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Stability of thermoelastic Timoshenko beam with suspenders and time-varying feedback

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Abstract

This paper considers a one-dimensional thermoelastic Timoshenko beam system with suspenders, general weak internal damping with time varying coefficient, and time-varying delay terms. Under suitable conditions on the nonlinear terms, we prove a general stability result for the beam model, where exponential and polynomial decay are special cases. We also gave some examples to illustrate our theoretical finding.

MSC: 35D30; 74K10; 74H20; 74H40; 65M60; 65M15

Keywords: General decay; Timoshenko beam; Thermoelasticity; Suspenders; Time delay

1 Introduction

In this paper, we consider a thermoelastic Timoshenko beam with suspension cables, weak internal damping, and a time-varying delay damping of the form

$$\begin{cases} \rho u_{tt}(x,t) - \alpha u_{xx}(x,t) - \lambda(\varphi - u)(x,t) \\ \quad + \gamma_1 a(t) g_1(u_t(x,t)) + \gamma_2 a(t) g_2(u_t(x,t-\tau(t))) = 0, \\ \rho_1 \varphi_{tt}(x,t) - k(\varphi_x + \psi)_x(x,t) + \lambda(\varphi - u)(x,t) + \gamma_3 \varphi_t(x,t) = 0, \\ \rho_2 \psi_{tt}(x,t) - b\psi_{xx}(x,t) + k(\varphi_x + \psi)(x,t) - m\theta_x(x,t) = 0, \\ \rho_3 \theta_t(x,t) - \beta\theta_{xx}(x,t) - m\psi_{xt}(x,t) = 0, \end{cases} \tag{1.1}$$

where $(x,t) \in (0,1) \times (0,\infty)$, φ represents the transverse displacement (in vertical direction) of the beam cross section, ψ is the angle of rotation of a cross-section. The vertical displacement of the vibrating spring (the cable) is represented by the function u, θ depicts the thermal moment of the beam, $\lambda > 0$ is the common stiffness of the string, and $\alpha > 0$ is the elastic modulus of the string (holding the cable to the deck). The positive constants ρ, ρ_1, ρ_2 are the density of the mass material of the cable, the mass density, and the moment of mass inertia of the beam, respectively. Also, b, k, β, m represent the rigidity coefficient of the cross-section, the shear modulus of elasticity, the thermal diffusivity, and the coupling coefficient which depends on the material properties, respectively. The function $\tau(t) > 0$ is the time-varying delay, γ_1 and γ_2 are real positive damping constants, g_1 and g_2 are the damping functions, and $a(t)$ is a nonlinear weight function. We supplement (1.1) with the boundary conditions

$$\begin{cases} u(0,t) = \varphi_x(0,t) = \psi(0,t) = \theta_x(0,t) = 0, & t > 0, \\ u(1,t) = \varphi(1,t) = \psi_x(1,t) = \theta(1,t) = 0, & t > 0, \end{cases} \tag{1.2}$$

and the initial data

$$\begin{cases} u(x,0) = u_0(x), \qquad \varphi(x,0) = \varphi_0(x), & \\ \psi(x,0) = \psi_0(x), \qquad \theta(x,0) = \theta_0(x), & \text{in } (0,1), \\ u_t(x,0) = u_1(x), \qquad \varphi_t(x,0) = \varphi_1(x), & \\ \psi_t(x,0) = \psi_1(x), & \text{in } (0,1), \\ u_t(x,t-\tau(0)) = f_0(x,t-\tau(0)), \quad \text{in } (0,1)\times(0,\tau(0)). & \end{cases} \tag{1.3}$$

The stability of the above thermoelastic Timoshenko system with suspension cables would be our penultimate focus in this work. The Timoshenko beam model is arguably very popular and most used when the vibration of a beam exhibits significant transverse shear strain. A model to describe this phenomenon was introduced by Timoshenko [35] in 1921, see also [15, 18]. The nonlinear vibration of suspension bridges have captured the attention of different researchers and a number of research articles were written on the topic. The somewhat unpredictable large oscillations of suspension bridges have been modeled in diverse ways, one may see [1, 14, 25]. In any attempt to adequately describe the complicated dynamics of a suspension bridge, a robust model would be one with a considerable amount of degrees of freedom. Without prejudice, some simplified models have been considered, but do not account for a number of realistic behavior of suspension bridges, e.g., torsional oscillations. Of an advantage is the fact that rigorous mathematical analysis is easily carried out with such simpler models. A typical simplified model is the one-dimensional vibrating beam model, where torsional motion is neglected by ignoring sectional dimensions of the beam when they are negligible compared to length of the beam. The emergence of string-beam systems which model a nonlinear coupling of a beam and main cable (the string) were born out of the pioneering works of Lazer, McKenna, and Walter [23, 25, 26] (see also [7] and its references). Though initially modeled through the classic Euler–Bernoulli beam theory, the Timoshenko beam theory is also proven to perform better in predicting a beam response to vibrations than a model based on the classical Euler–Bernoulli beam theory. Indeed, the Timoshenko beam theory takes into account both rotary inertia and shear deformation effects, these are often neglected when applying Euler–Bernoulli beam theory.

In the Timoshenko beam with suspenders which is modeled by (1.1), the suspenders are cables which are elastic in nature and are attached to the beam with elastic springs. The temperature dissipation here is assumed to be governed by the Fourier law of heat conduction. For $a(t) \equiv 1$, $g_1(s) \equiv s$ and $\gamma_2 \equiv 0$, $g_2 \equiv 0$ in system (1.1), Bochichio et al. [6] proved a well-posedness and an exponential stability result. A number of works have been done on different thermoelastic Timoshenko models without suspenders (see [10, 12, 16, 17, 28] and references in them). Time delays occur in systems modeling many phenomena in areas such as biosciences, medicine, physics, robotics, economics, chemical, thermal, and structural engineering. These phenomena depend on both present and some past history of occurrences, see [8, 9, 13, 21, 34] and the examples therein. In the case of constant delay and constant weight, the delay term usually accounts for the past history of strain, only up to some finite time $\tau(t) \equiv \tau$.

A step further involves results in the literature about constant weights ($\gamma_1 a(t) \equiv \gamma_1$, $\gamma_2 a(t) \equiv \gamma_2$ constants) and time-varying delay $\tau(t)$. Works presenting the exponential stability result for wave equation with boundary or internal time-varying delay appeared in Nicaise et al. [32, 33]. Enyi and Mukiawa in [11] presented a general decay result for a viscoelastic plate equation under the condition $|\gamma_2| < |\gamma_1|\sqrt{(1-d)}$. Furthermore, in [4, 24], the authors presented some existence and stability results for wave equation with internal time-varying delay and time-varying weights; and for suspension bridge models, see Mukiawa [3, 27, 29, 30].

Motivated by the works in [3, 6, 29], in the current paper, we are concerned with the stability result for the thermoelastic Timoshenko system with suspension cables, time-varying internal feedback, and time-varying weight given in (1.1)–(1.3). The result in [6] is a particular case of our result in this paper.

We arrange this paper in the following manner. In Sect. 2, we state the needed assumptions. In Sect. 3, we present the proof of some technical and needed lemmas for our main result. In the last Sect. 4, we present and prove our main stability result. Throughout this paper, c and c_i, $i = 1, 2, \dots$, are generic positive constants, which are not necessarily the same from line to line.

