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Exponential stabilisation of highly nonlinear neutral stochastic systems by variable-delay feedback control

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Abstract

This paper is devoted to dealing with the exponential stabilisation of highly nonlinear hybrid neutral stochastic differential equation (NSDE) by variable delay feedback control. There have been so far few results on the stabilization of variable delay stochastic systems by variable delay feedback control, although the stabilisation of neutral stochastic systems on the current state has been investigated. Finally, an illustrative example is provided for verifying our theoretical results.

Keywords: Neutral stochastic systems; Highly nonlinear systems; Delay feedback control; Exponential stability; Lyapunov functional

1 Introduction

Many dynamical systems not only depend on present and past states but also involve derivatives with delays. Neutral stochastic differential equations (NSDEs) are often used to describe such systems. Motivated by wide applications in engineering and sciences, problems of stability of NSDEs have attracted extensive attention from researchers. For example, Kolmanovskii et al. [1] established some fundamental theories for NSDEs with Markovian switching. The almost surely asymptotic stability for a class of hybrid neutral stochastic differential delay equations (NSDDEs) was investigated by Mao et al. [2]. Chen et al. [3] studied exponential stability for NSDEs with time-varying delay. Mao and Mao [4] investigated the existence and uniqueness of solutions to neutral stochastic functional differential equations with Lévy jumps. Shen et al. [5] explored the boundedness and stability of highly nonlinear NSDEs with multiple delays. Li and Deng [6] discussed almost sure stability with general decay rate of highly nonlinear NSDEs with Lévy noise. Some new criteria for the mean square exponential stability of neutral stochastic functional differential equations were given by Ngoc [7]. One of the important issues in the study of the stability of NSDEs is the design of feedback control. There are also a large number of results on stabilisation for stochastic delay systems in the previous literature. The pioneering work of delay feedback control was due to Mao et al. [8]. Since then some further developments have been made (see, e.g., [9–16]). A common feature of these existing in this area is that the coefficients are either linear or nonlinear but bounded by linear functions.

However, in the real world, many stochastic differential equations (SDEs) do not satisfy linear growth condition (see, e.g., [17–24]). Recently, the work of Lu et al. [25] is the first to study delay feedback control for highly nonlinear hybrid SDEs. It was later extended to the stochastic differential delay equations (SDDEs) by Li et al. [26]. Shen et al. [27] is the first to discuss delay feedback control of highly nonlinear NSDEs. However, the results in the paper of Shen et al. [27] only can be applied to neutral stochastic systems with constant delay. Moreover, they did not discuss the convergence rate of the solution. In this paper, we will extend the work of Shen et al. [27] to highly nonlinear NSDEs with variable delays for obtaining the exponential stabilisation criterion.

The structure of the paper is arranged as follows. In Sect. 2, some hypotheses are given. The main results are discussed in Sect. 3. An example is given to illustrate the effectiveness of our theory in Sect. 4, while the conclusion is made in Sect. 5.

Notations Throughout this paper, unless otherwise specified, we use the following notation. If A is a vector or matrix, then its transpose is denoted by A^T. If $x \in R^n$, then $|x|$ is its Euclidean norm. If A is a matrix, then we let $|A| = \sqrt{\operatorname{trace}(A^TA)}$ be its trace norm. Let $R_+ = [0,\infty)$. For $\delta > 0$, denote by $C([-\delta,0];R^n)$ the family of continuous functions φ from $[-\delta,0] \to R^n$ with the norm $\|\varphi\| = \sup_{-\delta\le s\le 0}|\varphi(s)|$. Let $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\ge 0},P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\ge 0}$ satisfying the usual conditions. Let $B(t) = (B_1(t),\dots,B_m(t))^T$ be an m-dimensional Brownian motion defined on the probability space. Let $\delta(t)$ be a differentiable function from $R_+ \to [0,\delta]$ such that $\dot{\delta}(t) := d\delta(t)/dt \le \bar{\delta}$ for all $t \ge 0$, where $\bar{\delta} \in [0,1)$. Let $r(t)$ be a right-continuous Markov chain on the probability space taking values in a finite state space $S = \{1,2,\dots,N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$, given by

$$P\{r(t+\Delta) = j \mid r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & \text{if } i \ne j, \\ 1 + \gamma_{ii}\Delta + o(\Delta) & \text{if } i = j, \end{cases}$$

where $\Delta > 0$. Here $\gamma_{ij} \ge 0$ is the transition rate from i to j if $i \ne j$ while $\gamma_{ii} = -\sum_{j\ne i}\gamma_{ij}$. We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $B(\cdot)$.

Let $C^{2,1}(R^n \times S \times R_+;R_+)$ denote the family of nonnegative functions $U(x,i,t)$ defined on $(x,i,t) \in R^n \times S \times R_+$, which are continuously twice differentiable in x and once in t. For such a function U, we will let $U_t = \frac{\partial U}{\partial t}$, $U_x = (\frac{\partial U}{\partial x_1},\dots,\frac{\partial U}{\partial x_n})$, and $U_{xx} = (\frac{\partial^2 U}{\partial x_k \partial x_l})_{n\times n}$.

2 Hypotheses

Suppose that the unstable system is described by the hybrid NSDE

$$\begin{aligned} d\big[X(t) - D\big(X\big(t-\delta(t)\big)\big)\big] ={}& f\big(X(t), X\big(t-\delta(t)\big), r(t), t\big)\,dt \\ &+ g\big(X(t), X\big(t-\delta(t)\big), r(t), t\big)\,dB(t) \end{aligned}$$

on $t \ge 0$ with the initial data

$$X(0) = \xi \in C\big([-\delta,0];R^n\big) \quad \text{and} \quad r(0) = i_0 \in S, \tag{2.1}$$

where $f : R^n \times R^n \times S \times R_+ \to R^n$ and $g : R^n \times R^n \times S \times R_+ \to R^{n\times m}$ are Borel measurable functions. We are required to design a variable delay feedback control $u(X(t-\tau(t)), r(t), t)$

so that the corresponding controlled system

$$d\big[X(t)-D\big(X\big(t-\delta(t)\big)\big)\big]=\big[f\big(X(t),X\big(t-\delta(t)\big),r(t),t\big)+u\big(X\big(t-\tau(t)\big),r(t),t\big)\big]\,dt \\ +g\big(X(t),X\big(t-\delta(t)\big),r(t),t\big)\,dB(t) \tag{2.2}$$

becomes exponentially stable. We assume that the controller function $u: R^n \times S \times R_+ \to R^n$ is Borel measurable and $\tau(t)$ is a differentiable function from $R_+ \to [0,\tau]$. To make our feedback control analysis more understandable, we will only consider the case where $d\tau(t)/dt \le \bar{\delta}$ and $\tau \le \delta$.

