(t)|^q| \\
&\quad \times ||\gamma(s_{n,i}) - \gamma(s_{n,j})|^q - |\gamma(s) - \gamma(t)|^q| \\
&\stackrel{(28)}{\leq} q(|s_{n,i} - s_{n,j}|_{\mathbb{R}/L\mathbb{Z}}^q + |t - s|_{\mathbb{R}/L\mathbb{Z}}^q) ||\gamma(s_{n,i}) - \gamma(s_{n,j})| - |\gamma(s) - \gamma(t)|| \\
&\quad \times \max\{|\gamma(s) - \gamma(t)|, |\gamma(s_{n,i}) - \gamma(s_{n,j})|\}^{q-1} \\
&\stackrel{(27)}{\leq} 2q \left(1 + 2\frac{c_2}{c_1}\right)^q |s_{n,i} - s_{n,j}|_{\mathbb{R}/L\mathbb{Z}}^q |\gamma(s_{n,i}) - \gamma(s) + \gamma(t) - \gamma(s_{n,j})| \\
&\quad \times \max\{|t - s|_{\mathbb{R}/L\mathbb{Z}}, |s_{n,i} - s_{n,j}|_{\mathbb{R}/L\mathbb{Z}}\}^{q-1} \\
&\stackrel{(27)}{\leq} 2q \left(1 + 2\frac{c_2}{c_1}\right)^{2q-1} |s_{n,i} - s_{n,j}|_{\mathbb{R}/L\mathbb{Z}}^{2q-1} (|\gamma(s) - \gamma(s_{n,i})| + |\gamma(t) - \gamma(s_{n,j})|) \\
&\leq 2q \left(1 + 2\frac{c_2}{c_1}\right)^{2q-1} |s_{n,i} - s_{n,j}|_{\mathbb{R}/L\mathbb{Z}}^{2q-1} (|s - s_{n,i}|_{\mathbb{R}/L\mathbb{Z}} + |t - s_{n,j}|_{\mathbb{R}/L\mathbb{Z}})
\end{aligned}$$

$$\leq 4q \left(1 + 2 \frac{c_2}{c_1}\right)^{2q-1} |s_{n,i} - s_{n,j}|_{\mathbb{R}/L\mathbb{Z}}^{2q-1} \max_{k=0,\dots,n-1} |s_{n,k+1} - s_{n,k}|. \quad (41)$$

Thus, by (30), (41), and (39),

$$\begin{aligned} |B_{i,j}| &\stackrel{(30)}{\leq} K^q \left(1 + 2 \frac{c_2}{c_1}\right)^{2q} |s_{n,i} - s_{n,j}|_{\mathbb{R}/L\mathbb{Z}}^{2q} \\ &\quad \times \int_{s_{n,j}}^{s_{n,j+1}} \int_{s_{n,i}}^{s_{n,i+1}} \left| \frac{|\gamma(s_{n,i}) - \gamma(s_{n,j})|^{2q} - |\gamma(s) - \gamma(t)|^{2q}}{|\gamma(s_{n,i}) - \gamma(s_{n,j})|^{2q} |\gamma(s) - \gamma(t)|^{2q}} \right| ds dt \\ &\stackrel{(41)}{\leq} 4q K^q \left(1 + 2 \frac{c_2}{c_1}\right)^{4q-1} |s_{n,i} - s_{n,j}|_{\mathbb{R}/L\mathbb{Z}}^{4q-1} \max_{k=0,\dots,n-1} |s_{n,k+1} - s_{n,k}| \\ &\quad \times \int_{s_{n,j}}^{s_{n,j+1}} \int_{s_{n,i}}^{s_{n,i+1}} \frac{1}{|\gamma(s_{n,i}) - \gamma(s_{n,j})|^{2q} |\gamma(s) - \gamma(t)|^{2q}} ds dt \\ &\stackrel{(39)}{\leq} \left(\frac{2c_2 c_\gamma}{c_1}\right)^{2q} c_\gamma^{2q} 4q K^q \left(1 + 2 \frac{c_2}{c_1}\right)^{4q-1} |s_{n,i} - s_{n,j}|_{\mathbb{R}/L\mathbb{Z}}^{4q-1} \max_{k=0,\dots,n-1} |s_{n,k+1} - s_{n,k}| \\ &\quad \times \int_{s_{n,j}}^{s_{n,j+1}} \int_{s_{n,i}}^{s_{n,i+1}} \frac{1}{|s_{n,i} - s_{n,j}|_{\mathbb{R}/L\mathbb{Z}}^{4q}} ds dt \\ &\leq \left(\frac{2c_2 c_\gamma}{c_1}\right)^{2q} c_\gamma^{2q} 4q K^q \left(1 + 2 \frac{c_2}{c_1}\right)^{4q-1} \left(\max_{k=0,\dots,n-1} |s_{n,k+1} - s_{n,k}|\right)^3 \frac{1}{|s_{n,i} - s_{n,j}|_{\mathbb{R}/L\mathbb{Z}}} \\ &\stackrel{(17)}{\leq} (c_2)^3 \left(\frac{2c_2 c_\gamma}{c_1}\right)^{2q} c_\gamma^{2q} 4q K^q \left(1 + 2 \frac{c_2}{c_1}\right)^{4q-1} \frac{1}{n^3} \frac{1}{|i-j|_n \min_{k=0,\dots,n-1} |s_{n,k+1} - s_{n,k}|} \\ &\stackrel{(17)}{\leq} \frac{C_B}{n^2} \frac{1}{|i-j|_n} \end{aligned} \quad (42)$$

with $C_B := \frac{(c_2)^3}{c_1} \left(\frac{2c_2 c_\gamma}{c_1}\right)^{2q} c_\gamma^{2q} 4q K^q \left(1 + 2 \frac{c_2}{c_1}\right)^{4q-1}$.

Step 7: The expression $\sum_{k=1}^n \frac{1}{k} - \ln(n)$ converges for $n \rightarrow \infty$ to the Euler–Mascheroni constant; see [18, p. xix]. Thus, there exists a constant $c_l \in (0, \infty)$ such that $|\sum_{k=1}^n \frac{1}{k} - \ln(n)| \leq c_l$ for all $n \in \mathbb{N}$. This leads for $n \geq 1$ to

$$\begin{aligned} &\sum_{i=0}^{n-1} \sum_{j, |i-j|_n > \Upsilon} (|A_{i,j}| + |B_{i,j}|) \\ &\stackrel{(40), (42)}{\leq} \frac{2 \max\{C_A, C_B\}}{n^2} \sum_{i=0}^{n-1} \sum_{j, |i-j|_n > \Upsilon} \frac{1}{|i-j|_n} \\ &\leq \frac{4 \max\{C_A, C_B\}}{n^2} \sum_{i=0}^{n-1} \sum_{k=1}^n \frac{1}{k} \\ &= \frac{4 \max\{C_A, C_B\}}{n} \left(\sum_{k=1}^n \frac{1}{k} - \ln(n)\right) + 4 \max\{C_A, C_B\} \frac{\ln(n)}{n} \leq \frac{C_{AB} \ln(n)}{n} \end{aligned} \quad (43)$$

with $C_{AB} := 8 \max\{c_l, 1\} \max\{C_A, C_B\}$.

Step 8: Recall from Step 3 that

$$|\lambda_{n,j} - |s_{n,j+1} - s_{n,j}|| \leq c_K |s_{n,j+1} - s_{n,j}|^3 \leq c_K \left(\max_{k=0,\dots,n-1} |s_{n,k+1} - s_{n,k}|\right)^3 \quad (44)$$

holds for all $n \geq N$ and $j = 0, \dots, n-1$, where N depends on the sequence $(\beta_n)_{n \in \mathbb{N}}$. From (22) and (24) we obtain from (44)

$$\begin{aligned}
 |C_{ij}| &\leq \frac{\text{dist}(l(\gamma(s_{nj})), \gamma(s_{ni}))^q}{|\gamma(s_{ni}) - \gamma(s_{nj})|^{2q}} |s_{n,i+1} - s_{ni}| |s_{n,j+1} - s_{nj}| - \lambda_{ni} \lambda_{nj}| \\
 &\stackrel{(22)}{\leq} (c_\gamma^2 K)^q [|s_{n,i+1} - s_{ni}| |s_{n,j+1} - s_{nj}| - \lambda_{nj}| \\
 &\quad + \lambda_{nj} |s_{n,i+1} - s_{ni}| - \lambda_{ni}|] \\
 &\stackrel{(24)}{\leq} (c_\gamma^2 K)^q d_K [|s_{n,i+1} - s_{ni}| |s_{n,j+1} - s_{nj}| - \lambda_{nj}| \\
 &\quad + |s_{n,j+1} - s_{nj}| |s_{n,i+1} - s_{ni}| - \lambda_{ni}|] \\
 &\stackrel{(44)}{\leq} (c_\gamma^2 K)^q d_K c_K \left(\max_{k=0, \dots, n-1} |s_{n,k+1} - s_{n,k}| \right)^3 \\
 &\quad \times [|s_{n,i+1} - s_{ni}| + |s_{n,j+1} - s_{nj}|] \\
 &\leq 2(c_\gamma^2 K)^q d_K c_K \left(\max_{k=0, \dots, n-1} |s_{n,k+1} - s_{n,k}| \right)^4 \stackrel{(17)}{\leq} \frac{C_C}{n^4}, \tag{45}
 \end{aligned}$$

with $C_C := 2(c_2)^4 (c_\gamma^2 K)^q d_K c_K$ for all $n \geq N$. We then conclude that

$$\sum_{i=0}^{n-1} \sum_{j, |i-j|_n > \Upsilon} |C_{ij}| \stackrel{(45)}{\leq} \frac{C_C}{n^4} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} 1 = \frac{C_C}{n^2}. \tag{46}$$

Step 9: Inserting (23), (25), (43), and (46) into (19) yields

$$|\text{TP}_q(\gamma) - \tilde{\mathcal{E}}_q^n(\beta_n)| \leq \frac{C_1}{n} + \frac{C_2}{n} + 2^q \left(\frac{C_{AB} \ln(n)}{n} + \frac{C_C}{n^2} \right) \leq \frac{C \ln(n)}{n} \quad \text{for } n \geq N,$$

with $C := 4 \max\{C_1, C_2, 2^q C_{AB}, 2^q C_C\}$, which gives the desired result. \square

4 Γ -convergence to the continuous tangent-point energy

In the present section, we show that the continuous tangent-point energy TP_q is the Γ -limit of the discrete tangent-point energies \mathcal{E}_q^n as $n \rightarrow \infty$ (see Theorem 1.1). As a consequence, we deduce that the limits of discrete almost minimizers are minimizers of the continuous tangent-point energy; see Corollary 1.2.