2 Functional settings and assumptions

In this section, we state some needed assumptions on the damping coefficients, nonlinear functions, and the time-varying delay. As in [5, 32, 33], we assume the following conditions:

(A_1) Function $a : [0, +\infty) \to (0, +\infty)$ is a nonincreasing C^1-function such that there exists a positive constant C satisfying

$$\left|a'(t)\right| \le Ca(t), \qquad \int_0^{+\infty} a(t)\,dt = +\infty. \tag{2.1}$$

(A_2) Fuction $g_1 : \mathbb{R} \to \mathbb{R}$ is a nondecreasing C^0-function such that there exist positive constants C_1, C_2, r and a convex increasing function $\chi \in C^1([0, +\infty)) \cap C^2((0, +\infty))$ satisfying $\chi(0) = 0$ or χ is a nonlinear strictly convex C^2-function on $(0, r]$ with $\chi'(0), \chi'' > 0$ such that

$$s^2 + g_1^2(s) \le \chi^{-1}\bigl(sg_1(s)\bigr), \quad \text{for all } |s| \le r, \tag{2.2}$$

$$C_1 s^2 \le sg_1(s) \le C_2 s^2, \quad \text{for all } |s| \ge r. \tag{2.3}$$

Function $g_2 : \mathbb{R} \to \mathbb{R}$ is an increasing and odd C^1-function such that for some positive constants C_3, α_1, α_2,

$$\left|g_2'(s)\right| \le C_3, \tag{2.4}$$

$$\alpha_1\bigl(sg_2(s)\bigr) \le G(s) \le \alpha_2\bigl(sg_1(s)\bigr), \tag{2.5}$$

where

$$G(s) = \int_0^s g_2(r)\,dr. \tag{2.6}$$

(A_3) There exist $\tau_0, \tau_1 > 0$ such that

$$0 < \tau_0 \le \tau(t) \le \tau_1, \quad \forall t > 0, \tag{2.7}$$

$$\tau \in W^{2,\infty}(0,T), \quad \forall T > 0, \tag{2.8}$$

$$\tau'(t) \le d < 1, \quad \forall t > 0. \tag{2.9}$$

(A_4) The damping coefficients satisfy

$$\gamma_2\alpha_2(1 - d\alpha_1) < \alpha_1(1-d)\gamma_1. \tag{2.10}$$

Remark 2.1 Using the monotonicity of g_2 and the mean value theorem for integrals, we deduce that

$$G(s) = \int_0^s g_2(r)\,dr < sg_2(s). \tag{2.11}$$

It follows from (2.5) that $\alpha_1 < 1$.

Similarly, as in Nicaise and Pignotti [31], we introduce the following change of variable:

$$z(x,\sigma,t) = u_t\big(x, t - \tau(t)\sigma\big), \quad \text{for } (x,\sigma,t) \in (0,1)\times(0,1)\times(0,\infty). \tag{2.12}$$

It follows that

$$\tau(t)z_t(x,\sigma,t) + \big(1 - \tau'(t)\sigma\big)z_\sigma(x,\sigma,t) = 0. \tag{2.13}$$

Therefore, system (1.1) becomes

$$\begin{cases} \rho u_{tt}(x,t) - \alpha u_{xx}(x,t) - \lambda(\varphi - u)(x,t) + \gamma_1 a(t) g_1(u_t(x,t)) \\ \quad + \gamma_2 a(t) g_2(z(x,1,t)) = 0, \\ \rho_1\varphi_{tt}(x,t) - k(\varphi_x + \psi)_x(x,t) + \lambda(\varphi - u)(x,t) + \gamma_3\varphi_t(x,t) = 0, \\ \rho_2\psi_{tt}(x,t) - b\psi_{xx}(x,t) + k(\varphi_x + \psi)(x,t) - m\theta_x(x,t) = 0, \\ \rho_3\theta_t(x,t) - \beta\theta_{xx}(x,t) - m\psi_{xt}(x,t) = 0, \\ \tau(t)z_t(x,\sigma,t) + (1 - \tau'(t)\sigma)z_\sigma(x,\sigma,t) = 0, \end{cases} \tag{2.14}$$

subjected to the boundary conditions

$$\begin{cases} u(0,t) = \varphi_x(0,t) = \psi(0,t) = \theta_x(0,t) = 0, \quad t > 0, \\ u(1,t) = \varphi(1,t) = \psi_x(1,t) = \theta(1,t) = 0, \quad t > 0, \\ z(x,0,t) = u_t(x,t), \quad x \in (0,1), t > 0, \end{cases} \tag{2.15}$$

and initial data

$$\begin{cases} u(x,0) = u_0(x), & \varphi(x,0) = \varphi_0(x), \\ \psi(x,0) = \psi_0(x), & \theta(x,0) = \theta_0(x), \qquad \text{in } (0,1), \\ u_t(x,0) = u_1(x), & \varphi_t(x,0) = \varphi_1(x), \\ \psi_t(x,0) = \psi_1(x), & \qquad\qquad\qquad\quad \text{in } (0,1), \\ z(x,\sigma,0) = u_t(x,-\tau(0)\sigma) = f_0(x,-\tau(0)\sigma), & \text{in } (0,1)\times(0,1). \end{cases} \tag{2.16}$$

We introduce the following spaces:

$$\begin{aligned} H_a^1(0,1) &= \{\phi \in H^1(0,1) : \phi(0) = 0\}, \\ H_b^1(0,1) &= \{\phi \in H^1(0,1) : \phi(1) = 0\}, \\ H_a^2(0,1) &= \{\phi \in H^2(0,1) : \phi_x \in H_a^1(0,1)\}, \\ H_b^2(0,1) &= \{\phi \in H^2(0,1) : \phi_x \in H_b^1(0,1)\}. \end{aligned}$$

For completeness, we state without proof the existence and uniqueness result for problem (1.1)–(1.3). The result can be established using the Faedo–Galerkin approximation method, see [5] or standard nonlinear semigroup method, see [19, 20].