The classical conditions for the existence and uniqueness of the global solution are the local Lipschitz condition and the linear growth condition. In this paper, we need the local Lipschitz condition. Moreover, we impose the following polynomial growth condition instead of the linear growth condition.

Assumption 2.1 Assume that for any $h>0$ there exists a positive constant K_h such that

$$\big|f(x,y,i,t)-f(\bar{x},\bar{y},i,t)\big| \vee \big|g(x,y,i,t)-g(\bar{x},\bar{y},i,t)\big| \le K_h\big(|x-\bar{x}|+|y-\bar{y}|\big)$$

for all $x,\bar{x},y,\bar{y}\in R^n$ with $|x|\vee|\bar{x}|\vee|y|\vee|\bar{y}|\le h$ and all $t\in R_+$. Assume, moreover, that there exist three constants $K>0$, $q_1\ge 1$, and $q_2\ge 1$ such that

$$\begin{aligned} &\big|f(x,y,i,t)\big| \le K\big(|x|+|x|^{q_1}+|y|+|y|^{q_1}\big),\\ &\big|g(x,y,i,t)\big| \le K\big(|x|+|x|^{q_2}+|y|+|y|^{q_2}\big) \end{aligned} \tag{2.3}$$

for all $(x,y,i,t)\in R^n\times R^n\times S\times R_+$.

Assumption 2.2 We assume that there exists a constant $\kappa\in(0,1)$ such that

$$\big|D(a)-D(b)\big| \le \kappa|a-b| \tag{2.4}$$

for all $a,b\in R^n$ and $D(0)=0$.

We emphasise that we are interested in highly nonlinear NSDEs which have either $q_1>1$ or $q_2>1$ in Assumption 2.1 in this paper. We will refer to condition (2.3) as the polynomial growth condition.

As a standing hypothesis of this paper, we assume that both coefficients f and g are sufficiently smooth so that NSDE (2.2) with the initial data (2.1) has the unique global solution $X(t)$ on $t\ge-\delta$ and, moreover, there is a constant $q\ge 2$ such that

$$\sup_{-\delta\le t<\infty} E\big|X(t)\big|^q<\infty. \tag{2.5}$$

For further information on this hypothesis, we refer the reader to the work of Shen et al. [5].

Assumption 2.3 Assume that there exists a positive number β such that

$$\big|u(x,i,t)-u(y,i,t)\big| \le \beta|x-y| \tag{2.6}$$

for all $(x,i,t) \in R^n \times S \times R_+$. Moreover, for the purpose of stability, we require that $u(0,i,t) = 0$.

3 Main results

In this section, we define two segments $\hat{X}_t := \{X(t+s) : -2\delta \le s \le 0\}$ and $\hat{r}_t := \{r(t+s) : -2\delta \le s \le 0\}$ for $t \ge 0$. For $\hat{X}_t$ and $\hat{r}_t$ to be well defined for $0 \le t < 2\delta$, we set $X(s) = \xi(-\delta)$ for $s \in [-2\delta, -\delta)$ and $r(s) = r_0$ for $s \in [-2\delta, 0)$. The Lyapunov functional defined in this paper will be in the form of

$$V(\hat{X}_t, \hat{r}_t, t) = \bar{U}\bigl(X(t) - D\bigl(X\bigl(t-\delta(t)\bigr)\bigr), r(t), t\bigr) + \theta \int_{-\tau}^{0} \int_{t+s}^{t} \Phi(v)\, dv\, ds \tag{3.1}$$

for $t \ge 0$, where $\bar{U} \in C^{2,1}(R^n \times S \times R_+; R_+)$ such that

$$\lim_{|x|\to\infty} \Bigl[\inf_{(i,t)\in S\times R_+} \bar{U}(x,i,t)\Bigr] = \infty,$$

θ is a positive number to be determined later, and

$$\begin{aligned}\Phi(t) = {}& \tau \bigl|f\bigl(X(t), X\bigl(t-\delta(t)\bigr), r(t), t\bigr) + u\bigl(X\bigl(t-\tau(t)\bigr), r(t), t\bigr)\bigr|^2 \\ & + \bigl|g\bigl(X(t), X\bigl(t-\delta(t)\bigr), r(t), t\bigr)\bigr|^2.\end{aligned}$$

By the generalized Itô formula (see, e.g., [28]) and the fundamental theory of calculus, we can have the following lemma.

Lemma 3.1 *With the notation above, $V(\hat{X}_t, \hat{r}_t, t)$ is an Itô process on $t \ge 0$ with its Itô differential*

$$dV(\hat{X}_t, \hat{r}_t, t) = LV(\hat{X}_t, \hat{r}_t, t)\, dt + dM(t), \tag{3.2}$$

where $M(t)$ is a continuous local martingale with $M(0) = 0$ [28, Theorem 1.45 on page 48], and

$$\begin{aligned} & LV(\hat{X}_t, \hat{r}_t, t) \\ &\quad = \bar{U}_t\bigl(X(t) - D\bigl(X(t) - \delta(t)\bigr), r(t), t\bigr) \\ &\qquad + \bar{U}_x\bigl(X(t) - D\bigl(X(t) - \delta(t)\bigr), r(t), t\bigr)\bigl[u\bigl(X\bigl(t-\tau(t)\bigr), r(t), t\bigr) - u\bigl(X(t), r(t), t\bigr)\bigr] \\ &\qquad + \mathcal{L}\bar{U}\bigl(X(t), X\bigl(t-\delta(t)\bigr), r(t), t\bigr) + \theta\tau\Phi(t) - \theta \int_{t-\tau}^{t} \Phi(v)\, dv, \end{aligned} \tag{3.3}$$

in which $\mathcal{L}\bar{U} : R^n \times R^n \times S \times R_+ \to R$ is defined by

$$\begin{aligned} \mathcal{L}\bar{U}(x,y,i,t) = {}& \bar{U}_t\bigl(x - D(y), i, t\bigr) + \bar{U}_x\bigl(x - D(y), i, t\bigr)\bigl[f(x,y,i,t) + u(x,i,t)\bigr] \\ & + \frac{1}{2}\operatorname{trace}\bigl[g^T(x,y,i,t)\bar{U}_{xx}\bigl(x - D(y), i, t\bigr)g(x,y,i,t)\bigr] \\ & + \sum_{j=1}^{N} \gamma_{ij}\bar{U}\bigl(x - D(y), j, t\bigr). \end{aligned} \tag{3.4}$$

The following inequality will be frequently used when we derive the main results. We list it here and refer, for example, to [1, 5].