4.1 Γ -convergence

In order to prove Theorem 1.1 we need to verify the liminf and limsup inequalities, see [8, Definition 1.5]. Here, the liminf inequality is verified in a rather straightforward manner (Theorem 4.1), whereas the proof of the limsup inequality requires more work; see Theorem 4.4 below. Similarly as the notation C_{ia} used before, we equip a function space \mathcal{S} with the index a if we take arclength-parametrized curves in that space.

Theorem 4.1 (Liminf inequality) *Let $\gamma, \gamma_n \in C_a^1(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^3)$ with $\gamma_n \xrightarrow{C^1} \gamma$ as $n \rightarrow \infty$. Then, $\text{TP}_q(\gamma) \leq \liminf_{n \rightarrow \infty} \mathcal{E}_q^n(\gamma_n)$.*

Proof We may assume that $\liminf_{n \rightarrow \infty} \mathcal{E}_q^n(\gamma_n) < \infty$. Then, there exists a subsequence $(\gamma_{n_k})_{k \in \mathbb{N}}$ satisfying $\liminf_{n \rightarrow \infty} \mathcal{E}_q^n(\gamma_n) = \lim_{k \rightarrow \infty} \mathcal{E}_q^{n_k}(\gamma_{n_k}) < \infty$. By definition of $\mathcal{E}_q^{n_k}$ we deduce $\gamma_{n_k} \in \mathcal{B}_{n_k}$ for all $k \in \mathbb{N}$; see (8) in the introduction. Denote the point-tangent pairs that are interpolated by γ_{n_k} as $(([q_{n_k,i}, t_{n_k,i}], [q_{n_k,i+1}, t_{n_k,i+1}]))_{i=0, \dots, n_k-1}$, with $q_{n_k,0} = q_{n_k,n_k}$ and $t_{n_k,0} = t_{n_k,n_k}$ for each $k \in \mathbb{N}$. Furthermore, we denote by $a_{n_k,0}, \dots, a_{n_k,n_k}$ the arclength parameters satisfying $\gamma_{n_k}(a_{n_k,i}) = q_{n_k,i}$ and $|a_{n_k,i+1} - a_{n_k,i}| = \lambda_{n_k,i}$ for all $i = 0, \dots, n_k - 1$. Define for all $s, t \in \mathbb{R}/L\mathbb{Z}$ with $s \neq t$ the function

$$f_{n_k}(s, t) := \sum_{i=0}^{n_k-1} \sum_{j=0, j \neq i}^{n_k-1} \left(2 \frac{\text{dist}(l(\gamma_{n_k}(a_{n_k,j})), \gamma_{n_k}(a_{n_k,i}))}{|\gamma_{n_k}(a_{n_k,i}) - \gamma_{n_k}(a_{n_k,j})|^2} \right)^q \chi_{[a_{n_k,i}, a_{n_k,i+1}) \times [a_{n_k,j}, a_{n_k,j+1})}(s, t),$$

where χ_A denotes the characteristic function of a set $A \subset \mathbb{R}/L\mathbb{Z} \times \mathbb{R}/L\mathbb{Z}$. Easy calculations show that

$$\lim_{k \rightarrow \infty} f_{n_k}(s, t) = \left(\frac{2 \text{dist}(l(\gamma(t)), \gamma(s))}{|\gamma(s) - \gamma(t)|^2} \right)^q \quad \text{for any } s \neq t.$$

The functions f_{n_k} are nonnegative and measurable since they are piecewise constant. We can rewrite the discrete tangent-point energies as $\mathcal{E}_q^n(\gamma_{n_k}) = \int_{\mathbb{R}/L\mathbb{Z}} \int_{\mathbb{R}/L\mathbb{Z}} f_{n_k}(s, t) \, ds \, dt$, which allows us to apply Fatou's lemma to obtain the desired liminf inequality. \square

An important first ingredient in the proof of the limsup inequality is the use of convolutions

$$\gamma_\varepsilon(x) := (\gamma * \eta_\varepsilon)(x) = \int_{\mathbb{R}} \gamma(x - y) \eta_\varepsilon(y) \, dy \quad \text{for } x \in \mathbb{R}/L\mathbb{Z} \quad (47)$$

that approximate γ in the C^1 -norm. Here, $\eta \in C^\infty(\mathbb{R})$ is a nonnegative mollifier with $\text{supp } \eta \subset [-1, 1]$ and $\int_{\mathbb{R}} \eta(x) \, dx = 1$, and for any $\varepsilon > 0$ we set $\eta_\varepsilon(x) := \frac{1}{\varepsilon} \eta(\frac{x}{\varepsilon})$.

In general, the convolutions are not parametrized by arclength even if γ is, and they do not need to have the same length as γ . Thus, we rescale the convolutions to have the same length as γ and reparametrize then according to arclength. The following theorem extends [6, Theorem 1.3] to the case $\rho \geq \frac{1}{s}$. A proof can be found in Appendix A.

Theorem 4.2 *Let $s \in (0, 1)$, $\rho \in [\frac{1}{s}, \infty)$, and $\gamma \in W_a^{1+s, \rho}(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^3)$. For $\varepsilon > 0$ denote by $\tilde{\gamma}_\varepsilon$ be the arclength parametrization of the rescaled convolutions $\frac{\mathcal{L}(\gamma)}{\mathcal{L}(\gamma_\varepsilon)} \gamma_\varepsilon$ with $\tilde{\gamma}_\varepsilon(0) = \frac{\mathcal{L}(\gamma)}{\mathcal{L}(\gamma_\varepsilon)} \gamma_\varepsilon(0)$, where $\mathcal{L}(\gamma)$ is the length of γ and $\mathcal{L}(\gamma_\varepsilon)$ is the length of γ_ε . Then, $\tilde{\gamma}_\varepsilon \rightarrow \gamma$ in $W^{1+s, \rho}$ as $\varepsilon \rightarrow 0$.*

The following abstract lemma is a specialization of [19, Lemma 6.1.1] and provides sufficient conditions to transfer the limsup inequality from approximating elements to the limit element. This result applied to smooth convolutions approximating a given C^1 -curve γ will be the second ingredient in the proof of the limsup inequality, Theorem 4.4 below.

Lemma 4.3 (Limsup inequality by approximation) *Let (X, d) be a metric space and $\mathcal{F}_n, \mathcal{F} : X \rightarrow [-\infty, \infty]$. If a sequence $(x^m)_{m \in \mathbb{N}} \subset X$ satisfies*

1. $d(x, x^m) \rightarrow 0$ as $m \rightarrow \infty$ for an element $x \in X$;
2. $\limsup_{m \rightarrow \infty} \mathcal{F}(x^m) \leq \mathcal{F}(x)$;

3. for every $m \in \mathbb{N}$ there exists a sequence $(x_n^m)_{n \in \mathbb{N}}$ with $d(x^m, x_n^m) \rightarrow 0$ as $n \rightarrow \infty$ and $\limsup_{n \rightarrow \infty} \mathcal{F}_n(x_n^m) \leq \mathcal{F}(x^m)$,
 then there exists a sequence $(y_n)_{n \in \mathbb{N}} \subset X$ with

$$d(x, y_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad \text{and} \quad \limsup_{n \rightarrow \infty} \mathcal{F}_n(y_n) \leq \mathcal{F}(x).$$

The proof of the limsup inequality is inspired by Blatt's improvement of Scholtes' Γ -convergence result for the Möbius energy [6, Theorem 4.8].

Theorem 4.4 (Limsup inequality) *For every $\gamma \in C_{\text{ia}}^1(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^3)$ there exists a sequence $(b_n)_{n \in \mathbb{N}} \subset C_{\text{ia}}^1(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^3)$ such that*

$$b_n \xrightarrow{C^1} \gamma \quad \text{and} \quad \limsup_{n \rightarrow \infty} \mathcal{E}_q^n(b_n) \leq \text{TP}_q(\gamma).$$

Proof If $\text{TP}_q(\gamma) = \infty$, choose $b_n = \gamma$ for all $n \in \mathbb{N}$. Then, $b_n \rightarrow \gamma$ in C^1 and the limsup inequality follows trivially. From now on let $\text{TP}_q(\gamma) < \infty$. Thus, we have $\gamma \in W^{2-\frac{1}{q}, q}(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^3)$ by [5, Theorem 1.1]. Moreover, Lemma B.1 yields a $c_\gamma > 0$ such that

$$c_\gamma |\gamma(s) - \gamma(t)| \geq |s - t|_{\mathbb{R}/L\mathbb{Z}} \quad \text{for any } t, s \in \mathbb{R}. \quad (48)$$

We now consider a sequence of suitably rescaled and reparametrized convolutions of γ and prove the limsup inequality for these convolutions. Applying Lemma 4.3 then yields the limsup inequality for γ .

Step 1: For $n \in \mathbb{N}$ define $s_{n,i} := \frac{il}{n}$ for $i = 0, \dots, n$. Then, for all $i = 0, \dots, n-1$ we have $|s_{n,i+1} - s_{n,i}| = h_n = \tilde{h}_n = \frac{L}{n}$, so that the $\mathcal{M}_n := \{s_{n,0}, \dots, s_{n,n}\}$ form a $(c_1 - c_2)$ -distributed sequence of partitions with $c_1 = c_2 = L$ for $n \geq 2$; see Definition 2.8.