Theorem 2.1 *Let*

$$\begin{aligned} (u_0,\varphi_0,\psi_0,\theta_0) \in H^2(0,1)\cap H_0^1(0,1) \times H_a^2(0,1)\cap H_b^1(0,1)\times H_b^2(0,1) \\ \cap H_a^1(0,1)\times H_a^2(0,1)\cap H_b^1(0,1) \end{aligned}$$

and

$$(u_1,\varphi_1,\psi_1) \in H_0^1(0,1)\times H_a^1(0,1)\times H_b^1(0,1), \qquad f_0(\cdot,-\tau(0)) \in H_0^1((0,1);H^1(0,1))$$

be given such that

$$f_0(\cdot,0) = u_1.$$

Suppose conditions (A_1)–(A_4) *hold. Then, problem* (1.1)–(1.3) *has a unique global weak solution in the class*

$$\begin{aligned} & u \in L^\infty([0,+\infty);H^2(0,1)\cap H_0^1(0,1)), \qquad u_t \in L^\infty([0,+\infty);H_0^1(0,1)), \\ & u_{tt} \in L^\infty((0,+\infty);L^2(0,1)), \\ & \varphi \in L^\infty([0,+\infty);H_a^2(0,1)\cap H_b^1(0,1)), \qquad \varphi_t \in L^\infty([0,+\infty);H_b^1(0,1)), \\ & \varphi_{tt} \in L^\infty((0,+\infty);L^2(0,1)), \\ & \psi \in L^\infty([0,+\infty);H_b^2(0,1)\cap H_a^1(0,1)), \qquad \psi_t \in L^\infty([0,+\infty);H_a^1(0,1)), \\ & \psi_{tt} \in L^\infty((0,+\infty);L^2(0,1)), \\ & \theta \in L^\infty([0,+\infty);H_a^2(0,1)\cap H_b^1(0,1)), \qquad \theta_t \in L^\infty((0,+\infty);L^2(0,1)). \end{aligned}$$

3 Technical lemmas

In this section, we prove some important lemmas which will be essential in establishing the main result. Let $\bar{\mu}$ be a positive constant satisfying

$$\frac{\gamma_2(1-\alpha_1)}{\alpha_1(1-d)} < \bar{\mu} < \frac{\gamma_1-\gamma_2\alpha_2}{\alpha_2} \tag{3.1}$$

and set

$$\mu(t) = \bar{\mu}a(t).$$

The energy functional of system (2.14)–(2.16) is defined by

$$\begin{aligned} E(t) = {} & \frac{1}{2}\int_0^1 \left[\rho u_t^2 + \rho_1\varphi_t^2 + \rho_2\psi_t^2 + \alpha u_x^2 + k(\varphi_x+\psi)^2 + b\psi_x^2 + \lambda(\varphi-u)^2\right] dx \\ & + \frac{1}{2}\int_0^1 \rho_3\theta^2\,dx + \mu(t)\tau(t)\int_0^1\int_0^1 G\big(z(x,\sigma,t)\big)\,d\sigma\,dx. \end{aligned} \tag{3.2}$$

Lemma 3.1 *Let $(u,\varphi,\psi,\theta,z)$ be the solution of system (2.14)–(2.16). Then, the energy functional (3.2) satisfies*

$$\begin{aligned} \frac{dE(t)}{dt} \leq {} & -a(t)[\gamma_1-\bar{\mu}\alpha_2-\gamma_2\alpha_2]\int_0^1 u_t g_1(u_t)\,dx \\ & -a(t)\left[\bar{\mu}\big(1-\tau'(t)\big)\alpha_1-\gamma_2(1-\alpha_1)\right]\int_0^1 z(x,1,t)g_2\big(z(x,1,t)\big)\,dx \\ & -\gamma_3\int_0^1 \varphi_t^2\,dx - \beta\int_0^1 \theta_x^2\,dx \\ \leq {} & 0, \quad \forall t\geq 0. \end{aligned} \tag{3.3}$$

Proof Multiplying $(2.14)_1$ by u_t, $(2.14)_2$ by φ_t, $(2.14)_3$ by ψ_t, and $(2.14)_4$ by θ, integrating the outcome over $(0,1)$, and applying integration by parts and the boundary conditions, we get

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\int_0^1 \left[\rho u_t^2 + \alpha u_x^2 + \lambda(\varphi-u)^2\right] dx \\ & \quad = \lambda\int_0^1 \varphi_t(\varphi-u)\,dx - \gamma_1 a(t)\int_0^1 u_t g_1(u_t)\,dx - \gamma_2 a(t)\int_0^1 u_t g_2\big(z(x,1,t)\big)\,dx, \end{aligned} \tag{3.4}$$

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\int_0^1 \left[\rho_1\varphi_t^2 + k(\varphi_x+\psi)^2\right] dx \\ & \quad = -\gamma_3\int_0^1 \varphi_t^2\,dx - \lambda\int_0^1 \varphi_t(\varphi-u)\,dx + k\int_0^1 \psi_t(\varphi_x+\psi)\,dx, \end{aligned} \tag{3.5}$$

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \left[\rho_2\psi_t^2 + b\psi_x^2\right] dx = m\int_0^1 \psi_t\theta_x\,dx - k\int_0^1 \psi_t(\varphi_x+\psi)\,dx, \tag{3.6}$$

$$\frac{1}{2}\int_0^1 \rho_3\theta^2\,dx = -\beta\int_0^1 \theta_x^2\,dx - m\int_0^1 \psi_t\theta_x\,dx. \tag{3.7}$$

Adding (3.4)–(3.7), we arrive at

$$\begin{aligned}
&\frac{1}{2}\int_0^1 \left(\rho u_t^2 + \alpha u_x^2 + \lambda(\varphi - u)^2 + \rho_1\varphi_t^2 + k(\varphi_x + \psi)^2 + \rho_2\psi_t^2 + b\psi_x^2 + \rho_3\theta^2\right) dx \\
&\quad = -\gamma_1 a(t)\int_0^1 u_t g_1(u_t)\,dx - \gamma_2 a(t)\int_0^1 u_t g_2\bigl(z(x,1,t)\bigr)\,dx \\
&\qquad - \gamma_3\int_0^1 \varphi_t^2\,dx - \beta\int_0^1 \theta_x^2\,dx.
\end{aligned} \tag{3.8}$$

Now, multiplying equation $(2.14)_5$ by $\mu(t)g_2(z(x,\sigma,t))$ and integrating over $(0,1)\times(0,1)$, we obtain

$$\begin{aligned}
&\mu(t)\tau(t)\int_0^1\int_0^1 z_t(x,\sigma,t)g_2\bigl(z(x,\sigma,t)\bigr)\,d\sigma\,dx \\
&\quad + \mu(t)\int_0^1\int_0^1 \bigl(1-\tau'(t)\sigma\bigr)z_\sigma(x,\sigma,t)g_2\bigl(z(x,\sigma,t)\bigr)\,d\sigma\,dx = 0.
\end{aligned} \tag{3.9}$$

On account of (2.6), we can write

$$\frac{\partial}{\partial\sigma}\bigl[G\bigl(z(x,\sigma,t)\bigr)\bigr] = z_\sigma(x,\sigma,t)g_2\bigl(z(x,\sigma,t)\bigr). \tag{3.10}$$

Therefore, (3.9) becomes

$$\begin{aligned}
&\mu(t)\tau(t)\int_0^1\int_0^1 z_t(x,\sigma,t)g_2\bigl(z(x,\sigma,t)\bigr)\,d\sigma\,dx \\
&\quad = -\mu(t)\int_0^1\int_0^1 \bigl(1-\tau'(t)\sigma\bigr)\frac{\partial}{\partial\sigma}\bigl[G\bigl(z(x,\sigma,t)\bigr)\bigr]\,d\sigma\,dx.
\end{aligned} \tag{3.11}$$