Lemma 3.2 *For $p \geq 1$, the following classical inequality holds*:

$$\left|x - D(y)\right|^p \leq (1-\kappa)^{1-p}|x|^p + \kappa|y|^p. \tag{3.5}$$

Assumption 3.3 For the function $Q \in C(R^n; R_+)$, there exist two positive constants α_1, α_2 such that $Q(x - D(y)) \leq \alpha_1 Q(x) + \alpha_2 Q(y)$.

Remark 3.4 Obviously, when λ_1, λ_2, p, q are positive constants, the function $Q(x) = \lambda_1|x|^p + \lambda_2|x|^q$ satisfies Assumption 3.3.

To study the exponential stability of NSDE (2.2), we need to impose a new assumption.

Assumption 3.5 Let Assumption 3.3 hold. Assume that there exist functions $\bar{U} \in C^{2,1}(R^n \times S \times R_+; R_+)$, $u : R^n \times S \times R_+ \to R^n$ and positive numbers $\bar{\theta}$, β_j $(j = 1, 2, \ldots, 7)$ such that

$$\beta_5 < \beta_4(1-\bar{\delta}), \qquad \beta_7 < \beta_6(1-\bar{\delta}), \qquad \bar{\theta}|x|^2 \leq \bar{U}(x,i,t) \leq Q(x), \tag{3.6}$$

$$\begin{aligned}
&\mathcal{L}\bar{U}(x,y,i,t) + \beta_1\left|\bar{U}_x\big(x - D(y), i, t\big)\right|^2 + \beta_2\left|f(x,y,i,t)\right|^2 + \beta_3\left|g(x,y,i,t)\right|^2 \\
&\quad \leq -\beta_4|x|^2 + \beta_5|y|^2 - \beta_6 Q(x) + \beta_7 Q(y)
\end{aligned} \tag{3.7}$$

for all $(x,y,i,t) \in R^n \times R^n \times S \times R_+$.

Theorem 3.6 *Let Assumptions 2.1, 2.3, and 3.5 hold. Assume that*

$$\tau \leq \frac{(1-\kappa)^2\beta_1\beta_3}{\beta^2} \wedge \frac{(1-\kappa)\sqrt{\beta_1\beta_2}}{\beta} \quad \textit{and} \quad \tau < \frac{(1-\kappa)\sqrt{2\beta_1(\beta_4(1-\bar{\delta}) - \beta_5)}}{2\beta^2}. \tag{3.8}$$

Then, for any given initial data (2.1), the solution of NSDE (2.2) has the property that

$$\lim_{t\to\infty} \sup \frac{1}{t} \log E\bar{U}\big(X(t) - D\big(X\big(t - \delta(t)\big)\big), r(t), t\big) < 0. \tag{3.9}$$

Proof Fix the initial data $\xi \in C([-\delta, 0]; R^n)$ and $r_0 \in S$ arbitrarily. Let $k_0 > 0$ be a sufficiently large integer such that $\|\xi\| < k_0$. For each integer $k \geq k_0$, define the stopping time

$$\sigma_k = \inf\big\{t \geq 0 : \left|X(t)\right| \geq k\big\},$$

where throughout this paper we set $\inf \emptyset = \infty$ (as usual, $\emptyset$ denotes the empty set). It is easy to see that σ_k is increasing as $k \to \infty$ and, by condition (2.5), $\lim_{k\to\infty} \sigma_k = \infty$ a.s. By the generalised Itô formula, we obtain from Lemma 3.1 that

$$\begin{aligned}
&E\big[e^{\lambda(t\wedge\sigma_k)} V(\hat{X}_{t\wedge\sigma_k}, \hat{r}_{t\wedge\sigma_k}, t\wedge\sigma_k)\big] \\
&\quad = V(\hat{X}_0, \hat{r}_0, 0) + E\int_0^{t\wedge\sigma_k} e^{\lambda s}\big(\lambda V(\hat{X}_s, \hat{r}_s, s) + LV(\hat{X}_s, \hat{r}_s, s)\big)\,ds
\end{aligned} \tag{3.10}$$

for any $t \geq 0$ and $k \geq k_0$, where λ is a sufficiently small positive number to be determined later.

We now let $\theta = \beta^2/(\beta_1(1-\kappa)^2)$. By Assumption 3.5 and the Hölder inequality, it is easy to show that

$$\begin{aligned} &\bar{U}_x\big(X(t) - D\big(X(t)-\delta(t)\big), r(t), t\big)\big[u\big(X\big(t-\tau(t)\big), r(t), t\big) - u\big(X(t), r(t), t\big)\big] \\ &\quad\leq \beta_1\big|\bar{U}_x\big(X(t) - D\big(X(t)-\delta(t)\big), r(t), t\big)\big|^2 + \frac{\beta^2}{4\beta_1}\big|X(t) - X\big(t-\tau(t)\big)\big|^2. \end{aligned} \tag{3.11}$$

By condition (3.8), we have $\theta\tau^2 \leq \beta_2$ and $\theta\tau \leq \beta_3$,

$$\begin{aligned} LV(\hat{X}_s, \hat{r}_s, s) \leq{}& \mathcal{L}\bar{U}\big(X(s), X\big(s-\delta(s)\big), r(s), t\big) + \beta_1\big|\bar{U}_X\big(X(s) - D\big(X\big(s-\delta(s)\big)\big), r(t), t\big)\big|^2 \\ &+ \beta_2\big|f\big(X(s), X\big(s-\delta(s)\big), r(t), t\big)\big|^2 + \beta_3\big|g\big(X(s), X\big(s-\delta(s)\big), r(t), t\big)\big|^2 \\ &+ 2\theta\tau^2\beta^2\big|X\big(s-\tau(s)\big)\big|^2 + \frac{\beta^2}{4\beta_1}\big|X(s) - X\big(s-\tau(s)\big)\big|^2 \\ &- \frac{\beta^2}{\beta_1(1-\kappa)^2}\int_{t-\tau}^{t}\Phi(v)\,dv \\ \leq{}& 2\theta\tau^2\beta^2\big|X\big(s-\tau(s)\big)\big|^2 - \beta_4\big|X(s)\big|^2 + \beta_5\big|X\big(s-\delta(s)\big)\big|^2 - \beta_6 Q\big(X(s)\big) \\ &+ \beta_7 Q\big(X\big(s-\delta(s)\big)\big) + \frac{\beta^2}{4\beta_1}\big|X(s) - X\big(s-\tau(s)\big)\big|^2 \\ &- \frac{\beta^2}{\beta_1(1-\kappa)^2}\int_{s-\tau}^{s}\Phi(v)\,dv. \end{aligned} \tag{3.12}$$