Step 2: For $k \in \mathbb{N}$ we set $\varepsilon_k := \frac{1}{k}$. Let γ_{ε_k} be the convolution as in (47) and $\mathcal{L}(\gamma_{\varepsilon_k})$ the length of γ_{ε_k} . We then define $\tilde{\gamma}_k$ as the arclength parametrization of the rescalings $L\gamma_{\varepsilon_k}/\mathcal{L}(\gamma_{\varepsilon_k})$. Thus, $\tilde{\gamma}_k$ has the same length as γ for every $k \in \mathbb{N}$. Furthermore, $\tilde{\gamma}_k$ is on $[0, L)$ injective for k sufficiently large, which follows from the bi-Lipschitz property (48) of γ together with the C^1 -convergence of the convolutions $\gamma_{\varepsilon_k} \rightarrow \gamma$ as $k \rightarrow \infty$. By omitting finitely many indices we may assume that $\tilde{\gamma}_k \in C_{\text{ia}}^\infty(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^3)$ for all $k \in \mathbb{N}$. For every $k \in \mathbb{N}$ there is by Lemma 2.9 some index $N_0(k) \in \mathbb{N}$ such that there exist proper $\tilde{\gamma}_k$ -interpolating balanced biarc curves $\tilde{\beta}_n^k$ parametrized by arclength that interpolate the point-tangent pairs $(([\tilde{\gamma}_k(s_{n,i}), \tilde{\gamma}_k'(s_{n,i})], [\tilde{\gamma}_k(s_{n,i+1}), \tilde{\gamma}_k'(s_{n,i+1})]))_{i=0, \dots, n-1}$ for all $n \geq N_0(k)$, such that the matching points $m_{n,i}^k \in \Sigma_{++}^{n,i,k}$ satisfy (see Definitions 2.7 and 2.5(iv)) $|\tilde{\gamma}_k(s_{n,i}) - m_{n,i}^k| = |\tilde{\gamma}_k(s_{n,i+1}) - m_{n,i}^k|$ for all $n \geq N_0(k)$, $i = 0, \dots, n-1$. Let $L_n^k := \mathcal{L}(\tilde{\beta}_n^k)$ be the length of $\tilde{\beta}_n^k$, and note that Theorem 2.11 implies

$$L_n^k \rightarrow L = \mathcal{L}(\tilde{\gamma}_k) \quad \text{for each } k \in \mathbb{N} \text{ as } n \rightarrow \infty. \quad (49)$$

Step 3: For $k \in \mathbb{N}$, let φ_n^k be Smutny's reparametrization [31, Appendix A] and define $\tilde{B}_n^k := \tilde{\beta}_n^k \circ \varphi_n^k$, so that Theorem 2.12 implies

$$\|\tilde{\gamma}_k - \tilde{B}_n^k\|_{C^1} \rightarrow 0 \quad \text{for each } k \in \mathbb{N} \text{ as } n \rightarrow \infty. \quad (50)$$

Now, define $B_n^k(s) := L(L_n^k)^{-1}\tilde{B}_n^k(s)$ for all $s \in \mathbb{R}$. Then, B_n^k obviously has length L . However, B_n^k is not parametrized by arclength. Nevertheless, by means of (49) we find $\|B_n^k - \tilde{B}_n^k\|_{C^1} = |\frac{L}{L_n^k} - 1| \|\tilde{B}_n^k\|_{C^1} \rightarrow 0$ for each $k \in \mathbb{N}$ as $n \rightarrow \infty$. Consequently, by (50), one has $\|\tilde{\gamma}_k - B_n^k\|_{C^1} \rightarrow 0$ for each $k \in \mathbb{N}$ as $n \rightarrow \infty$, and therefore by means of Lemma B.3,

$$\|\tilde{\gamma}_k - \beta_n^k\|_{C^1} \rightarrow 0 \quad \text{for each } k \in \mathbb{N} \text{ as } n \rightarrow \infty, \quad (51)$$

where β_n^k is the reparametrization of B_n^k by arclength. Again, since $\tilde{\gamma}_k$ is injective for k sufficiently large, this implies that $\beta_n^k \in C_{\text{ia}}^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ for n and k sufficiently large.

Step 4: We now show that $\beta_n^k \in \mathcal{B}_n$ holds if n is sufficiently large, such that the values $\mathcal{E}_q^n(\beta_n^k)$ are finite by definition (8) in the introduction. We need to show that the length $\lambda_{n,i}^k$ of the i th biarc of β_n^k satisfies (7). For that we apply Lemma 2.10 to the length $\tilde{\lambda}_{n,i}^k$ of the i th biarc of $\tilde{\beta}_n^k$. More precisely, we take the limit $n \rightarrow \infty$ in the following inequality that holds for each $k \in \mathbb{N}$, $n \geq N_0(k)$, $i = 0, \dots, n-1$,

$$-\max_{j=0,\dots,n-1} \left| \frac{\tilde{\lambda}_{n,j}^k}{(L/n)} - 1 \right| + 1 \leq \frac{\tilde{\lambda}_{n,i}^k}{(L/n)} \leq \max_{j=0,\dots,n-1} \left| \frac{\tilde{\lambda}_{n,j}^k}{(L/n)} - 1 \right| + 1,$$

to obtain

$$1 \leftarrow \min_{i=0,\dots,n-1} \frac{\tilde{\lambda}_{n,i}^k}{(L/n)} \leq \max_{i=0,\dots,n-1} \frac{\tilde{\lambda}_{n,i}^k}{(L/n)} \rightarrow 1 \quad \text{as } n \rightarrow \infty. \quad (52)$$

Since the image of β_n^k is just the image of $\tilde{\beta}_n^k$ scaled by the factor $L(L_n^k)^{-1}$ we deduce $\lambda_{n,j}^k = L(L_n^k)^{-1}\tilde{\lambda}_{n,j}^k$. Combining this with (49) we find for each $k \in \mathbb{N}$ and index $N_1(k) \geq N_0(k)$ such that

$$\frac{L}{2n} \leq \min_{j=0,\dots,n-1} \lambda_{n,j}^k \leq \max_{j=0,\dots,n-1} \lambda_{n,j}^k \leq \frac{2L}{n} \quad \text{for any } n \geq N_1(k), \quad (53)$$

which is (7) for $\lambda_i := \lambda_i^k$. Thus, $\beta_n^k \in \mathcal{B}_n$ for all $n \geq N_1(k)$.

Step 5: The scaling property (9) and the parameter invariance of the discrete tangent-point energies yields

$$\mathcal{E}_q^n(\beta_n^k) = (L(L_n^k)^{-1})^{2-q} \tilde{\mathcal{E}}_q^n(\tilde{\beta}_n^k) \quad \text{for all } k \in \mathbb{N} \text{ and } n \geq N_1(k),$$

so that we obtain by (49) and Theorem 3.1 applied to $\gamma := \tilde{\gamma}_k$ and $\beta_n := \tilde{\beta}_n^k$

$$|\text{TP}_q(\tilde{\gamma}_k) - \mathcal{E}_q^n(\beta_n^k)| \leq |\text{TP}_q(\tilde{\gamma}_k) - \tilde{\mathcal{E}}_q^n(\tilde{\beta}_n^k)| + |\tilde{\mathcal{E}}_q^n(\tilde{\beta}_n^k)| |1 - (L(L_n^k)^{-1})^{2-q}| \rightarrow 0 \quad (54)$$

for each $k \in \mathbb{N}$ as $n \rightarrow \infty$.

Step 6: In this final step we check the assumptions of Lemma 4.3. The space $C_{\text{ia}}^1(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^3)$ is a metric space with the metric induced by the C^1 -norm. By Morrey–Sobolev embedding (see [21, Theorem A.2] in the setting of periodic functions) there exists a constant $c_E > 0$, such that

$$\|\tilde{\gamma}_k - \gamma\|_{C^1} \leq c_E \|\tilde{\gamma}_k - \gamma\|_{W^{2-\frac{1}{q},q}}.$$

According to Theorem 4.2 applied to $\rho = q$ and $s = 1 - \frac{1}{q}$ for $q > 2$ the right-hand side converges to 0 as $k \rightarrow \infty$. Thus, $\tilde{\gamma}_k$ converges in the C^1 -norm to γ , which verifies condition (i) in Lemma 4.3. Furthermore, [34, (4.2) Satz] implies that TP_q is continuous on $W_{\text{ia}}^{2-\frac{1}{q},q}$ since $q > 2$. Thus, we obtain $\lim_{k \rightarrow \infty} \text{TP}_q(\tilde{\gamma}_k) = \text{TP}_q(\gamma)$, which gives us condition (ii) of Lemma 4.3. Combining (51) with (54) verifies condition (iii) of Lemma 4.3. Hence, Lemma 4.3 yields the limsup inequality for γ . \square

Remark 4.5 For the proof of Theorem 1.3 in Sect. 5 (see in particular Lemma 5.2) it is important to note that the actual recovery sequence for the limsup inequality in the previous proof is a subsequence of the (doubly subscripted) arclength parametrized biarc curves $\beta_n^k \in C_{\text{ia}}^{1,1}(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^3)$ for $k \in \mathbb{N}$ and $n \geq N_1(k)$; see the choice of the abstract recovery sequence towards the end of the proof of [19, Lemma 6.1.1].

Proof of Theorem 1.1 According to [8, Definition 1.5] it suffices to verify two fundamental inequalities. Indeed, the liminf inequality is the content of Theorem 4.1, whereas the limsup inequality is established in Theorem 4.4. \square

4.2 Convergence of discrete almost minimizers

In this subsection, we prove the convergence of discrete almost minimizers of the discrete tangent-point energies in the metric space defined before. The following lemma can be found in [12, Corollary 7.20].

Lemma 4.6 (Convergence of minimizers) *Let (X, d) be a metric space and $\mathcal{F}_n, \mathcal{F} : X \rightarrow [-\infty, \infty]$. Assume that $\mathcal{F}_n \xrightarrow[n \rightarrow \infty]{\Gamma} \mathcal{F}$. Let $|\mathcal{F}_n(z_n) - \inf_X \mathcal{F}_n| \rightarrow 0$ and $z_n \rightarrow z \in X$ as n tends to infinity. Then, $\mathcal{F}(z) = \min_X \mathcal{F}$ and $\lim_{n \rightarrow \infty} \mathcal{F}_n(z_n) = \mathcal{F}(z)$.*

Proof of Corollary 1.2. The proof follows immediately from Lemma 4.6 with the metric space $X = C_{\text{ia}}^{1,1}(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^3) \cap \mathcal{K}$, with the metric induced by the C^1 -norm. Note that the knot class \mathcal{K} is stable under C^1 -convergence; see, e.g., [25]. Since $\text{TP}_q(\gamma) < \infty$ holds, we obtain $\gamma \in W^{2-\frac{1}{q},q}(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^3)$ by [5, Theorem 1.1]. \square

5 Γ convergence to the Ropelength functional

As a first step towards the proof of Theorem 1.3 we show that the continuous tangent-point energies $(\text{TP}_k)^{\frac{1}{k}}$ Γ -converge to the ropelength \mathcal{R} on $C_{\text{ia}}^{1,1}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ equipped with the C^1 -norm as $k \rightarrow \infty$. We follow the proof of [14, Theorem 6.11], where Gilsbach showed Γ -convergence of integral Menger curvatures towards ropelength.