It follows that

$$\begin{aligned}
&\frac{d}{dt}\left(\mu(t)\tau(t)\int_0^1\int_0^1 G\bigl(z(x,\sigma,t)\bigr)\,d\sigma\,dx\right) \\
&\quad = -\mu(t)\int_0^1\int_0^1 \frac{\partial}{\partial\sigma}\bigl[\bigl(1-\tau'(t)\sigma\bigr)G\bigl(z(x,\sigma,t)\bigr)\bigr]\,d\sigma\,dx \\
&\qquad + \mu'(t)\tau(t)\int_0^l\int_0^1 G\bigl(z(x,\sigma,t)\bigr)\,d\sigma\,dx \\
&\quad = \mu(t)\int_0^1 \bigl(G\bigl(z(x,0,t)\bigr) - G\bigl(z(x,1,t)\bigr)\bigr)\,dx + \mu(t)\tau'(t)\int_0^1 G\bigl(z(x,1,t)\bigr)\,dx \\
&\qquad + \mu'(t)\tau(t)\int_0^1\int_0^1 G\bigl(z(x,\sigma,t)\bigr)\,d\sigma\,dx \\
&\quad = \mu(t)\int_0^1 G\bigl(u_t(x,t)\bigr)\,dx - \mu(t)\bigl(1-\tau'(t)\bigr)\int_0^1 G(z(x,1,t)\,dx \\
&\qquad + \mu'(t)\tau(t)\int_0^1\int_0^1 G\bigl(z(x,\sigma,t)\bigr)\,d\sigma\,dx.
\end{aligned} \tag{3.12}$$

Recalling the definition of the energy functional (3.2), and adding (3.8) and (3.12), we obtain

$$\begin{aligned}\frac{dE(t)}{dt} = &-\gamma_1 a(t)\int_0^1 u_t g_1(u_t)\,dx-\gamma_2 a(t)\int_0^1 u_t g_2\big(z(x,1,t)\big)\,dx\\&+\mu(t)\int_0^1 G\big(u_t(x,t)\big)\,dx-\mu(t)\big(1-\tau'(t)\big)\int_0^1 G(z(x,1,t)\,dx\\&-\gamma_3\int_0^1\varphi_t^2\,dx-\beta\int_0^1\theta_x^2\,dx+\mu'(t)\tau(t)\int_0^1\int_0^1 G\big(z(x,\sigma,t)\big)\,d\sigma\,dx.\end{aligned}\tag{3.13}$$

On the account of (A_1) and (2.5), we get

$$\begin{aligned}\frac{dE(t)}{dt}\le &-\big(\gamma_1 a(t)-\mu(t)\alpha_2\big)\int_0^1 u_t g_1(u_t)\,dx-\gamma_2 a(t)\int_0^1 u_t g_2\big(z(x,1,t)\big)\,dx\\&-\mu(t)\big(1-\tau'(t)\big)\int_0^1 G(z(x,1,t)\,dx-\gamma_3\int_0^1\varphi_t^2\,dx-\beta\int_0^1\theta_x^2\,dx.\end{aligned}\tag{3.14}$$

Now, we consider the convex conjugate of G defined by

$$G^*(s)=s\big(G'\big)^{-1}(s)-G\big(\big(G'\big)^{-1}(s)\big),\quad \forall s\ge 0,\tag{3.15}$$

which satisfies the generalized Young inequality (see [2])

$$AB\le G^*(A)+G(B),\quad \forall A,B>0.\tag{3.16}$$

Using (2.5) and the definition of G, we get

$$G^*(s)=sg_2^{-1}(s)-G\big(g_2^{-1}(s)\big),\quad \forall s\ge 0.\tag{3.17}$$

Therefore, on account of (2.5) and (3.17), we have

$$\begin{aligned}G^*\big(g_2\big(z(x,1,t)\big)\big)&=z(x,1,t)g_2\big(z(x,1,t)\big)-G\big(z(x,1,t)\big)\\&\le(1-\alpha_1)z(x,1,t)g_2\big(z(x,1,t)\big).\end{aligned}\tag{3.18}$$

A combination of (3.14), (3.16), and (3.18) leads to

$$\begin{aligned}\frac{dE(t)}{dt}&\le-\big(\gamma_1 a(t)-\mu(t)\alpha_2\big)\int_0^1 u_t g_1(u_t)\,dx\\&\quad+\gamma_2 a(t)\int_0^1\big(G(u_t)+G^*\big(g_2\big(z(x,1,t)\big)\big)\big)\,dx\\&\quad-\mu(t)\big(1-\tau'(t)\big)\int_0^1 G(z(x,1,t)\,dx-\gamma_3\int_0^1\varphi_t^2\,dx-\beta\int_0^1\theta_x^2\,dx\\&\le-\big(\gamma_1 a(t)-\mu(t)\alpha_2\big)\int_0^1 u_t g_1(u_t)\,dx+\gamma_2 a(t)\alpha_2\int_0^1 u_t g_1(u_t)\,dx\\&\quad+\gamma_2 a(t)(1-\alpha_1)\int_0^1 z(x,1,t)g_2\big(z(x,1,t)\big)\,dx\end{aligned}\tag{3.19}$$

$$
\begin{aligned}
&-\mu(t)\big(1-\tau'(t)\big)\int_0^1 G(z(x,1,t)\,dx-\gamma_3\int_0^1\varphi_t^2\,dx-\beta\int_0^1\theta_x^2\,dx\\
\le{}&-\big(\gamma_1 a(t)-\mu(t)\alpha_2-\gamma_2 a(t)\alpha_2\big)\int_0^1 u_t g_1(u_t)\,dx\\
&-\Big(\mu(t)\big(1-\tau'(t)\big)\alpha_1-\gamma_2 a(t)(1-\alpha_1)\Big)\int_0^1 z(x,1,t)g_2\big(z(x,1,t)\big)\,dx\\
&-\gamma_3\int_0^1\varphi_t^2\,dx-\beta\int_0^1\theta_x^2\,dx.
\end{aligned}
$$

Recalling that $\mu(t)=\bar{\mu}a(t)$, it follows from (3.19) that

$$
\begin{aligned}
\frac{dE(t)}{dt}\le{}&-a(t)[\gamma_1-\bar{\mu}\alpha_2-\gamma_2\alpha_2]\int_0^1 u_t g_1(u_t)\,dx\\
&-a(t)\big[\bar{\mu}\big(1-\tau'(t)\big)\alpha_1-\gamma_2(1-\alpha_1)\big]\int_0^1 z(x,1,t)g_2\big(z(x,1,t)\big)\,dx\\
&-\gamma_3\int_0^1\varphi_t^2\,dx-\beta\int_0^1\theta_x^2\,dx.
\end{aligned}
\tag{3.20}
$$

Using (2.9) and (3.1), we obtain the desired result. This finishes the proof. □

Lemma 3.2 *The functional F_1, defined by*

$$
F_1(t):=-\rho_2\rho_3\int_0^1\psi_t\int_0^x\theta(y,t)\,dy\,dx,
$$

satisfies, along the solution of system (2.14)–(2.16) and for any $\epsilon_1,\epsilon_2>0$, the estimate

$$
\begin{aligned}
F_1'(t)\le{}&-\frac{m\rho_2}{2}\int_0^1\psi_t^2\,dx+\epsilon_1\int_0^1\psi_x^2\,dx+\epsilon_2\int_0^1(\varphi_x+\psi)^2\,dx\\
&+c\left(1+\frac{1}{\epsilon_1}+\frac{1}{\epsilon_2}\right)\int_0^1\theta_x^2\,dx.
\end{aligned}
\tag{3.21}
$$