Substituting this into (3.10) implies

$$\begin{aligned} E\big[e^{\lambda(t\wedge\sigma_k)}V(\hat{X}_{t\wedge\sigma_k}, \hat{r}_{t\wedge\sigma_k}, t\wedge\sigma_k)\big] \leq{}& V(\hat{X}_0, \hat{r}_0, 0) + \frac{\lambda\beta^2}{\beta_1(1-\kappa)^2}H_1 \\ &+ H_2 + H_3 - H_4 + H_5 - H_6 + H_7 + H_8 - H_9, \end{aligned}$$

where

$$\begin{aligned} &H_1 = E\int_0^{t\wedge\sigma_k} e^{\lambda s}\int_{-\tau}^{0}\int_{s+u}^{s}\Phi(v)\,dv\,du\,ds, \\ &H_2 = \lambda E\int_0^{t\wedge\sigma_k} e^{\lambda s}\bar{U}\big(X(s) - D\big(X\big(s-\delta(s)\big)\big), r(s), s\big)\,ds, \\ &H_3 = 2\theta\tau^2\beta^2 E\int_0^{t\wedge\sigma_k} e^{\lambda s}\big|X\big(t-\tau(t)\big)\big|^2\,ds, \\ &H_4 = \beta_4 E\int_0^{t\wedge\sigma_k} e^{\lambda s}\big|X(s)\big|^2\,ds, \qquad H_5 = \beta_5 E\int_0^{t\wedge\sigma_k} e^{\lambda s}\big|X\big(s-\delta(s)\big)\big|^2\,ds, \\ &H_6 = \beta_6 E\int_0^{t\wedge\sigma_k} e^{\lambda s}Q\big(X(s)\big)\,ds, \qquad H_7 = \beta_7 E\int_0^{t\wedge\sigma_k} Q\big(X\big(s-\delta(s)\big)\big)\,ds, \\ &H_8 = \frac{\beta^2}{4\beta_1}E\int_0^{t\wedge\sigma_k} e^{\lambda s}\big|X(s) - X\big(s-\tau(s)\big)\big|^2\,ds, \\ &H_9 = \frac{\beta^2}{\beta_1(1-\kappa)^2}E\int_0^{t\wedge\sigma_k} e^{\lambda s}\int_{s-\tau}^{s}\Phi(v)\,dv\,ds. \end{aligned}$$

It is easy to see that

$$H_1 \le \tau E \int_0^{t\wedge\sigma_k} e^{\lambda s} \int_{s-\tau}^{s} \Phi(v)\,dv\,ds.$$

By inequality (3.5) and Assumption 3.3, we have

$$\begin{aligned} &E\int_0^{t\wedge\sigma_k} e^{\lambda s}\bar{U}\bigl(X(s)-D\bigl(X\bigl(s-\delta(s)\bigr)\bigr),r(s),s\bigr)\,ds \\ &\quad\le E\int_0^{t\wedge\sigma_k} e^{\lambda s}\bigl[\alpha_1 Q\bigl(X(s)\bigr)+\alpha_2 Q\bigl(X\bigl(s-\delta(s)\bigr)\bigr)\bigr]\,ds. \end{aligned}$$

On the other hand,

$$\begin{aligned} &E\int_0^{t\wedge\sigma_k} e^{\lambda s}Q\bigl(X\bigl(s-\delta(s)\bigr)\bigr)\,ds \le \frac{e^{\lambda\delta}}{1-\bar{\delta}}\int_{-\delta}^{0} Q\bigl(\xi(s)\bigr)\,ds+\frac{e^{\lambda\delta}}{1-\bar{\delta}}E\int_0^{t\wedge\sigma_k} e^{\lambda s}Q\bigl(X(s)\bigr)\,ds, \\ &E\int_0^{t\wedge\sigma_k} e^{\lambda s}\bigl|X\bigl(s-\delta(s)\bigr)\bigr|^2\,ds \le \frac{e^{\lambda\delta}}{1-\bar{\delta}}\int_{-\delta}^{0}\bigl|\xi(s)\bigr|^2\,ds+\frac{e^{\lambda\delta}}{1-\bar{\delta}}E\int_0^{t\wedge\sigma_k} e^{\lambda s}\bigl|X(s)\bigr|^2\,ds \end{aligned}$$

and

$$E\int_0^{t\wedge\sigma_k} e^{\lambda s}\bigl|X\bigl(s-\tau(t)\bigr)\bigr|^2\,ds \le \frac{e^{\lambda\delta}}{1-\bar{\delta}}\int_{-\delta}^{0}\bigl|\xi(s)\bigr|^2\,ds+\frac{e^{\lambda\delta}}{1-\bar{\delta}}E\int_0^{t\wedge\sigma_k} e^{\lambda s}\bigl|X(s)\bigr|^2\,ds.$$

Noting that $\tau < \frac{(1-\kappa)\sqrt{2\beta_1(\beta_4(1-\bar{\delta})-\beta_5)}}{2\beta^2}$, we can now choose a sufficiently small λ such that

$$\begin{aligned} &\frac{2\tau^2\beta^4 e^{\lambda\delta}}{\beta_1(1-\kappa)^2(1-\bar{\delta})}+\frac{\beta_5 e^{\lambda\delta}}{1-\bar{\delta}}\le\beta_4, \\ &\frac{\beta_7}{1-\bar{\delta}}+\lambda\alpha_1+\frac{\lambda\alpha_2}{1-\bar{\delta}}\le\beta_6, \quad \text{and} \quad \lambda\tau\le\frac{1}{2}. \end{aligned} \tag{3.13}$$