Lemma 5.1 *For any $\gamma \in C_{\text{ia}}^{1,1}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ one has $(\text{TP}_k)^{\frac{1}{k}}(\gamma) \rightarrow \mathcal{R}(\gamma)$ as $k \rightarrow \infty$. Moreover, $(\text{TP}_k)^{\frac{1}{k}} \xrightarrow[k \rightarrow \infty]{\Gamma} \mathcal{R}$ on $(C_{\text{ia}}^{1,1}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3), \|\cdot\|_{C^1})$.*

Proof According to [30, Theorem 1(iii)] one has³ $\mathcal{R}(\gamma) < \infty$ for $\gamma \in C_{\text{ia}}^{1,1}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$. In addition, by [30, Lemma 2]

$$\|r_{\text{tp}}^{-1}(\gamma(\cdot), \gamma(\cdot))\|_{L^\infty(\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z})} = \sup_{s,t \in \mathbb{R}/\mathbb{Z}, s \neq t} r_{\text{tp}}^{-1}(\gamma(s), \gamma(t)) = \frac{1}{\Delta[\gamma]} = \mathcal{R}(\gamma). \quad (55)$$

³Be aware of the notation: In [30] the expression $\mathcal{R}[\cdot]$ was used for thickness $\Delta[\cdot]$, whereas $\mathcal{K}[\cdot]$ in [30] corresponds to $\Delta[\cdot]^{-1}$.

It is well known (see, e.g., [1, E3.4]) that the mapping

$$k \mapsto \|r_{\text{tp}}^{-1}(\gamma(\cdot), \gamma(\cdot))\|_{L^k(\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z})} = (\text{TP}_k)^{\frac{1}{k}}$$

is nondecreasing and satisfies by means of (55)

$$\lim_{k \rightarrow \infty} (\text{TP}_k)^{\frac{1}{k}}(\gamma) = \|r_{\text{tp}}^{-1}(\gamma(\cdot), \gamma(\cdot))\|_{L^\infty(\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z})} \stackrel{(55)}{=} \mathcal{R}(\gamma).$$

Furthermore, the continuous tangent-point energy is lower semicontinuous with respect to the C^1 -norm, see [32, Proof of Corollary 2.3] or [14, Lemma 1.41]. Then, by [8, Remark 1.40(ii)] the pointwise limit of $(\text{TP}_k)^{\frac{1}{k}}$ is also the Γ -limit and we obtain $(\text{TP}_k)^{\frac{1}{k}} \xrightarrow{\Gamma} \mathcal{R}$ as $k \rightarrow \infty$. \square

Lemma 5.2 $(\mathcal{E}_q^n)^{\frac{1}{q}} \xrightarrow[n \rightarrow \infty]{\Gamma} (\text{TP}_q)^{\frac{1}{q}}$ on $(C_{\text{ia}}^{1,1}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3), \|\cdot\|_{C^1})$ for all $q > 2$.

Proof By Theorem 1.1 we have $\mathcal{E}_q^n \xrightarrow{\Gamma} \text{TP}_q$ on $(C_{\text{ia}}^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3), \|\cdot\|_{C^1})$ for any $q > 2$ as $n \rightarrow \infty$. However, in the proof of the limsup inequality in Theorem 4.4 the recovery sequence is a sequence consisting only of biarc curves that are in $C_{\text{ia}}^{1,1}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$; see Remark 4.5. Therefore, we also have $\mathcal{E}_q^n \xrightarrow{\Gamma} \text{TP}_q$ on the space $(C_{\text{ia}}^{1,1}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3), \|\cdot\|_{C^1})$ as $n \rightarrow \infty$. Now, apply [12, Proposition 6.16] to $\mathcal{F}_n := \mathcal{E}_q^n$, $\mathcal{F} := \text{TP}_q$ and the continuous and nondecreasing function $g : (0, \infty) \rightarrow \mathbb{R}, x \mapsto x^{\frac{1}{q}}$ to infer $(\mathcal{E}_q^n)^{\frac{1}{q}} = g \circ \mathcal{F}_n \xrightarrow[n \rightarrow \infty]{\Gamma} g \circ \mathcal{F} = (\text{TP}_q)^{\frac{1}{q}}$. \square

Next, we compare two different discrete tangent-point energies.

Lemma 5.3 Let $n, m, k \in \mathbb{N}$, $2 \leq k \leq m$ and $\gamma \in C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ with length $\mathcal{L}(\gamma)$. Then, $(\mathcal{E}_k^n)^{\frac{1}{k}}(\gamma) \leq \left(\frac{4\mathcal{L}(\gamma)^2 n(n-1)}{n^2}\right)^{\frac{1}{k} - \frac{1}{m}} (\mathcal{E}_m^n)^{\frac{1}{m}}(\gamma)$.

Proof We only have to consider the case that $\gamma \in \mathcal{B}_n$ since otherwise both sides of the inequality are infinite by definition of the discrete energy \mathcal{E}_k^n ; see (8) in the introduction. Denote by $(([q_i, t_i], [q_{i+1}, t_{i+1}]))_{i=0, \dots, n-1}$ the point-tangent pairs that γ interpolates. For $i \neq j$ define $x_{i,j} := \frac{2 \text{dist}([q_i, t_i], [q_j, t_j])}{|q_i - q_j|^2} \geq 0$. Then, we estimate by means of the generalized mean inequality for finite sums, $(\frac{1}{\ell} \sum_{i=1}^{\ell} |a_i|^p)^{\frac{1}{p}} \leq (\frac{1}{\ell} \sum_{i=1}^{\ell} |a_i|^q)^{\frac{1}{q}}$ for $p \leq q$ (here, for $\ell := n(n-1)$, $p := k$, $q := m$), and by (7)

$$\begin{aligned} (\mathcal{E}_k^n)^{\frac{1}{k}}(\gamma) &= \left(\sum_{i=0}^{n-1} \sum_{j=0, j \neq i}^{n-1} x_{i,j}^k \lambda_i \lambda_j \right)^{\frac{1}{k}} \\ &= \left(\sum_{i=0}^{n-1} \sum_{j=0, j \neq i}^{n-1} [x_{i,j} (\lambda_i \lambda_j)^{\frac{1}{k}}]^k \right)^{\frac{1}{k}} \\ &\leq (n(n-1))^{\frac{1}{k} - \frac{1}{m}} \left(\sum_{i=0}^{n-1} \sum_{j=0, j \neq i}^{n-1} x_{i,j}^m \underbrace{(\lambda_i \lambda_j)^{\frac{m}{k}}}_{=(\lambda_i \lambda_j)(\lambda_i \lambda_j)^{\frac{m}{k} - 1}} \right)^{\frac{1}{m}} \end{aligned}$$

$$\leq (n(n-1))^{\frac{1}{k}-\frac{1}{m}} \max_{\substack{i,j=0,\dots,n-1 \\ i \neq j}} \underbrace{(\lambda_i \lambda_j)^{\frac{1}{k}-\frac{1}{m}}}_{\stackrel{(7)}{\leq} \left(\frac{4\mathcal{L}(\gamma)^2}{n^2}\right)^{\frac{1}{k}-\frac{1}{m}}} \underbrace{\left(\sum_{i=0}^{n-1} \sum_{j=0, j \neq i}^{n-1} x_{ij}^m \lambda_i \lambda_j\right)^{\frac{1}{m}}}_{=(\mathcal{E}_m^n)^{\frac{1}{m}}(\gamma)}.$$

□

Proof of Theorem 1.3 It suffices to prove the Γ -convergence for $L = 1$, since then the statement for general L follows from the scaling property and parametrization invariance of the energies involved. Indeed, assume the theorem was proven for $L = 1$. Now, take $L \neq 1$ and let $(\gamma_n)_{n \in \mathbb{N}} \subset C_{\text{ia}}^{1,1}(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^3)$ with $\gamma_n \rightarrow \gamma$ in C^1 as $n \rightarrow \infty$. Denote by $\tilde{\gamma}_n$ the arclength parametrization of $\frac{\gamma_n}{L}$. By Lemma B.3 this implies $\tilde{\gamma}_n \rightarrow \tilde{\gamma}$ in C^1 as $n \rightarrow \infty$, where $\tilde{\gamma}$ is the arclength parametrization of $\frac{\gamma}{L}$. Together with the fact that the ropelength functional is invariant under reparametrization and scaling, the liminf inequality for $L = 1$ yields the liminf equality for general L :

$$\mathcal{R}(\gamma) = \mathcal{R}(\tilde{\gamma}) \leq \liminf_{n \rightarrow \infty} (\mathcal{E}_n^n)^{\frac{1}{n}}(\tilde{\gamma}_n) = \liminf_{n \rightarrow \infty} (\mathcal{E}_n^n)^{\frac{1}{n}}\left(\frac{\gamma_n}{L}\right) \stackrel{(9)}{=} \liminf_{n \rightarrow \infty} L^{\frac{n-2}{n}} (\mathcal{E}_n^n)^{\frac{1}{n}}(\gamma_n).$$

For the limsup inequality let $\gamma \in C_{\text{ia}}^{1,1}(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^3)$. Then, $\tilde{\gamma}(x) := \frac{\gamma(Lx)}{L}$ is the arclength parametrization of γ scaled to unit length. Hence, there exists a recovery sequence $(\tilde{\gamma}_n)_{n \in \mathbb{N}} \subset C_{\text{ia}}^{1,1}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ such that

$$\tilde{\gamma}_n \xrightarrow{C^1} \tilde{\gamma} \quad \text{as } n \rightarrow \infty \quad \text{and} \quad \limsup_{n \rightarrow \infty} (\mathcal{E}_n^n)^{\frac{1}{n}}(\tilde{\gamma}_n) \leq \mathcal{R}(\tilde{\gamma}). \quad (56)$$

Define the reparametrization $\varphi : [0, L] \rightarrow [0, 1], x \mapsto \frac{x}{L}$ and set $\gamma_n(x) := L\tilde{\gamma}_n(\varphi(x))$ and $\hat{\gamma}(x) := L\tilde{\gamma}(\varphi(x))$. Note that γ_n is parametrized by arclength and that $\hat{\gamma} = \gamma$ holds. Then, $\gamma_n \rightarrow \hat{\gamma} = \gamma$ in C^1 for $n \rightarrow \infty$ by (56). Again, by the scaling property of the energies and the invariance under reparametrization we deduce with (56)

$$\limsup_{n \rightarrow \infty} L^{\frac{n-2}{n}} (\mathcal{E}_n^n)^{\frac{1}{n}}(\gamma_n) \stackrel{(9)}{=} \limsup_{n \rightarrow \infty} (\mathcal{E}_n^n)^{\frac{1}{n}}(\tilde{\gamma}_n) \leq \mathcal{R}(\tilde{\gamma}) = \mathcal{R}(\gamma).$$

Hence, it remains to prove the statement of Theorem 1.3 for $L = 1$, and for that we take a general sequence $(\gamma_n)_{n \in \mathbb{N}} \subset C_{\text{ia}}^{1,1}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ with $\gamma_n \rightarrow \gamma$ in C^1 as $n \rightarrow \infty$.