Proof Differentiating F_1, using $(2.14)_3$ and $(2.14)_4$, then integrating by parts and exploiting the boundary conditions lead to

$$
\begin{aligned}
F_1'(t)={}&b\rho_3\int_0^1\psi_x\theta\,dx+k\rho_3\int_0^1(\varphi_x+\psi)\int_0^x\theta(y,t)\,dy\,dx\\
&+m\rho_3\int_0^1\theta^2\,dx-\rho_2\beta\int_0^1\psi_t\theta_x\,dx-\rho_2 m\int_0^1\psi_t^2\,dx.
\end{aligned}
\tag{3.22}
$$

Making use of Cauchy–Schwarz, Young's, and Poincaré's inequalities, we get (3.21). □

Lemma 3.3 *The functional F_2, defined by*

$$
F_2(t):=\int_0^1\left(\rho u u_t+\rho_1\varphi\varphi_t+\rho_2\psi\psi_t+\frac{\gamma_3}{2}\varphi^2\right)dx,
$$

satisfies, along the solution of system (2.14)–(2.16), *the estimate*

$$\begin{aligned} F_2'(t) \leq & -\int_0^1 \left(\frac{\alpha}{2}u_x^2 + \lambda(\varphi - u)^2 + k(\varphi_x + \psi)^2 + \frac{b}{2}\psi_x^2\right) dx \\ & + \int_0^1 \left(\rho u_t^2 + \rho_1 \varphi_t^2 + \rho_2 \psi_t^2\right) dx + c\int_0^1 \theta_x^2 \, dx \\ & + c\int_0^1 \left|g_1(u_t)\right|^2 dx + c\int_0^1 \left|g_2\big(z(x,1,t)\big)\right|^2 dx, \quad \forall t \geq 0. \end{aligned} \tag{3.23}$$

Proof Directly differentiating F_2, using $(2.14)_1$, $(2.14)_2$, and $(2.14)_3$, then applying integration by parts and boundary conditions, we obtain

$$\begin{aligned} F_2'(t) = & -\int_0^1 \left(\alpha u_x^2 + \lambda(\varphi - u)^2 + k(\varphi_x + \psi)^2 + b\psi_x^2\right) dx \\ & + \int_0^1 \left(\rho u_t^2 + \rho_1 \varphi_t^2 + \rho_2 \psi_t^2\right) dx + m\int_0^1 \psi\theta_x \, dx \\ & - \gamma_1 a(t)\int_0^1 u g_1(u_t)\, dx - \gamma_2 a(t)\int_0^1 u g_2\big(z(x,1,t)\big)\, dx. \end{aligned} \tag{3.24}$$

Using (A_1), Young's and Poincaré's inequalities, we obtain (3.23). □

Lemma 3.4 *The functional*

$$F_3(t) := \bar{\mu}\tau(t)\int_0^1 \int_0^1 e^{-2\tau(t)\sigma} G\big(z(x,\sigma,t)\big)\, d\sigma\, dx,$$

satisfies, along the solution of system (2.14)–(2.16), *the estimate*

$$F_3'(t) \leq -2F_3(t) + \frac{\bar{\mu}\alpha_2}{2}\int_0^1 \left(u_t^2 + \left|g_1(u_t)\right|^2\right) dx, \quad \forall t \geq 0. \tag{3.25}$$

Proof Differentiating F_3, we get

$$\begin{aligned} F_3'(t) = & \bar{\mu}\tau'(t)\int_0^1 \int_0^1 e^{-2\tau(t)\sigma} G\big(z(x,\sigma,t)\big)\, d\sigma\, dx \\ & - 2\bar{\mu}\tau(t)\tau'(t)\int_0^1 \int_0^1 \sigma e^{-2\tau(t)\sigma} G\big(z(x,\sigma,t)\big)\, d\sigma\, dx \\ & + \bar{\mu}\tau(t)\int_0^1 \int_0^1 e^{-2\tau(t)\sigma} z_t(x,\sigma,t) g_2\big(z(x,\sigma,t)\big)\, d\sigma\, dx. \end{aligned} \tag{3.26}$$

Using the last equation in (2.14), we can express the last term on the right hand-side of (3.26) as

$$\begin{aligned} & \tau(t)\int_0^1 \int_0^1 e^{-2\tau(t)\sigma} z_t(x,\sigma,t) g_2\big(z(x,\sigma,t)\big)\, d\sigma\, dx \\ & = \int_0^1 \int_0^1 e^{-2\tau(t)\sigma}\big(\tau'(t)\sigma - 1\big) z_\sigma(x,\sigma,t) g_2\big(z(x,\sigma,t)\big)\, d\sigma\, dx \end{aligned}$$

$$\begin{aligned}
&= \int_0^1\int_0^1 \frac{\partial}{\partial\sigma}\bigl[e^{-2\tau(t)\sigma}\bigl(\tau'(t)\sigma-1\bigr)G\bigl(z(x,\sigma,t)\bigr)\bigr]\,d\sigma\,dx \\
&\quad + 2\tau(t)\int_0^1\int_0^1 e^{-2\tau(t)\sigma}\bigl(\tau'(t)\sigma-1\bigr)G\bigl(z(x,\sigma,t)\bigr)\,d\sigma\,dx \\
&\quad - \tau'(t)\int_0^1\int_0^1 e^{-2\tau(t)\sigma}G\bigl(z(x,\sigma,t)\bigr)\,d\sigma\,dx \\
&= -\bigl(1-\tau'(t)\bigr)e^{-2\tau(t)}\int_0^1 G\bigl(z(x,1,t)\bigr)\,dx + \int_0^l G(u_t)\,dx \\
&\quad + 2\tau(t)\int_0^1\int_0^1 e^{-2\tau(t)\sigma}\bigl(\tau'(t)\sigma-1\bigr)G\bigl(z(x,\sigma,t)\bigr)\,d\sigma\,dx \\
&\quad - \tau'(t)\int_0^1\int_0^1 e^{-2\tau(t)\sigma}G\bigl(z(x,\sigma,t)\bigr)\,d\sigma\,dx.
\end{aligned} \tag{3.27}$$

Substituting (3.27) into (3.26), we arrive at

$$\begin{aligned}
F_3'(t) = {}& -2\bar{\mu}\tau(t)\int_0^1\int_0^1 e^{-2\tau(t)\sigma}G\bigl(z(x,\sigma,t)\bigr)\,d\sigma\,dx + \bar{\mu}\int_0^1 G(u_t)\,dx \\
& - \bigl(1-\tau'(t)\bigr)e^{-2\tau(t)}\int_0^1 G\bigl(z(x,1,t)\bigr)\,dx.
\end{aligned} \tag{3.28}$$