Then we can obtain

$$E\bigl[e^{\lambda(t\wedge\sigma_k)}V(\hat{X}_{t\wedge\sigma_k},\hat{r}_{t\wedge\sigma_k},t\wedge\sigma_k)\bigr]\le C_1+H_8-\frac{1}{2}H_9, \tag{3.14}$$

where C_1 is a positive constant. By the well-known Fatou lemma, we can let $k\to\infty$ in (3.14) to obtain

$$e^{\lambda t}EV(\hat{X}_t,\hat{r}_t,t)\le C_1+\bar{H}_8-\frac{1}{2}\bar{H}_9, \tag{3.15}$$

where

$$\begin{aligned} \bar{H}_8 &= \frac{\beta^2}{4\beta_1}E\int_0^t e^{\lambda s}\bigl|X(s)-X\bigl(s-\tau(s)\bigr)\bigr|^2\,ds, \\ \bar{H}_9 &= \frac{\beta^2}{\beta_1(1-\kappa)^2}E\int_0^t e^{\lambda s}\int_{s-\tau}^{s}\Phi(v)\,dv\,ds. \end{aligned}$$

By the well-known Fubini theorem,

$$\bar{H}_8=\frac{\beta^2}{4\beta_1}\int_0^t e^{\lambda s}E\bigl|X(s)-X\bigl(s-\tau(s)\bigr)\bigr|^2\,ds.$$

For $t \in [0, \tau]$, we clearly have

$$\bar{H}_8 \le \frac{\tau e^{\lambda\tau}\beta^2}{\beta_1}\Big(\sup_{-\tau\le v\le\tau} E\big|X(v)\big|^2\Big) =: C_2.$$

For $t > \tau$, we have

$$\bar{H}_8 \le C_2 + \frac{\beta^2}{4\beta_1}\int_\tau^t e^{\lambda s}E\big|X(s) - X\big(s-\tau(s)\big)\big|^2\,ds.$$

Note that

$$\begin{aligned}
&\big|X(s) - X\big(s-\tau(s)\big)\big| \\
&\quad\le \big|X(s) - D\big(X\big(s-\delta(s)\big)\big) - X\big(s-\tau(s)\big) + D\big(X\big(s-\delta(s)-\tau(s)\big)\big)\big| \\
&\qquad + \big|D\big(X\big(s-\delta(s)\big)\big) - D\big(X\big(s-\delta(s)-\tau(s)\big)\big)\big| \\
&\quad\le \kappa\big|X\big(s-\delta(s)\big) - X\big(s-\delta(s)-\tau(s)\big)\big| + \bigg|\int_{s-\tau}^s \big[f\big(X(v), X\big(v-\delta(v)\big), r(v), v\big) \\
&\qquad + u\big(X\big(v-\tau(v)\big), r(v), v\big)\big]\,dv + \int_{s-\tau}^s g\big(X(v), X\big(v-\delta(v)\big), r(v), v\big)\,dB(v)\bigg|.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&E\big|X(s) - X\big(s-\tau(s)\big)\big|^2 \\
&\quad\le (1+\varrho)\kappa^2 E\big|X\big(s-\delta(s)\big) - X\big(s-\delta(s)-\tau(s)\big)\big|^2 \\
&\qquad + \left(1+\frac{1}{\varrho}\right)E\bigg|\int_{s-\tau}^s \big[f\big(X(v), X\big(v-\delta(v)\big), r(v), v\big) + u\big(X\big(v-\tau(v)\big), r(v), v\big)\big]\,dv \\
&\qquad + \int_{s-\tau}^s g\big(X(v), X\big(v-\delta(v)\big), r(v), v\big)\,dB(v)\bigg|^2 \\
&\quad\le (1+\varrho)\kappa^2 E\big|X\big(s-\delta(s)\big) - X\big(s-\delta(s)-\tau(s)\big)\big|^2 + 2\left(1+\frac{1}{\theta}\right)E\int_{s-\tau}^s H(v)\,dv.
\end{aligned}$$

Setting $\varrho = \frac{1}{\kappa} - 1$, then we have

$$\begin{aligned}
&\int_\tau^t E\big|X(s) - X\big(s-\tau(s)\big)\big|^2\,ds \\
&\quad\le \kappa\int_\tau^t E\big|X\big(s-\delta(s)\big) - X\big(s-\delta(s)-\tau(s)\big)\big|^2\,ds + \frac{2}{1-\kappa}E\int_\tau^t\int_{s-\tau}^s H(v)\,dv\,ds \\
&\quad\le \kappa\int_{\tau-\delta}^t E\big|X(s) - X\big(s-\tau(s)\big)\big|^2\,ds + \frac{2}{1-\kappa}E\int_\tau^t\int_{s-\tau}^s H(v)\,dv\,ds.
\end{aligned}$$

Noting that $0 < \kappa < 1$, it follows that

$$\begin{aligned}
&\int_\tau^t E\big|X(s) - X\big(s-\tau(s)\big)\big|^2\,ds \\
&\quad\le \frac{\kappa}{1-\kappa}\int_{\tau-\delta}^\tau E\big|X(s) - X\big(s-\tau(s)\big)\big|^2\,ds + \frac{2}{(1-\kappa)^2}E\int_\tau^t\int_{s-\tau}^s H(v)\,dv\,ds.
\end{aligned}$$

Noting that

$$\int_{\tau-\delta}^{\tau} E\big|X(s)-X\big(s-\tau(s)\big)\big|^2\,ds \le 2E\int_{\tau-\delta}^{\tau}\big|X(s)\big|^2\,ds+\big|X\big(s-\tau(s)\big)\big|^2\,ds$$
$$\le 4E\int_{-\delta}^{\delta}\big|X(s)\big|^2\,ds \le 8\delta \sup_{-\delta\le v\le\delta} E\big|X(v)\big|^2 := C_3.$$

Hence

$$\bar{H}_8 \le \frac{1}{2}\bar{H}_9 + C_2 + C_3. \tag{3.16}$$

Substituting (3.16) into (3.15), we get

$$EV(\hat{X}_t,\hat{r}_t,t) \le (C_1+C_2+C_3)e^{-\lambda t},\quad t\ge 0.$$

It follows from the definition of $V(\hat{X}_t,\hat{r}_t,t)$ that

$$E\bar{U}\big(X(t)-D\big(X\big(t-\delta(t)\big),r(t),t\big) \le (C_1+C_2+C_3)e^{-\lambda t}, \tag{3.17}$$

which implies assertion (3.9). Thus the proof is complete. □

Corollary 3.7 *Let the conditions of Theorem 3.6 hold. Assume, moreover, that $\kappa e^{\delta} < 1$. Then, for any given initial data (2.1), the solution of NSDE (2.2) satisfies*

$$\lim_{t\to\infty}\sup \frac{1}{t}\log\big(E\big|X(t)\big|^{\bar{q}}\big) < 0,\quad \forall \bar{q}\in[2,q). \tag{3.18}$$

That is, NSDE (2.2) is exponentially stable in $L^{\bar{q}}$ for $\bar{q}\in[2,q)$.