By Lemma 5.2 for $q := k$

$$(\text{TP}_k)^{\frac{1}{k}}(\gamma) \leq \liminf_{n \rightarrow \infty} (\mathcal{E}_k^n)^{\frac{1}{k}}(\gamma_n) = \lim_{n \rightarrow \infty} \inf_{n \geq k} (\mathcal{E}_k^n)^{\frac{1}{k}}(\gamma_n). \quad (57)$$

For $k \leq n$ we apply Lemma 5.3 to $\gamma := \gamma_n$ and $m := n$ to find

$$(\mathcal{E}_k^n)^{\frac{1}{k}}(\gamma_n) \leq \left(\frac{4\mathcal{L}(\gamma_n)^2 n(n-1)}{n^2}\right)^{\frac{1}{k}-\frac{1}{n}} (\mathcal{E}_n^n)^{\frac{1}{n}}(\gamma_n).$$

Together with (57) and $\mathcal{L}(\gamma_n) = 1$ for all $n \in \mathbb{N}$, this yields

$$(\text{TP}_k)^{\frac{1}{k}}(\gamma) \leq \lim_{n \rightarrow \infty} \inf_{n \geq k} \left(\frac{4n(n-1)}{n^2}\right)^{\frac{1}{k}-\frac{1}{n}} (\mathcal{E}_n^n)^{\frac{1}{n}}(\gamma_n). \quad (58)$$

Now, we have

$$\lim_{n \rightarrow \infty} \left(\frac{4n(n-1)}{n^2} \right)^{\frac{1}{k} - \frac{1}{n}} = \lim_{n \rightarrow \infty} \exp \left(\left(\frac{1}{k} - \frac{1}{n} \right) \log \left(\frac{4n(n-1)}{n^2} \right) \right) = \exp \left(\frac{1}{k} \log(4) \right) = 4^{\frac{1}{k}}.$$

Combining this with the pointwise convergence in Lemma 5.1 and (58) we arrive at the desired liminf inequality:

$$\begin{aligned} \mathcal{R}(\gamma) &= \lim_{k \rightarrow \infty} (\text{TP}_k)^{\frac{1}{k}}(\gamma) \stackrel{(58)}{\leq} \lim_{k \rightarrow \infty} \liminf_{n \rightarrow \infty} \inf_{n \geq k} \left(\frac{4n(n-1)}{n^2} \right)^{\frac{1}{k} - \frac{1}{n}} (\mathcal{E}_n^n)^{\frac{1}{n}}(\gamma_n) \\ &= \lim_{k \rightarrow \infty} 4^{\frac{1}{k}} \liminf_{n \rightarrow \infty} (\mathcal{E}_n^n)^{\frac{1}{n}}(\gamma_n) = \liminf_{n \rightarrow \infty} (\mathcal{E}_n^n)^{\frac{1}{n}}(\gamma_n). \end{aligned}$$

To verify the limsup inequality let $\gamma \in C_{\text{ia}}^{1,1}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$, and for $n \in \mathbb{N}$ set $s_{n,i} := \frac{i}{n}$ for $i = 0, \dots, n$. Then, we have $|s_{n,i+1} - s_{n,i}| = \frac{1}{n}$ for all $i = 0, \dots, n-1$, and therefore a sequence of $(c_1 - c_2)$ -distributed partitions with $c_1 = c_2 = 1$; see Definition 2.8. Now, we follow the proof of Theorem 4.4. However, since γ is now a $C^{1,1}$ -curve, we do not have to work with convolutions, but can follow the proof for γ directly. By Lemma 2.9 there exists for n sufficiently large a γ -interpolating, proper, and balanced biarc curve $\tilde{\beta}_n$ interpolating the point-tangent pairs $([\gamma(s_{n,i}), \gamma'(s_{n,i})], [\gamma(s_{n,i+1}), \gamma'(s_{n,i+1})])_{i=0, \dots, n-1}$. Then, we obtain by Theorem 2.11 that $\mathcal{L}(\tilde{\beta}_n) \rightarrow \mathcal{L}(\gamma) = 1$ as $n \rightarrow \infty$. Let φ_n be the reparametrization function from [31, Appendix A] and set $\tilde{B}_n := \tilde{\beta}_n \circ \varphi_n$. Then, by Theorem 2.12, we have $\tilde{B}_n \rightarrow \gamma$ in C^1 for $n \rightarrow \infty$. Setting $B_n := \mathcal{L}(\tilde{\beta}_n)^{-1} \tilde{B}_n$ we obtain as in the proof of Theorem 4.4 that $B_n \rightarrow \gamma$ in C^1 for $n \rightarrow \infty$. Let β_n be the arclength parametrization of B_n . By Lemma B.3 we finally arrive at $\beta_n \rightarrow \gamma$ in C^1 for $n \rightarrow \infty$. The biarc curves β_n are only reparametrized versions of $\tilde{\beta}_n$ rescaled by the factor $\mathcal{L}(\tilde{\beta}_n)^{-1}$, so that we can show exactly as in the proof of Theorem 4.4 that $\beta_n \in \mathcal{B}_n$ for n sufficiently large. Moreover, due to the C^1 -convergence towards γ , the β_n are also injective for n large enough. Since β_n is scaled to unit length and parametrized by arclength, we have $\beta_n \in C_{\text{ia}}^{1,1}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ for n sufficiently large. Set $L_n := \mathcal{L}(\tilde{\beta}_n)$. By the scaling property of the discrete tangent-point energy (9) and its parameter invariance we have

$$(\mathcal{E}_k^n)^{\frac{1}{k}}(\beta_n) = L_n^{1-\frac{2}{k}} (\mathcal{E}_k^n)^{\frac{1}{k}}(\tilde{\beta}_n) \quad \text{for any } k > 2. \quad (59)$$

Abbreviating $x_{i,j} := \frac{2 \text{dist}(\gamma(s_{n,i}) + \mathbb{R}\gamma'(s_{n,i}), \gamma(s_{n,j}))}{|\gamma(s_{n,i}) - \gamma(s_{n,j})|^2}$ for $i, j = 0, \dots, n-1$ with $i \neq j$ we can write and estimate for sufficiently large $n \in \mathbb{N}$

$$\begin{aligned} (\mathcal{E}_k^n)^{\frac{1}{k}}(\beta_n) &\stackrel{(59)}{=} L_n^{1-\frac{2}{k}} (\mathcal{E}_k^n)^{\frac{1}{k}}(\tilde{\beta}_n) = L_n^{1-\frac{2}{k}} \left(\sum_{i=0}^{n-1} \sum_{j=0, j \neq i}^{n-1} x_{i,j}^k \lambda_i \lambda_j \right)^{\frac{1}{k}} \\ &= L_n^{1-\frac{2}{k}} \left(\frac{1}{n(n-1)} \sum_{i=0}^{n-1} \sum_{j=0, j \neq i}^{n-1} n(n-1) x_{i,j}^k \lambda_i \lambda_j \right)^{\frac{1}{k}} \\ &\leq L_n^{1-\frac{2}{k}} \left(\frac{4L_n^2 n(n-1)}{n^2} \right)^{\frac{1}{k}} \left(\frac{1}{n(n-1)} \sum_{i=0}^{n-1} \sum_{j=0, j \neq i}^{n-1} x_{i,j}^k \right)^{\frac{1}{k}}. \end{aligned} \quad (60)$$

Here, we used (7) for $\tilde{\beta}_n \in \mathcal{B}_n$, which can be verified for n sufficiently large exactly as for the $\tilde{\beta}_n^k$ in Step 4 of the proof of Theorem 4.4; see in particular (52). Now, observe that by [1, E3.4] applied to the discrete measure $\mu := \sum_{i=0}^{n-1} \sum_{j=0, j \neq i}^{n-1} \delta_{(s_{n,i}, s_{n,j})}$ with $\mu((\mathbb{R}/L\mathbb{Z})^2) = n(n-1)$ we have

$$\lim_{k \rightarrow \infty} \left(\frac{1}{n(n-1)} \sum_{i=0}^{n-1} \sum_{j=0, j \neq i}^{n-1} x_{i,j}^k \right)^{\frac{1}{k}} = \max_{i,j=0, \dots, n-1, i \neq j} x_{i,j}. \quad (61)$$

With $\lim_{k \rightarrow \infty} L_n^{1-\frac{2}{k}} \left(\frac{4L_n^2 n(n-1)}{n^2} \right)^{\frac{1}{k}} = L_n$, we obtain by means of (60), (61), and [30, Lemma 2.5] for n sufficiently large

$$\begin{aligned} \limsup_{k \rightarrow \infty} (\mathcal{E}_k^n)^{\frac{1}{k}}(\beta_n) &\stackrel{(60), (61)}{\leq} L_n \max_{i,j=0, \dots, n-1, i \neq j} x_{i,j} \leq L_n \sup_{s,t \in \mathbb{R}/L\mathbb{Z}, s \neq t} \frac{2 \operatorname{dist}(\gamma(s) + \mathbb{R}\gamma'(s), \gamma(t))}{|\gamma(s) - \gamma(t)|^2} \\ &= L_n \sup_{s,t \in \mathbb{R}/L\mathbb{Z}, s \neq t} \frac{1}{r_{\text{tp}}(\gamma(s), \gamma(t))} = L_n \mathcal{R}(\gamma). \end{aligned} \quad (62)$$