Using condition (2.5) and Young's inequality, we obtain (3.26). □

Lemma 3.5 *Let $(u,\varphi,\psi,\theta,z)$ be the solution of system (2.14)–(2.16). Then, for $N,N_1,N_2>0$ sufficiently large, the Lyapunov functional L, defined by*

$$L(t) := NE(t) + N_1F_1(t) + N_2F_2(t) + F_3(t), \tag{3.29}$$

satisfies, for some positive constants c_1, c_2, η,

$$c_1E(t) \le L(t) \le c_2E(t), \quad \forall t\ge 0, \tag{3.30}$$

and

$$L'(t) \le -\eta E(t) + c\int_0^1\bigl(u_t^2 + \bigl|g_1(u_t)\bigr|^2\bigr)\,dx + c\int_0^1\bigl|g_2\bigl(z(x,1,t)\bigr)\bigr|^2\,dx, \quad \forall t\ge 0. \tag{3.31}$$

Proof Applying Cauchy–Schwarz, Young's, and Poincaré's inequalities, we have

$$\begin{aligned}
\bigl|L(t)-NE(t)\bigr| \le {}& N_1\left|-\rho_2\int_0^1 \psi_t\int_0^x \theta(y,t)\,dy\,dx\right| \\
& + N_2\left|\int_0^1\left(\rho u u_t + \rho_1\varphi\varphi_t + \rho_2\psi\psi_t + \frac{\gamma_3}{2}\varphi^2\right)dx\right| \\
& + \left|\bar{\mu}\tau(t)\int_0^1\int_0^1 e^{-2\tau(t)\sigma}G\bigl(z(x,\sigma,t)\bigr)\,d\sigma\,dx\right| \\
\le {}& \frac{N_2\rho}{2}\int_0^1 u_t^2\,dx + \frac{N_2\rho_1}{2}\int_0^1\varphi_t^2\,dx + \frac{(N_1+N_2)\rho_2}{2}\int_0^1\psi_t^2\,dx
\end{aligned} \tag{3.32}$$

$$
\begin{aligned}
&+\frac{N_2\gamma_3}{2}\int_0^1 \varphi^2\,dx+\frac{N_2\rho}{2}\int_0^1 u_x^2\,dx+\frac{N_2\rho_1}{2}\int_0^1 \varphi_x^2\,dx\\
&+\frac{N_2\rho_2}{2}\int_0^1 \psi_x^2\,dx+\frac{N_1\rho_2}{2}\int_0^1\left(\int_0^x \theta(y,t)\,dy\right)^2 dx\\
&+\bar{\mu}\tau(t)\int_0^1\int_0^1 G\bigl(z(x,\sigma,t)\bigr)\,d\sigma\,dx.
\end{aligned}
$$

Using the relations

$$
\begin{aligned}
&\int_0^1 \varphi^2\,dx\le 2\int_0^l(\varphi-u)^2\,dx+2\int_0^1 u_x^2\,dx,\\
&\int_0^1 \varphi_x^2\,dx\le 2\int_0^1(\varphi_x+\psi)^2\,dx+2\int_0^1 \psi_x^2\,dx,
\end{aligned}
$$

we arrive at

$$
\begin{aligned}
\bigl|L(t)-NE(t)\bigr|\le{}&\frac{N_2\rho}{2}\int_0^1 u_t^2\,dx+\frac{N_2\rho_1}{2}\int_0^1 \varphi_t^2\,dx+\frac{(N_1+N_2)\rho_2}{2}\int_0^1 \psi_t^2\,dx\\
&+N_2\gamma_3\int_0^l(\varphi-u)^2\,dx+\left(N_2\gamma_3+\frac{N_2\rho}{2}\right)\int_0^1 u_x^2\,dx\\
&+N_2\rho_1\int_0^1(\varphi_x+\psi)^2\,dx+\left(\frac{N_2(\rho_1+\rho_2)}{2}\right)\int_0^1 \psi_x^2\,dx\\
&+\frac{N_1\rho_2}{2}\int_0^1 \theta^2\,dx+\bar{\mu}\tau(t)\int_0^1\int_0^1 G\bigl(z(x,\sigma,t)\bigr)\,d\sigma\,dx.
\end{aligned}
\tag{3.33}
$$

From (3.33), we obtain

$$
\bigl|L(t)-NE(t)\bigr|\le\bar{c}E(t). \tag{3.34}
$$

By choosing N large enough such that

$$
c_1=N-\bar{c}>0,\qquad c_2=N+\bar{c}>0, \tag{3.35}
$$

estimate (4.14) follows. Next, we establish (3.31). Using Lemmas 3.1–3.4, we get

$$
\begin{aligned}
L'(t)\le{}&-\rho\int_0^1 u_t^2\,dx-[N\gamma_3-N_2\rho_1]\int_0^1 \varphi_t^2\,dx-\left[N_1\frac{m\rho_2}{2}-N_2\rho_2\right]\int_0^1 \psi_t^2\,dx\\
&-\frac{N_2\alpha}{2}\int_0^1 u_x^2\,dx-N_2\lambda\int_0^1(\varphi-u)^2\,dx-[N_2k-N_1\epsilon_2]\int_0^1(\varphi_x+\psi)^2\,dx\\
&-\left[N_2\frac{b}{2}-N_1\epsilon_1\right]\int_0^1 \psi_x^2\,dx-\left[N\beta-N_1c\left(1+\frac{1}{\epsilon_1}+\frac{1}{\epsilon_2}\right)-N_2c\right]\int_0^1 \theta_x^2\,dx\\
&-\frac{2e^{-2\tau_1}}{a(0)}\mu(t)\tau(t)\int_0^1\int_0^1 G\bigl(z(x,\sigma,t)\bigr)\,d\sigma\,dx+\left[\rho+N_2\rho+\frac{\bar{\mu}\alpha_2}{2}\right]\int_0^1 u_t^2\,dx\\
&+\left[cN_2+\frac{\bar{\mu}\alpha_2}{2}\right]\int_0^1\bigl|g_1(u_t)\bigr|^2\,dx+cN_2\int_0^1\bigl|g_2\bigl(z(x,1,t)\bigr)\bigr|^2\,dx.
\end{aligned}
$$

Choosing

$$N_2 = 1, \qquad \epsilon_1 = \frac{N_2 b}{4N_1}, \qquad \epsilon_2 = \frac{N_2 k}{2N_1},$$

we arrive at

$$\begin{aligned} L'(t) \leq & -\rho \int_0^1 u_t^2 \, dx - [N\gamma_3 - \rho_1] \int_0^1 \varphi_t^2 \, dx - \left[N_1 \frac{m\rho_2}{2} - \rho_2\right] \int_0^1 \psi_t^2 \, dx \\ & - \frac{\alpha}{2} \int_0^1 u_x^2 \, dx - \lambda \int_0^1 (\varphi - u)^2 \, dx - \frac{k}{2} \int_0^1 (\varphi_x + \psi)^2 \, dx \\ & - \frac{b}{4} \int_0^1 \psi_x^2 \, dx - \left[N\beta - N_1 c\left(1 + \frac{4N_1}{b} + \frac{2N_1}{k}\right) - c\right] \int_0^1 \theta_x^2 \, dx \\ & - \frac{2e^{-2\tau_1}}{a(0)} \mu(t)\tau(t) \int_0^1 \int_0^1 G\big(z(x,\sigma,t)\big) \, d\sigma \, dx + \left[2\rho + \frac{\bar{\mu}\alpha_2}{2}\right] \int_0^1 u_t^2 \, dx \\ & + \left[c + \frac{\bar{\mu}\alpha_2}{2}\right] \int_0^1 \left|g_1(u_t)\right|^2 dx + c \int_0^1 \left|g_2\big(z(x,1,t)\big)\right|^2 dx. \end{aligned} \tag{3.36}$$

Now, we choose N_1 large such that

$$N_1 \frac{m\rho_2}{2} - \rho_2 > 0.$$

Next, we select N very large so that (4.14) remains true and

$$N\gamma_3 - \rho_1 > 0, \qquad N\beta - N_1 c\left(1 + \frac{4N_1}{b} + \frac{2N_1}{k}\right) - c > 0.$$

Therefore, using the energy functional defined by (3.2), we obtain (3.31). □

4 Stability result

In this section, we are concerned with the main stability result, and is stated as follows.