Proof By condition (2.5),

$$C_4 := \sup_{-\delta\le t<\infty} E\big|X(t)\big|^q < \infty.$$

For $T>\delta$, by (3.5), we have

$$\begin{aligned}
&\sup_{0\le t\le T} e^{\lambda t}E\big|X(t)\big|^2\\
&\quad\le \frac{1}{1-\kappa}\sup_{0\le t\le T} e^{\lambda t}E\big|X(t)-D\big(X\big(t-\delta(t)\big)\big)\big|^2 + \sup_{0\le t\le T}\kappa e^{\lambda t}\big|X\big(t-\delta(t)\big)\big|^2\\
&\quad\le \frac{1}{1-\kappa}\sup_{0\le t\le T} e^{\lambda t}E\big|X(t)-D\big(X\big(t-\delta(t)\big)\big)\big|^2\\
&\qquad+\kappa e^{\lambda\delta}\Big(\sup_{0\le t\le T} e^{\lambda t}E\big|X(t)\big|^2 + \sup_{-\delta\le t\le 0} E\big|\xi(t)\big|^2\Big).
\end{aligned}$$

This implies

$$\sup_{0\le t\le T} e^{\lambda t}E\big|X(t)\big|^2 \le \frac{1}{(1-\kappa)(1-\kappa e^{\delta})}\sup_{0\le t\le T} e^{\lambda t}E\big|X(t)-D\big(X\big(t-\delta(t)\big)\big)\big|^2$$
$$+\frac{\kappa e^{\lambda\delta}}{1-\kappa e^{\delta}}\sup_{-\delta\le t\le 0} E\big|\xi(t)\big|^2.$$

Letting $T \to \infty$, it then follows from (3.6) and (3.9) that

$$E\left|X(t)\right|^2 \le C_5 e^{-\lambda t}, \tag{3.19}$$

where C_5 is a positive constant. Fix any $\bar{q} \in [2, q)$, for a constant $\epsilon \in (0,1)$, by the Hölder inequality, we can show

$$E\left|X(t)\right|^{\bar{q}} \le \left(E\left|X(t)\right|^2\right)^{\epsilon} \left(E\left|X(t)\right|^{(\bar{q}-2\epsilon)/(1-\epsilon)}\right)^{1-\epsilon}.$$

Letting $\epsilon = \frac{q-\bar{q}}{q-2}$, we can obtain

$$\begin{aligned} E\left|X(t)\right|^{\bar{q}} &\le \left(E\left|X(t)\right|^2\right)^{(q-\bar{q})/(q-2)} \left(E\left|X(t)\right|^{q}\right)^{(\bar{q}-2)/(q-2)} \\ &\le C_4^{(\bar{q}-2)/(q-2)} \left(E\left|X(t)\right|^2\right)^{(q-\bar{q})/(q-2)}. \end{aligned} \tag{3.20}$$

It follows from (3.19) that

$$E\left|X(t)\right|^{\bar{q}} \le C_4^{(\bar{q}-2)/(q-2)} C_5^{(q-\bar{q})/(q-2)} e^{-\bar{\lambda} t}, \tag{3.21}$$

where $\bar{\lambda} = \lambda(q-\bar{q})/(q-2)$, which implies assertion (3.18). Thus the proof is complete. □

Theorem 3.8 *Let the conditions of Corollary 3.7 hold. If, moreover,*

$$2q_1 \vee 2q_2 < q,$$

then the solution of NSDE (2.2) satisfies

$$\lim_{t\to\infty} \sup \frac{1}{t} \log\left|X(t)\right| < 0 \quad a.s.$$

That is, NSDE (2.2) is almost surely exponentially stable.

Proof Let k be any nonnegative integer. By the Hölder inequality and the Doob martingale inequality, we have

$$\begin{aligned} &E\Big(\sup_{k\le t\le k+1} \left|X(t) - D\big(X\big(t-\delta(t)\big)\big)\right|^2\Big) \\ &\quad\le 3E\left|X(k) - D\big(X\big(k-\delta(k)\big)\big)\right|^2 \\ &\qquad + 3E\int_k^{k+1} \left|f\big(X(t), X\big(t-\delta(t)\big), r(t), t\big) + u(X\big(t-\tau(t), r(t), t\big)\right|^2 dt \\ &\qquad + 12E\int_k^{k+1} \left|g\big(X(t), X\big(t-\delta(t)\big), r(t), t\big)\right|^2 dt \\ &\quad\le C_6 \int_k^{k+1} E\big(\left|X(t)\right|^2 + \left|X\big(t-\delta(t)\big)\right|^2 + \left|X\big(t-\tau(t)\big)\right|^2 + \left|X(t)\right|^{\bar{q}} + \left|X\big(t-\delta(t)\big)\right|^{\bar{q}}\big)\, dt \\ &\qquad + 6\kappa^2 E\left|X\big(k-\delta(k)\big)\right|^2 + 6E\left|X(k)\right|^2, \end{aligned}$$

where C_6 is a positive constant and $\bar{q} = 2q_1 \vee 2q_2 \in [2, q)$. By (3.19) and (3.21), we have

$$E\Big(\sup_{k\le t\le k+1}\big|X(t) - D\big(X\big(t-\delta(t)\big)\big)\big|^2\Big) \le C_7 e^{-\bar{\lambda}k},$$

where $\bar{\lambda} = \lambda(q-\bar{q})/(q-2)$ and C_7 is a positive constant. Consequently,

$$\sum_{k=0}^{\infty} P\Big(\sup_{k\le t\le k+1}\big|X(t) - D\big(X\big(t-\delta(t)\big)\big)\big|^2 > e^{-0.5\bar{\lambda}k}\Big) \le \sum_{k=0}^{\infty} C_7 e^{-0.5\bar{\lambda}k} < \infty.$$

By the Borel–Cantelli lemma, we can show that, for almost all $\omega \in \Omega$, there is a positive integer $k_1 = k_1(\omega)$ such that

$$\sup_{k\le t\le k+1}\big|X(t) - D\big(X\big(t-\delta(t)\big)\big)\big|^2 \le e^{-0.5\bar{\lambda}k}, \quad k \ge k_1.$$