Now, let $k \geq n$. By virtue of Lemma 5.3 applied to β_n , $m := k$ and replacing the index k in that lemma by n here, we have $(\mathcal{E}_n^n)^{\frac{1}{n}}(\beta_n) \leq \left(\frac{4L_n n(n-1)}{n^2} \right)^{\frac{1}{n} - \frac{1}{k}} (\mathcal{E}_k^n)^{\frac{1}{k}}(\beta_n)$, which leads to

$$(\mathcal{E}_n^n)^{\frac{1}{n}}(\beta_n) \leq \limsup_{k \rightarrow \infty} \left(\frac{4L_n n(n-1)}{n^2} \right)^{\frac{1}{n} - \frac{1}{k}} (\mathcal{E}_k^n)^{\frac{1}{k}}(\beta_n) \stackrel{(62)}{\leq} \left(\frac{4L_n n(n-1)}{n^2} \right)^{\frac{1}{n}} L_n \mathcal{R}(\gamma) \quad (63)$$

for n sufficiently large. Finally, taking the limsup yields the desired limsup inequality

$$\limsup_{n \rightarrow \infty} (\mathcal{E}_n^n)^{\frac{1}{n}}(\beta_n) \stackrel{(63)}{\leq} \limsup_{n \rightarrow \infty} \underbrace{\left(\frac{4L_n n(n-1)}{n^2} \right)^{\frac{1}{n}}}_{\rightarrow 1} \underbrace{L_n}_{\mathcal{L}(\gamma)=1} \mathcal{R}(\gamma) = \mathcal{R}(\gamma). \quad \square$$

Proof of Corollary 1.4. Apply Lemma 4.6 to the metric space $X = C_{\text{ia}}^{1,1}(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^3) \cap \mathcal{K}$ with the metric induced by the C^1 -norm. Note as in the proof of Corollary 1.2 that according to [25] the knot class \mathcal{K} is stable under C^1 -convergence. Since $\mathcal{R}(\gamma) < \infty$ holds, we obtain by [16, Lemma 2] that $\gamma \in C_{\text{ia}}^{1,1}(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^3)$. \square

Appendix A: Convergence of convolutions in $W^{2-\frac{1}{q}, q}(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^3)$

For fixed $L > 0$, $s \in (0, 1)$ and $\rho \in [1, \infty)$ define the seminorm $[f]_{s, \rho}$ of an L -periodic locally ρ -integrable function $f: \mathbb{R} \rightarrow \mathbb{R}^n$ as

$$[f]_{s, \rho} := \int_{\mathbb{R}/L\mathbb{Z}} \int_{\mathbb{R}/L\mathbb{Z}} \frac{|f(x) - f(y)|^\rho}{|x - y|_{\mathbb{R}/L\mathbb{Z}}^{1+s\rho}} dx dy, \quad (64)$$

where $|x - y|_{\mathbb{R}/L\mathbb{Z}}$ denotes the periodic distance on \mathbb{R} defined in (13). Then, the *periodic fractional⁴ Sobolev space* $W^{1+s, \rho}(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^n)$ consists of those Sobolev functions $f \in$

⁴Also known as periodic Sobolev–Slobodeckii space.

$W^{1,\rho}(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^n)$ whose weak derivatives f' have a finite seminorm $[f']_{s,\rho}$. The norm on $W^{1+s,\rho}(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^n)$ is given by $(\|f\|_{W^{1,\rho}} + [f']_{s,\rho})^{\frac{1}{\rho}}$.

Proof of Theorem 4.2 The case $\rho = \frac{1}{s}$ is treated in [6, Theorem 1.3], so we may assume from now on that $\rho > \frac{1}{s}$.

Step 1: According to Morrey–Sobolev embedding [21, Theorem A.2] we have $\gamma \in C^1(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^3)$, which implies that γ' is of *vanishing mean oscillation*, in short $\gamma' \in \text{VMO}(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^3)$, that is $\lim_{r \rightarrow 0} (\sup_{x \in \mathbb{R}/L\mathbb{Z}} \frac{1}{2r} (\int_{B_r(x)} |\gamma'(y) - \gamma'_{x,r}| dy)) = 0$, where $\gamma'_{x,r} := \frac{1}{2r} \int_{B_r(x)} \gamma'(z) dz$ denotes the integral mean. Indeed, γ' is uniformly continuous so that for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that $|\gamma'(x) - \gamma'(y)| < \frac{\varepsilon}{2}$ for all $x \in \mathbb{R}/L\mathbb{Z}$ and $y \in B_\delta(x)$. Let $0 < r < \delta$ and $x \in \mathbb{R}/L\mathbb{Z}$. Then,

$$\begin{aligned} \frac{1}{2r} \int_{B_r(x)} |\gamma'(y) - \gamma'_{x,r}| dy &\leq \sup_{y \in B_r(x)} |\gamma'(y) - \gamma'(x)| + |\gamma'(x) - \gamma'_{x,r}| \\ &\leq \frac{\varepsilon}{2} + \frac{1}{2r} \int_{B_r(x)} |\gamma'(x) - \gamma'(z)| dz < \varepsilon \quad \text{for any } x \in \mathbb{R}/L\mathbb{Z}, \end{aligned}$$

thus $\sup_{x \in \mathbb{R}/L\mathbb{Z}} \frac{1}{2r} (\int_{B_r(x)} |\gamma'(y) - \gamma'_{x,r}| dy) < \varepsilon$, which implies that $\gamma' \in \text{VMO}(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^3)$ since $\varepsilon > 0$ was arbitrary.

Step 2: For the lengths $L_\varepsilon := \mathcal{L}(\gamma_\varepsilon)$ and $L := \mathcal{L}(\gamma)$ we estimate

$$|L_\varepsilon - L| \leq \int_0^L ||\gamma'_\varepsilon(x)| - |\gamma'(x)|| dx \leq ||\gamma'_\varepsilon| - |\gamma'|||_{C^0} L \xrightarrow{\varepsilon \rightarrow 0} 0, \quad (65)$$

since $\gamma' \in \text{VMO}(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^3)$ allows us to apply [6, Theorem 1.1] to deduce that $|\gamma'_\varepsilon|$ converges uniformly to $|\gamma'| = 1$ as ε tends to 0. Therefore, there is an $\varepsilon_0 > 0$ such that

$$\frac{1}{2} \leq \left| \frac{L}{L_\varepsilon} \gamma'_\varepsilon(x) \right| \leq 2 \quad \text{for any } x \in \mathbb{R}/L\mathbb{Z}, \varepsilon \in (0, \varepsilon_0]. \quad (66)$$

Step 3: Since the convolutions γ_ε converge to γ in C^1 we obtain by means of (65) that also the rescalings $L\gamma_\varepsilon/L_\varepsilon$ converge towards γ in C^1 . According to Lemma B.3 we obtain

$$\|\tilde{\gamma}_\varepsilon - \gamma\|_{C^1} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0. \quad (67)$$

Step 4: It remains to show that $[\tilde{\gamma}'_\varepsilon - \gamma']_{s,\rho} \rightarrow 0$ holds as $\varepsilon \rightarrow 0$, since then, together with (67), we have established $\|\tilde{\gamma}_\varepsilon - \gamma\|_{W^{1+s,\rho}} \rightarrow 0$ as $\varepsilon \rightarrow 0$. Abbreviating the integrand of the seminorm by $I_\varepsilon(x, y) := \frac{|(\tilde{\gamma}'_\varepsilon(x) - \gamma'(x)) - (\tilde{\gamma}'_\varepsilon(y) - \gamma'(y))|^\rho}{|x - y|^{\frac{1+s\rho}{\rho}}}$ we want to apply Vitali's theorem (see, e.g., [1, 3.23]) to prove $\|I_\varepsilon\|_{L^1} \rightarrow 0$ as $\varepsilon \rightarrow 0$. Since we have a compact domain it suffices to show that the sequence $(I_\varepsilon)_{\varepsilon > 0}$ is uniformly integrable and converges pointwise to 0 a.e. on $\mathbb{R}/L\mathbb{Z} \times \mathbb{R}/L\mathbb{Z}$. The pointwise convergence $I_\varepsilon(x, y) \rightarrow 0$ (even for all $x \neq y$) follows from the C^1 -convergence (67).

Hence, we need to show the uniform integrability. In the obvious inequality

$$I_\varepsilon(x, y) \leq 2^{\rho-1} \left[\frac{|\tilde{\gamma}'_\varepsilon(x) - \tilde{\gamma}'_\varepsilon(y)|^\rho}{|x - y|^{\frac{1+s\rho}{\rho}}} + \frac{|\gamma'(x) - \gamma'(y)|^\rho}{|x - y|^{\frac{1+s\rho}{\rho}}} \right] \quad (68)$$

we estimate both summands on the right-hand side separately.