Theorem 4.1 *Let $(u, \varphi, \psi, \theta, z)$ be the solution of system (2.14)–(2.16) and assume (A_1)–(A_4) hold. Then, for some positive constants δ_1, δ_2, δ_3, and r_0, the energy functional (3.2) satisfies*

$$E(t) \leq \delta_1 \chi_1^{-1}\left(\delta_2 \int_0^t a(s) \, ds + \delta_3\right), \quad t \geq 0, \tag{4.1}$$

where

$$\chi_1(t) = \int_t^1 \frac{1}{\chi_0(s)} \, ds \quad \text{and} \quad \chi_0(t) = t\chi'(r_0 t).$$

Proof We divide the proof into two cases:

Case I: χ is linear. Using (A_2), we get

$$C_1|s| \leq \left|g_1(s)\right| \leq C_2|s|, \quad \forall s \in \mathbb{R}.$$

Thus,

$$g_1^2(s) \leq C_2 s g_1(s), \quad \forall s \in \mathbb{R}. \tag{4.2}$$

Therefore, multiplying (3.31) by $a(t)$ and using (3.3) and (4.2), we conclude that

$$\begin{aligned} a(t)L'(t) &\leq -\eta a(t)E(t) + ca(t)\int_0^1 u_t g_1(u_t)\,dx + ca(t)\int_0^1 z(x,1,t)g_2\big(z(x,1,t)\big)\,dx \\ &\leq -\eta a(t)E(t) - cE'(t), \quad \forall t \in \mathbb{R}^+. \end{aligned}$$

Exploiting (A_2) and (3.30), it follows that

$$L_0(t) := a(t)L(t) + cE(t) \sim E(t) \tag{4.3}$$

and, for some constant $\eta_1 > 0$, the functional L_0 satisfies

$$L_0'(t) \leq -\eta_1 a(t)L_0(t), \quad \forall t \geq 0. \tag{4.4}$$

A simple integration of (4.4) over $(0,t)$, using (4.3), yields

$$E(t) \leq \delta_1 \exp\left(-\delta_2 \int_0^t a(s)\,ds\right) = \delta_1 \chi_1^{-1}\left(\delta_2 \int_0^t a(s)\,ds\right), \quad \forall t \geq 0. \tag{4.5}$$

Case II: χ is nonlinear on $[0,r]$. Here, as in [22], we select $0 < r_1 \leq r$ so that

$$sg_1(s) \leq \min\{r, \chi(r)\}, \quad \forall |s| \leq r_1. \tag{4.6}$$

On account of (A_2) and the continuity of g_1 with the fact that $|g_1(s)| > 0$, for $s \neq 0$, we conclude that

$$\begin{cases} s^2 + g_1^2(s) \leq \chi^{-1}(sg_1(s)), & \forall |s| \leq r_1, \\ C_1'|s| \leq |g_1(s)| \leq C_2'|s|, & \forall |s| \geq r_1. \end{cases} \tag{4.7}$$

Now, we introduce the following partitions:

$$\begin{aligned} &I_1 = \{x \in (0,1) : |u_t| \leq r_1\}, \qquad I_2 = \{x \in (0,1) : |u_t| > r_1\}, \\ &\tilde{I}_1 = \{x \in (0,1) : |z(x,1,t)| \leq r_1\}, \qquad \tilde{I}_2 = \{x \in (0,1) : |z(x,1,t)| > r_1\} \end{aligned}$$

and the functional h, defined by

$$h(t) = \int_{I_1} u_t g_1(u_t)\,dx.$$

Using the fact that χ^{-1} is concave and Jensen's inequality, it follows that

$$\chi^{-1}\big(h(t)\big) \geq c\int_{I_1} \chi^{-1}\big(u_t g_1(u_t)\big)\,dx. \tag{4.8}$$

Combining (4.7) and (4.8), we have

$$\begin{aligned} a(t)\int_0^1 \big(u_t^2+g_1^2(u_t)\big)\,dx &= a(t)\int_{I_1}\big(u_t^2+g_1^2(u_t)\big)\,dx + a(t)\int_{I_2}\big(u_t^2+g_1^2(u_t)\big)\,dx \\ &\le a(t)\int_{I_1}\chi^{-1}\big(u_t g_1(u_t)\big)\,dx + ca(t)\int_{I_2} u_t g_1(u_t)\,dx \\ &\le ca(t)\chi^{-1}\big(h(t)\big) - cE'(t). \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} a(t)\int_0^1 g_2^2\big(z(x,1,t)\big)\,dx &= a(t)\int_{\bar{I}_1} g_2^2\big(z(x,1,t)\big)\,dx + a(t)\int_{\bar{I}_2} g_2^2\big(z(x,1,t)\big)\,dx \\ &\le ca(t)\int_{\bar{I}_1} z(x,1,t)g_2\big(z(x,1,t)\big)\,dx \\ &\quad + a(t)\int_{\bar{I}_2} z(x,1,t)g_2\big(z(x,1,t)\big)\,dx \\ &\le -cE'(t). \end{aligned} \tag{4.10}$$

Multiplying (3.31) by $a(t)$ and using (4.9) and (4.10), we obtain

$$a(t)L'(t) + cE'(t) \le -\eta a(t)E(t) + ca(t)\chi^{-1}\big(h(t)\big). \tag{4.11}$$

It follows from (A_1) that

$$L_1'(t) \le -\eta a(t)E(t) + ca(t)\chi^{-1}\big(h(t)\big), \tag{4.12}$$

where

$$L_1(t) = a(t)L(t) + cE(t) \sim E(t) \quad \text{by virtue of (3.30).} \tag{4.13}$$

Let $r_0 < r$ and $\eta_0 > 0$ to be specified later. Then, combining (4.12) and the fact that

$$E' \le 0, \qquad \chi' > 0, \qquad \chi'' > 0 \quad \text{on } (0,r],$$

the functional L_2, defined by

$$L_2(t) := \chi'\left(r_0\frac{E(t)}{E(0)}\right)L_1(t) + \eta_0 E(t),$$

satisfies

$$\kappa_1 L_2(t) \le E(t) \le \kappa_2 L_2(t) \tag{4.14}$$

for some positive constants κ_1, κ_2, and

$$L_2'(t) = r_0\frac{E'(t)}{E(0)}\chi''\left(r_0\frac{E(t)}{E(0)}\right)L_1(t) + \chi'\left(r_0\frac{E(t)}{E(0)}\right)L_1'(t) + \eta_0 E'(t)$$

$$\leq -\eta a(t)E(t)\chi'\left(r_0\frac{E(t)}{E(0)}\right) + \underbrace{ca(t)\chi'\left(r_0\frac{E(t)}{E(0)}\right)\chi^{-1}\big(h(t)\big)}_{A} + \eta_0 E'(t). \tag{4.15}$$