In other words, for almost all $\omega \in \Omega$,

$$\big|X(t) - D\big(X\big(t-\delta(t)\big)\big)\big|^2 \le e^{-0.5\bar{\lambda}t}, \quad t \ge k_1.$$

However, $|X(t) - D(X(t-\delta(t)))|^2$ is finite on $[0, k_1]$. Therefore, for almost all $\omega \in \Omega$, there exists a finite number $M = M(\omega)$ such that

$$\big|X(t) - D\big(X\big(t-\delta(t)\big)\big)\big|^2 \le Me^{-0.5\bar{\lambda}t} \quad \text{for all } t \ge 0.$$

Choose any $\varpi \in (\kappa^2, 1)$. By the inequality

$$\big|X(t)\big|^2 \le \frac{1}{1-\varpi}\big|X(t) - D\big(X\big(t-\delta(t)\big)\big)\big|^2 + \frac{\kappa^2}{\varpi}\big|X\big(t-\delta(t)\big)\big|^2,$$

we can show that, for any $T > 0$,

$$\begin{aligned}\sup_{0\le t\le T} e^{0.5\bar{\lambda}t}\big|X(t)\big|^2 &\le \frac{M}{1-\varpi} + \frac{\kappa^2}{\varpi}\sup_{0\le t\le T} e^{0.5\bar{\lambda}t}\big|X\big(t-\delta(t)\big)\big|^2 \\ &\le \frac{M}{1-\varpi} + \frac{\kappa^2}{\varpi}\sup_{-\delta\le t\le T} e^{0.5\bar{\lambda}t}\big|X(t)\big|^2.\end{aligned}$$

This implies

$$\lim_{t\to\infty}\sup\frac{1}{t}\log\big|X(t)\big| < 0 \quad \text{a.s.},$$

which is the required assertion. Thus the proof is complete. □

4 An example

In this section we will discuss an example to illustrate our theory.

Example 4.1 Consider a scalar hybrid NSDE

$$\begin{aligned} d\big[X(t) - D\big(X\big(t-\delta(t)\big)\big)\big] &= f\big(X(t), X\big(t-\delta(t)\big), r(t), t\big)\,dt \\ &\quad + g\big(X(t), X\big(t-\delta(t)\big), r(t), t\big)\,dB(t), \end{aligned} \tag{4.1}$$

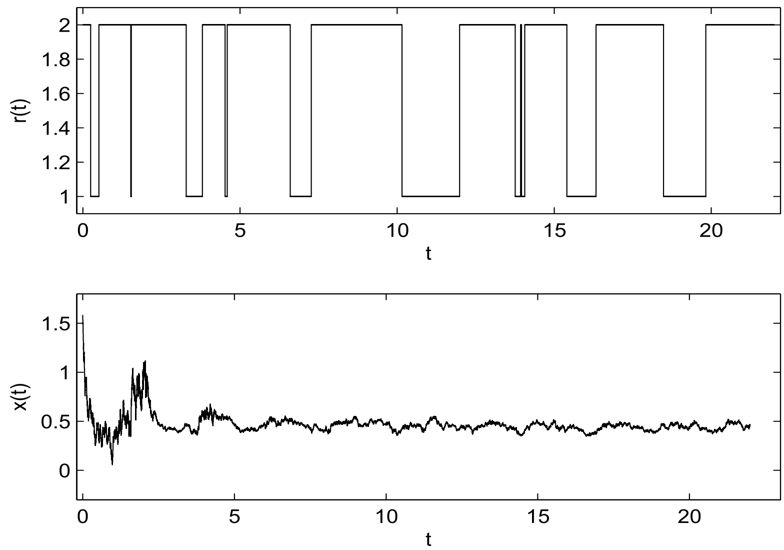

Figure 1 The computer simulation of the sample paths of the Markovian chain and NSDE (4.1) with $\delta(t) = 2 - 0.2\cos(t)$ using the Euler–Maruyama method

where $B(t)$ is a scalar Brownian motion, $r(t)$ is a Markov chain on the state space $S = \{1, 2\}$ with its generator

$$\Gamma = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{4.2}$$

and, moreover, the coefficients f and g are defined by

$$\begin{aligned} &f(x,y,1,t) = y - 4x^3, \qquad f(x,y,2,t) = y - 5x^3, \\ &g(x,y,1,t) = g(x,y,2,t) = 0.5y^2, \qquad D\big(x - D(y)\big) = 0.1y. \end{aligned} \tag{4.3}$$

Before applying our theory, we set $\delta(t) = 2 - 0.2\cos(t)$ for all $t \geq 0$, the initial data $X(v) = 2 + \cos(v)$ for $v \in [-2, 0]$, $r(0) = 1$. The sample paths of the Markov chain and the solution of the NSDE are plotted in Fig. 1, which indicates that the NSDE is unstable.

Let the control function $u : R \times S \times R_+ \to R$ as follows:

$$u(x,t,1) = -2x, \qquad u(x,t,2) = -3x. \tag{4.4}$$

Then the controlled hybrid NSDE has the form

$$\begin{aligned} d\big[X(t) - D\big(X\big(t - \delta(t)\big)\big)\big] &= \big[f\big(X(t), X\big(t - \delta(t)\big), r(t), t\big) + u\big(X\big(t - \tau(t)\big), r(t), t\big)\big]\,dt \\ &\quad + g\big(X(t), X\big(t - \delta(t)\big), r(t), t\big)\,dB(t). \end{aligned} \tag{4.5}$$

Define $U(x,i,t) = |x|^6$ for $(x,i,t) \in R \times S \times R_+$. It is easy to show that

$$\begin{aligned} &LU(x,y,i,t) + U_x\big(x - D(y), i, t\big)u(z,i,t) \\ &\quad = 6(x - 0.1y)^5 f(x,y,i,t) + 15(x - 0.1y)^4 \big|g(x,y,i,t)\big|^2 + 6(x - 0.1y)^5 u(z,i,t) \end{aligned}$$

for $(x, y, i, t) \in R \times R \times S \times R_+$. Applying the inequalities $(a+b)^p \leq (1+\epsilon)^{p-1}(a^p + \epsilon^{1-p} b^p)$ and $a^\beta b^{1-\beta} \leq \beta a + (1-\beta)b$, we can obtain

$$
\begin{aligned}
& LU(x,y,i,t) + U_x\bigl(x - D(y), i, t\bigr)u(z,i,t) \\
& \quad \leq -10.891x^8 + 4.928y^8 + 22.876x^6 + 3.13y^6 + 2.2z^6 \\
& \quad \leq c_1 - 10\bigl(x^8 + 2.2x^6\bigr) + 5\bigl(y^8 + 2.2y^6\bigr) + \bigl(z^8 + 2.2z^6\bigr),
\end{aligned}
$$

where $c_1 = \sup_{x\in R}(45x^6 - 0.891x^8) < \infty$. Thus, we can conclude that the unique global solution of (4.1) has the property that

$$
\sup_{-\delta \leq t < \infty} E|X|^6 < \infty.
$$

Let

$$
\bar{U}(x,i,t) = \begin{cases} 1.5x^2 + x^4 & \text{if } i = 1, \\ x^2 + x^4 & \text{if } i = 2. \end{cases}
$$