This is easy for the second summand. Fix $\tilde{\varepsilon} > 0$. Since γ is in $W^{1+s,\rho}(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^3)$, we find a $\delta_1 = \delta_1(\tilde{\varepsilon}) > 0$ such that for every measurable subset $E \subset (\mathbb{R}/L\mathbb{Z})^2$ with $|E| < \delta_1$ we obtain

$$\int \int_E \frac{|\gamma'(x) - \gamma'(y)|^\rho}{|x - y|_{\mathbb{R}/L\mathbb{Z}}^{1+s\rho}} dx dy < \frac{\tilde{\varepsilon}}{2^\rho}. \quad (69)$$

Regarding the first summand in (68) we consider the arclength function $s_\varepsilon(x) := \int_0^x |\frac{L}{L_\varepsilon} \gamma'_\varepsilon(z)| dz$ such that $s_\varepsilon(0) = 0$. From (66) the derivative $s'_\varepsilon(z) = |\frac{L}{L_\varepsilon} \gamma'_\varepsilon(z)|$ is uniformly bounded away from 0 for all $\varepsilon \in (0, \varepsilon_0]$. As a consequence, s_ε is for $\varepsilon \in (0, \varepsilon_0]$ invertible. Let \tilde{s}_ε denote the inverse function of s_ε . As a next step, we will show $|s_\varepsilon(x) - s_\varepsilon(y)|_{\mathbb{R}/L\mathbb{Z}} \geq \frac{1}{2}|x - y|_{\mathbb{R}/L\mathbb{Z}}$ for $x, y \in \mathbb{R}/L\mathbb{Z}$ and $\varepsilon \in (0, \varepsilon_0]$. Let $0 \leq x < y < L$, so that by monotonicity $0 \leq s_\varepsilon(x) \leq s_\varepsilon(y) < L$. First, assume that $|s_\varepsilon(x) - s_\varepsilon(y)|_{\mathbb{R}/L\mathbb{Z}} = |s_\varepsilon(x) - s_\varepsilon(y)| = s_\varepsilon(y) - s_\varepsilon(x)$. Then, we estimate by means of (66)

$$|s_\varepsilon(x) - s_\varepsilon(y)|_{\mathbb{R}/L\mathbb{Z}} = \int_x^y \left| \frac{L}{L_\varepsilon} \gamma'_\varepsilon(z) \right| dz \stackrel{(66)}{\geq} \frac{1}{2}(y - x) \geq \frac{1}{2}|x - y|_{\mathbb{R}/L\mathbb{Z}}.$$

If $|s_\varepsilon(x) - s_\varepsilon(y)|_{\mathbb{R}/L\mathbb{Z}} = L - (s_\varepsilon(y) - s_\varepsilon(x))$, then again by (66)

$$\begin{aligned} |s_\varepsilon(x) - s_\varepsilon(y)|_{\mathbb{R}/L\mathbb{Z}} &= L - \int_x^y \left| \frac{L}{L_\varepsilon} \gamma'_\varepsilon(z) \right| dz = \int_0^x \left| \frac{L}{L_\varepsilon} \gamma'_\varepsilon(z) \right| dz + \int_y^L \left| \frac{L}{L_\varepsilon} \gamma'_\varepsilon(z) \right| dz \\ &\stackrel{(66)}{\geq} \frac{1}{2}(L - (y - x)) \geq \frac{1}{2}|x - y|_{\mathbb{R}/L\mathbb{Z}}. \end{aligned}$$

In particular, this yields for the inverse function

$$|\tilde{s}_\varepsilon(x) - \tilde{s}_\varepsilon(y)|_{\mathbb{R}/L\mathbb{Z}} \leq 2|x - y|_{\mathbb{R}/L\mathbb{Z}} \quad \text{for any } x, y \in \mathbb{R}/L\mathbb{Z}, \varepsilon \in (0, \varepsilon_0]. \quad (70)$$

Due to (65) there exists a constant $c > 0$ such that

$$2^{1+(1+s)\rho} \left| \frac{L}{L_\varepsilon} \right|^\rho \left[1 + 2^\rho \left| \frac{L}{L_\varepsilon} \right|^\rho \right] \leq c \quad \text{for any } \varepsilon > 0. \quad (71)$$

Now, we estimate pointwise for $x \neq y$ with Jensen's inequality and by (70)

$$\begin{aligned} J_\varepsilon(x, y) &:= \frac{|\tilde{\gamma}'_\varepsilon(x) - \tilde{\gamma}'_\varepsilon(y)|^\rho}{|x - y|_{\mathbb{R}/L\mathbb{Z}}^{1+s\rho}} = \frac{|\frac{L}{L_\varepsilon} \gamma'_\varepsilon(\tilde{s}_\varepsilon(x)) \tilde{s}'_\varepsilon(x) - \frac{L}{L_\varepsilon} \gamma'_\varepsilon(\tilde{s}_\varepsilon(y)) \tilde{s}'_\varepsilon(y)|^\rho}{|\tilde{s}_\varepsilon(x) - \tilde{s}_\varepsilon(y)|_{\mathbb{R}/L\mathbb{Z}}^{1+s\rho}} \frac{|\tilde{s}_\varepsilon(x) - \tilde{s}_\varepsilon(y)|_{\mathbb{R}/L\mathbb{Z}}^{1+s\rho}}{|x - y|_{\mathbb{R}/L\mathbb{Z}}^{1+s\rho}} \\ &\stackrel{(70)}{\leq} 2^{1+s\rho} \left| \frac{L}{L_\varepsilon} \right|^\rho \left[\frac{|\tilde{s}'_\varepsilon(x)|^\rho |\gamma'_\varepsilon(\tilde{s}_\varepsilon(x)) - \gamma'_\varepsilon(\tilde{s}_\varepsilon(y))|^\rho}{|\tilde{s}_\varepsilon(x) - \tilde{s}_\varepsilon(y)|_{\mathbb{R}/L\mathbb{Z}}^{1+s\rho}} + \frac{|\gamma'_\varepsilon(\tilde{s}_\varepsilon(y))|^\rho |\tilde{s}'_\varepsilon(x) - \tilde{s}'_\varepsilon(y)|^\rho}{|\tilde{s}_\varepsilon(x) - \tilde{s}_\varepsilon(y)|_{\mathbb{R}/L\mathbb{Z}}^{1+s\rho}} \right]. \end{aligned}$$

Together with $|\gamma'_\varepsilon| \leq 1$, $|s'_\varepsilon(x)| = |\frac{L}{L_\varepsilon} \gamma'_\varepsilon(x)| \in [\frac{1}{2}, 2]$ due to (66), $|\tilde{s}'_\varepsilon(x)| = |\frac{L}{L_\varepsilon} \gamma'_\varepsilon(\tilde{s}_\varepsilon(x))|^{-1} \in [\frac{1}{2}, 2]$ for $\varepsilon \in (0, \varepsilon_0]$, and the estimate

$$\begin{aligned} |\tilde{s}'_\varepsilon(x) - \tilde{s}'_\varepsilon(y)| &= \left| \left| \frac{L}{L_\varepsilon} \gamma'_\varepsilon(\tilde{s}_\varepsilon(x)) \right|^{-1} - \left| \frac{L}{L_\varepsilon} \gamma'_\varepsilon(\tilde{s}_\varepsilon(y)) \right|^{-1} \right| \\ &\leq \left| \frac{L}{L_\varepsilon} \right| \frac{|\gamma'_\varepsilon(\tilde{s}_\varepsilon(x)) - \gamma'_\varepsilon(\tilde{s}_\varepsilon(y))|}{|\gamma'_\varepsilon(\tilde{s}_\varepsilon(x))| |\gamma'_\varepsilon(\tilde{s}_\varepsilon(y))|} \end{aligned}$$

$$\stackrel{(66)}{\leq} 4 \left| \frac{L}{L_\varepsilon} \right| \left| \gamma'_\varepsilon(\tilde{s}_\varepsilon(x)) - \gamma'_\varepsilon(\tilde{s}_\varepsilon(y)) \right|,$$

we obtain for all $\varepsilon \in (0, \varepsilon_0]$ the inequality

$$J_\varepsilon(x, y) \leq 2^{1+s\rho} \left| \frac{L}{L_\varepsilon} \right|^\rho \left[2^\rho + 4^\rho \left| \frac{L}{L_\varepsilon} \right|^\rho \right] \frac{|\gamma'_\varepsilon(\tilde{s}_\varepsilon(x)) - \gamma'_\varepsilon(\tilde{s}_\varepsilon(y))|^\rho}{|\tilde{s}_\varepsilon(x) - \tilde{s}_\varepsilon(y)|_{\mathbb{R}/L\mathbb{Z}}^{1+s\rho}} \stackrel{(71)}{\leq} cA_\varepsilon(\psi_\varepsilon(x, y)) \quad (72)$$

for $A_\varepsilon(x, y) := \frac{|\gamma'_\varepsilon(x) - \gamma'_\varepsilon(y)|^\rho}{|x - y|_{\mathbb{R}/L\mathbb{Z}}^{1+s\rho}}$ and the transformation $\psi_\varepsilon : (\mathbb{R}/L\mathbb{Z})^2 \rightarrow (\mathbb{R}/L\mathbb{Z})^2$ sending (x, y) to $(\tilde{s}_\varepsilon(x), \tilde{s}_\varepsilon(y))$. Observe that by (66), ψ_ε is bi-Lipschitz, since $\det(D\psi_\varepsilon(x, y)) = |\tilde{s}'_\varepsilon(x)| |\tilde{s}'_\varepsilon(y)| \in [\frac{1}{4}, 4]$. This implies by (72) that

$$J_\varepsilon(x, y) \leq 4cA_\varepsilon(\psi_\varepsilon(x, y)) |\det(D\psi_\varepsilon(x, y))| \quad \text{for any } \varepsilon \in (0, \varepsilon_0].$$

Let now $E \subset (\mathbb{R}/L\mathbb{Z})^2$. By a change of variables

$$\begin{aligned} \int \int_E J_\varepsilon(x, y) \, dx \, dy &\leq 4c \int \int_E A_\varepsilon(\psi_\varepsilon(x, y)) |\det(D\psi_\varepsilon(x, y))| \, dx \, dy \\ &= 4c \int \int_{\psi_\varepsilon(E)} A_\varepsilon(x, y) \, dx \, dy \quad \text{for any } \varepsilon \in (0, \varepsilon_0]. \end{aligned} \quad (73)$$

It is well known that the standard convolution γ_ε converges in $W^{1+s, \rho}$ to γ ; see, e.g., [13, Lemma 11], which according to Vitali's theorem implies that the $A_\varepsilon(x, y)$ are uniformly integrable. In particular, for given $\tilde{\varepsilon} > 0$ there exists $\delta_2 = \delta_2(\tilde{\varepsilon}) > 0$ such that if $|\psi_\varepsilon(E)| < \delta_2$, we have

$$\int \int_{\psi_\varepsilon(E)} A_\varepsilon(x, y) \, dx \, dy < \frac{\tilde{\varepsilon}}{c2^{\rho+2}} \quad \text{for any } \varepsilon > 0. \quad (74)$$

Since ψ_ε is uniformly Lipschitz continuous for $\varepsilon \in (0, \varepsilon_0]$, there exists a $\delta_3 > 0$ such that $|E| < \delta_3$ implies $|\psi_\varepsilon(E)| < \delta_2$. Now, set $\tilde{\delta} := \min\{\delta_1, \delta_3\}$ so that for any set $E \subset (\mathbb{R}/L\mathbb{Z})^2$ with $|E| < \tilde{\delta}$ we infer by means of (68), (73), (69), and (74) that

$$\begin{aligned} \int \int_E I_\varepsilon(x, y) \, dx \, dy &\stackrel{(68), (73)}{\leq} 2^{\rho-1} \left[4c \int \int_{\psi_\varepsilon(E)} A_\varepsilon(x, y) \, dx \, dy + \int \int_E \frac{|\gamma'(x) - \gamma'(y)|^\rho}{|x - y|_{\mathbb{R}/L\mathbb{Z}}^{1+s\rho}} \, dx \, dy \right] \\ &\stackrel{(69), (74)}{<} 2^{\rho-1} \left[4c \frac{\tilde{\varepsilon}}{c2^{\rho+2}} + \frac{\tilde{\varepsilon}}{2^\rho} \right] = \tilde{\varepsilon} \quad \text{for any } \varepsilon \in (0, \varepsilon_0]. \end{aligned}$$