To estimate the term A in (4.15), we consider the convex conjugate of χ denoted by χ^*, defined by

$$\chi^*(y) = y\big(\chi'\big)^{-1}(y) - \chi\big[\big(\chi'\big)^{-1}(y)\big] \leq y\big(\chi'\big)^{-1}(y), \quad \text{if } y \in (0, \chi'(r)], \tag{4.16}$$

and which satisfies the generalized Young's inequality

$$XY \leq \chi^*(X) + \chi(Y), \quad \text{if } X \in \big(0, \chi'(r)\big], Y \in (0, r]. \tag{4.17}$$

Taking $X = \chi'(r_0\frac{E(t)}{E(0)})$ and $Y = \chi^{-1}(h(t))$ and recalling Lemma 3.1 and (4.6), then (4.15)–(4.17) lead to

$$\begin{aligned}
L_2'(t) \leq & -\eta a(t)E(t)\chi'\left(r_0\frac{E(t)}{E(0)}\right) + ca(t)\left[\chi * \left(\chi'\left(r_0\frac{E(t)}{E(0)}\right)\right) + \chi\big(\chi^{-1}\big(h(t)\big)\big)\right] \\
& + \eta_0 E'(t) \\
= & -\eta a(t)E(t)\chi'\left(r_0\frac{E(t)}{E(0)}\right) + ca(t)\chi * \left(\chi'\left(r_0\frac{E(t)}{E(0)}\right)\right) \\
& + ca(t)h(t) + \eta_0 E'(t) \\
\leq & -\eta a(t)E(t)\chi'\left(r_0\frac{E(t)}{E(0)}\right) + cr_0 a(t)\left(\frac{E(t)}{E(0)}\right)\chi'\left(r_0\frac{E(t)}{E(0)}\right) \\
& - cE'(t) + \eta_0 E'(t) \\
\leq & -\big(\eta E(0) - cr_0\big)a(t)\left(\frac{E(t)}{E(0)}\right)\chi'\left(r_0\frac{E(t)}{E(0)}\right) + (\eta_0 - c)E'(t).
\end{aligned} \tag{4.18}$$

By choosing $r_0 = \frac{\eta E(0)}{2c}$, $\eta_0 = 2c$, and recalling that $E'(t) \leq 0$, we arrive at

$$L_2'(t) \leq -\eta_1 a(t)\frac{E(t)}{E(0)}\chi'\left(r_0\frac{E(t)}{E(0)}\right) = -\eta_1 a(t)\chi_0\left(\frac{E(t)}{E(0)}\right), \tag{4.19}$$

where $\eta_1 > 0$ and $\chi_0(t) = t\chi'(r_0 t)$. Now, since χ is strictly convex on $(0, r]$, we conclude that $\chi_0(t) > 0$, $\chi_0'(t) > 0$ on $(0, 1]$. Using (4.14) and (4.19), it follows that the functional

$$L_3(t) = \frac{\kappa_1 L_2(t)}{E(0)}$$

satisfies

$$L_3(t) \sim E(t) \tag{4.20}$$

and, for some $\delta_2 > 0$,

$$L_3'(t) \leq -\delta_2 a(t)\chi_0\big(L_3(t)\big), \tag{4.21}$$

which yields

$$\left[\chi_1\left(L_3(t)\right)\right]' \ge \delta_2 a(t), \tag{4.22}$$

where

$$\chi_1(t) = \int_t^1 \frac{1}{\chi_0(s)}\,ds, \quad t \in (0,1].$$

Integrating (4.22) over $[0,t]$, keeping in mind the properties of χ_0, and the fact that χ_1 is strictly decreasing on $(0,1]$, we obtain

$$L_3(t) \le \chi_1^{-1}\left(\delta_2 \int_0^t a(s)\,ds + \delta_3\right), \quad \forall t \in \mathbb{R}^+, \tag{4.23}$$

for some $\delta_3 > 0$. Using (4.20) and (4.23), the proof of Theorem 4.1 is completed. □

5 Examples

We end this section by giving some examples to illustrate the obtained result.

Let

$$g_0 \in C^2\left([0,+\infty)\right)$$

be a strictly increasing function such that $g_0(0) = 0$ and, for some positive constants c_1, c_2 and r, the function g_1 satisfies

$$\begin{aligned} &g_0\left(|s|\right) \le \left|g_1(s)\right| \le g_0^{-1}\left(|s|\right), \quad \forall |s| \le r, \\ &c_1 s^2 \le s g_1(s) \le c_2 s^2, \quad \forall |s| \ge r. \end{aligned} \tag{5.1}$$

We consider the function

$$\chi(s) = \left(\sqrt{\frac{s}{2}}\right) g_0\left(\sqrt{\frac{s}{2}}\right). \tag{5.2}$$

It follows that χ is a C^2-strictly convex function on $(0,r]$ when g_0 is nonlinear and therefore satisfies condition (A_2). Now, we give some examples of g_0 such that g_1 satisfies (5.1) near 0.

1. Let $g_0(s) = \lambda s$, where $\lambda > 0$ a constant, then $\chi(s) = \bar{\lambda} s$, where $\bar{\lambda} = \frac{\lambda}{2}$ satisfies (A_2) near 0 and from (4.1), we get

$$E(t) \le \bar{\delta} \exp\left(-\delta_2 \int_0^t a(s)\,ds\right), \quad \forall t \ge 0.$$

2. Let $g_0(s) = \frac{1}{s} e^{-\frac{1}{s^2}}$, then $\chi(s) = e^{-\frac{2}{s}}$ satisfies (A_2) in the neighborhood of 0 and from (4.1), we obtain

$$E(t) \le \delta_1 \left(\ln\left(\delta_2 \int_0^t a(s)\,ds + \delta_3\right)\right)^{-1}, \quad \forall t \ge 0. \tag{5.3}$$

3. Let $g_0(s) = e^{-\frac{1}{s}}$, then $\chi(s) = \sqrt{\frac{s}{2}}e^{-\sqrt{\frac{2}{s}}}$ satisfies (A_2) near 0 and using (4.1), we obtain

$$E(t) \leq \delta_1 \left(\ln\left(\delta_2 \int_0^t a(s)\, ds + \delta_3 \right) \right)^{-2}, \quad \forall t \geq 0. \tag{5.4}$$

6 Conclusion

In this work, we obtained some general decay results for a thermoelastic Timoshenko beam system with suspenders, general weak internal damping, time-varying coefficient, and time-varying delay terms. The damping structure in system (2.14)–(2.16) is sufficient enough to stabilize the system without any additional conditions on the coefficient parameters as it is the case with many Timoshenko beam systems in the literature. The result of the present paper generalizes the one established in Bochichio et al. [6] and allows a large class of functions that satisfy condition (A_2). We also gave some examples to illustrate our theoretical finding.

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Declarations

Competing interests
The authors declare no competing interests.

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