Hence, $(I_\varepsilon)_{\varepsilon \in (0, \varepsilon_0]}$ is uniformly integrable. \square

Appendix B: Quantitative analysis of C^1 -curves

Lemma B.1 (Injective C^1 -curves are bi-Lipschitz) *For any curve $\gamma \in C^1_{\text{ia}}(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^3)$ there is a constant $c_\gamma > 0$ such that*

$$|t - s|_{\mathbb{R}/L\mathbb{Z}} \leq c_\gamma |\gamma(t) - \gamma(s)| \quad \text{for any } t, s \in \mathbb{R}. \quad (75)$$

Proof Using the Taylor expansion

$$\gamma(s + h) - \gamma(s) = \int_s^{s+h} \gamma'(\tau) \, d\tau = \gamma'(s)h + \int_s^{s+h} (\gamma'(\tau) - \gamma'(s)) \, d\tau \quad (76)$$

we choose $h_0 = h_0(\gamma) \in (0, \frac{L}{2}]$ such that $\omega_{\gamma'}(h_0) \leq \frac{1}{2}$ to infer for all $s \in \mathbb{R}$

$$|\gamma(s+h) - \gamma(s)| \geq (1 - \omega_{\gamma'}(h_0))|h| \geq \frac{1}{2}|h| = \frac{1}{2}|h|_{\mathbb{R}/L\mathbb{Z}} \quad \text{for any } |h| \leq h_0. \quad (77)$$

On the other hand, since γ is injective, we find a constant $\delta_0 = \delta_0(\gamma) > 0$ such that

$$|\gamma(s+h) - \gamma(s)| \geq \delta_0 \geq \frac{2\delta_0}{L}|h| = \frac{2\delta_0}{L}|h|_{\mathbb{R}/L\mathbb{Z}} \quad \text{for any } s \in \mathbb{R}, h_0 \leq |h| \leq \frac{L}{2}, \quad (78)$$

which implies (75) for $c_\gamma := \max\{2, \frac{L}{2\delta_0}\}$. \square

Lemma B.2 *Let $\gamma \in C^1_{\text{ia}}(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^3)$ and $h \in (0, \frac{L}{2}]$ such that $\omega_{\gamma'}(h) < 1$ and $h \leq \frac{L}{2}$. Then, for every $s \in \mathbb{R}$,*

$$1 - \omega_{\gamma'}(h) \leq \frac{\langle \gamma(s+h) - \gamma(s), \gamma'(s) \rangle}{|\gamma(s+h) - \gamma(s)|} \leq 1, \quad (79)$$

$$1 - \omega_{\gamma'}(h) \leq \frac{\langle \gamma(s+h) - \gamma(s), \gamma'(s+h) \rangle}{|\gamma(s+h) - \gamma(s)|} \leq 1. \quad (80)$$

Proof Since $0 < h \leq \frac{L}{2}$ we have $|s+h-s|_{\mathbb{R}/L\mathbb{Z}} = h$; see (13). We use the Taylor expansion (76) to estimate

$$\langle \gamma(s+h) - \gamma(s), \gamma'(s) \rangle \geq h \left[1 - \sup_{\tau \in [s, s+h]} |\gamma'(\tau) - \gamma'(s)| \right] \geq h[1 - \omega_{\gamma'}(h)] \quad (81)$$

and analogously

$$\langle \gamma(s+h) - \gamma(s), \gamma'(s+h) \rangle \geq h[1 - \omega_{\gamma'}(h)]. \quad (82)$$

Using the above estimate for the inner product and the Lipschitz estimate $|\gamma(s+h) - \gamma(s)| \leq h$ we can deduce

$$\frac{\langle \gamma(s+h) - \gamma(s), \gamma'(s) \rangle}{|\gamma(s+h) - \gamma(s)|} \stackrel{(81)}{\geq} \frac{h(1 - \omega_{\gamma'}(h))}{|\gamma(s+h) - \gamma(s)|} \geq \frac{h(1 - \omega_{\gamma'}(h))}{h} = 1 - \omega_{\gamma'}(h).$$

Applying the Cauchy–Schwarz inequality yields the right part of inequality (79). Thus, statement (79) is shown. In the same manner we can conclude the statement (80) with the Cauchy–Schwarz inequality and estimate (82). \square

Lemma B.3 *Let $\gamma \in C^1([a, b], \mathbb{R}^3)$ satisfy $|\gamma'| \geq v_\gamma > 0$ and $\mathcal{L}(\gamma) > 0$ and let $\Gamma \in C^1([0, \mathcal{L}(\gamma)], \mathbb{R}^3)$ be the arclength parametrization. Suppose that $\beta \in C^1([a, b], \mathbb{R}^3)$ has equal length, i.e. $\mathcal{L}(\gamma) = \mathcal{L}(\beta)$, and satisfies*

$$\|\gamma - \beta\|_{C^1([a, b], \mathbb{R}^3)} < \varepsilon \leq \frac{v_\gamma}{2}. \quad (83)$$

Then, β possesses an arclength parametrization $B \in C^1([0, \mathcal{L}(\gamma)], \mathbb{R}^3)$ with

$$\|\Gamma - B\|_{C^1([0, \mathcal{L}(\gamma)], \mathbb{R}^3)} \leq \frac{2}{v_\gamma} \omega_{\gamma'}\left(\frac{(b-a)\varepsilon}{v_\gamma}\right) + \omega_\gamma\left(\frac{(b-a)\varepsilon}{v_\gamma}\right) + \varepsilon\left(1 + \frac{2}{v_\gamma}\right), \quad (84)$$

where ω_γ denotes the modulus of continuity of γ and $\omega_{\gamma'}$ denotes the modulus of continuity of the tangent γ' of γ .

Proof Without loss of generality, we can assume $\Gamma(s) = \gamma(t(s))$ for $s \in [0, \mathcal{L}(\gamma)]$, where $t : [0, \mathcal{L}(\gamma)] \rightarrow [a, b]$ is the inverse function of the arclength function $s(t) := \int_a^t |\gamma'(u)| \, du$ for $t \in [a, b]$. Furthermore, the conditions $|\gamma'| \geq \nu_\gamma$ and (83) imply $|\beta'(t)| \geq \frac{\nu_\gamma}{2} > 0$ for all $t \in [a, b]$. Hence, the arclength function of β , $\sigma(t) := \int_a^t |\beta'(u)| \, du$ is therefore also invertible. Let $\tau : [0, \mathcal{L}(\gamma)] \rightarrow [a, b]$ be the inverse function of σ and define the arclength parametrization of β as $B(s) := \beta(\tau(s))$ for $s \in [0, \mathcal{L}(\gamma)]$. Now, fix an $s \in [0, \mathcal{L}(\gamma)]$. Then, there exist unique $t, \tau \in [a, b]$ such that $s = s(t) = \sigma(\tau)$. This leads to

$$0 = \sigma(\tau) - s(t) = \int_a^\tau (|\beta'(u)| - |\gamma'(u)|) \, du - \int_\tau^t |\gamma'(u)| \, du. \quad (85)$$

Thus, we can estimate

$$\nu_\gamma |t - \tau| \leq \left| \int_\tau^t |\gamma'(u)| \, du \right| \stackrel{(85)}{=} \left| \int_a^\tau (|\beta'(u)| - |\gamma'(u)|) \, du \right| \stackrel{(83)}{\leq} \varepsilon(b-a). \quad (86)$$

Now, we can use (83) and (86) to estimate the distance between Γ and B by

$$\begin{aligned} |\Gamma(s) - B(s)| &= |\gamma(t) - \beta(\tau)| \leq |\gamma(t) - \gamma(\tau)| + |\gamma(\tau) - \beta(\tau)| \\ &\stackrel{(83)}{\leq} \omega_\gamma(|t - \tau|) + \varepsilon \stackrel{(86)}{\leq} \omega_\gamma\left(\frac{(b-a)\varepsilon}{\nu_\gamma}\right) + \varepsilon. \end{aligned} \quad (87)$$

With $\tau'(s) = \frac{1}{\sigma'(\tau(s))} = \frac{1}{|\beta'(\tau(s))|}$ we obtain $B'(s) = \beta'(\tau(s))\tau'(s) = \frac{\beta'(\tau(s))}{|\beta'(\tau(s))|}$ and analogously $\Gamma'(s) = \frac{\gamma'(t(s))}{|\gamma'(t(s))|}$. This leads to

$$\begin{aligned} &|\Gamma'(s) - B'(s)| \\ &= \left| \frac{\gamma'(t)}{|\gamma'(t)|} - \frac{\beta'(\tau)}{|\beta'(\tau)|} \right| \\ &\leq \frac{|\gamma'(t) - \gamma'(\tau)|}{|\gamma'(t)|} + |\gamma'(\tau)| \left| \frac{1}{|\gamma'(t)|} - \frac{1}{|\gamma'(\tau)|} \right| \\ &\quad + \frac{|\gamma'(\tau) - \beta'(\tau)|}{|\gamma'(\tau)|} + |\beta'(\tau)| \left| \frac{1}{|\gamma'(\tau)|} - \frac{1}{|\beta'(\tau)|} \right| \\ &\leq \frac{1}{\nu_\gamma} \omega_{\gamma'}(|t - \tau|) + \frac{1}{\nu_\gamma} \omega_{\gamma'}(|t - \tau|) + \frac{\varepsilon}{\nu_\gamma} + \frac{\varepsilon}{\nu_\gamma} \stackrel{(86)}{\leq} \frac{2}{\nu_\gamma} \left(\omega_{\gamma'}\left(\frac{(b-a)\varepsilon}{\nu_\gamma}\right) + \varepsilon \right). \end{aligned} \quad (88)$$

With (87) and (88), we deduce (84). \square

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