Applying the above inequalities and the Young inequality, we can get

$$
\mathcal{L}\bar{U}(x,y,1,t) \leq -11.442x^6 - 13.592x^4 - 4.65x^2 + 2.073y^6 + 2.918y^4 + 1.555y^2
$$

and

$$
\mathcal{L}\bar{U}(x,y,2,t) \leq -14.44x^6 - 14.84x^4 - 4.15x^2 + 2.275y^6 + 3.047y^4 + 1.155y^2.
$$

Moreover,

$$
\begin{aligned}
\bigl|\bar{U}_x\bigl(x - D(y), i, t\bigr)\bigr|^2 &= \begin{cases} 9(x-D(y))^2 + 24(x-D(y))^4 + 16(x-D(y))^6 & \text{if } i = 1, \\ 4(x-D(y))^2 + 16(x-D(y))^4 + 16(x-D(y))^6 & \text{if } i = 2, \end{cases} \\
&\leq \begin{cases} 10x^2 + 32.922x^4 + 27.097x^6 + 0.9y^2 + 2.4y^4 + 16y^6 & \text{if } i = 1, \\ 4.445x^2 + 21.948x^4 + 27.097x^6 + 0.4y^2 + 1.6y^4 + 16y^6 & \text{if } i = 2, \end{cases} \\
\bigl|f(x,y,i,t)\bigr|^2 &\leq \begin{cases} y^2 + 2y^4 + 6x^4 + 16x^6, & \text{if } i = 1, \\ y^2 + 2.5y^4 + 7.5x^4 + 25x^6, & \text{if } i = 2, \end{cases} \\
\bigl|g(x,y,1,t)\bigr|^2 &= \bigl|g(x,y,2,t)\bigr|^2 = 0.25y^4.
\end{aligned}
$$

Set $\beta_1 = 0.1$, $\beta_2 = 0.2$, $\beta_3 = 4$. This implies

$$
\begin{aligned}
& \mathcal{L}\bar{U}(x,y,i,t) + \beta_1\bigl|\bar{U}_x\bigl(x - D(y), i, t\bigr)\bigr|^2 + \beta_2\bigl|f(x,y,i,t)\bigr|^2 + \beta_3\bigl|g(x,y,i,t)\bigr|^2 \\
& \quad \leq -3.5x^2 - 9.09x^4 - 5.54x^6 + 1.85y^2 + 4.71y^4 + 3.88y^6 \\
& \quad \leq -2.1x^2 + 0.85y^2 - 0.7\bigl(2x^2 + 10x^4 + 7x^6\bigr) + 0.5\bigl(2y^2 + 10y^4 + 7y^6\bigr).
\end{aligned}
$$

Letting $Q(x) = 2x^2 + 10x^4 + 7x^6$, we have $\bar{U}(x,i,t) \le Q(x)$. Noting that $\beta_4 = 2.1$, $\beta_5 = 0.85$, we get condition (3.7). Moreover, it is easy to check that condition (3.6) holds as well. In other words, Assumption 3.5 is satisfied. Noting that

$$\left|u(x,i,t) - u(y,i,t)\right| \le 3|x-y|,$$

we see that Assumption 2.3 is satisfied with $\beta = 3$. Furthermore, condition (3.8) becomes

$$\tau \le 0.0203.$$

By Theorem 3.6, we can therefore conclude that the solution of NSDE (4.1) has the properties that

$$\lim_{t\to\infty} \sup \frac{1}{t} \log E\left(X^2(t) + X^4(t)\right) < 0.$$

Moreover, as $X^2(t) \le X^2(t) + X^4(t)$, by Corollary 3.7, we have

$$\lim_{t\to\infty} \sup \frac{1}{t} \log E\left|X(t)\right|^{\bar{q}} < 0, \quad \forall \bar{q} \in [2,q) \quad \text{and}$$
$$\lim_{t\to\infty} \sup \frac{1}{t} \log\left|X(t)\right| < 0 \quad \text{a.s.}$$

That is, the solution of equation (4.5) is almost surely exponentially stable.

We set the initial data $X(v) = 2 + \cos(v)$ for $v \in [-0.02, 0]$ and $r(0) = 2$. Figure 2 shows the sample paths of the Markov chain and the solution of NSDE (4.5). The computer simulation shows that NSDE (4.5) is stable.

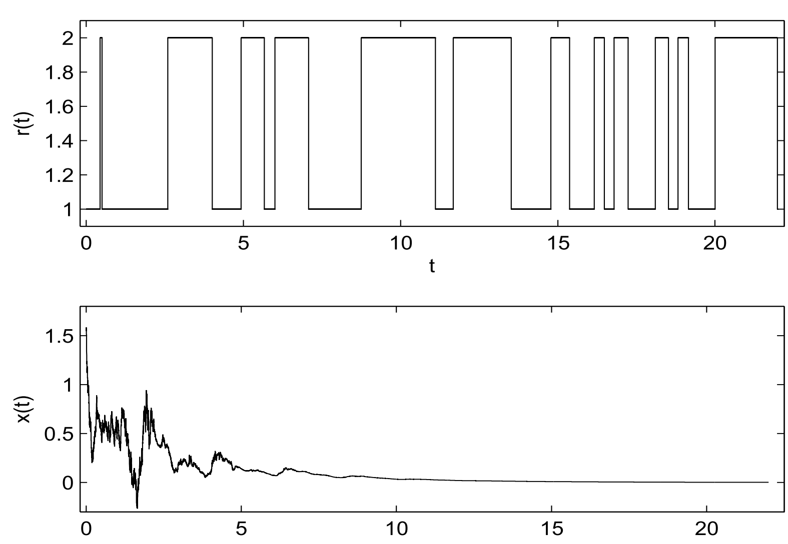

Figure 2 The computer simulation of the sample paths of the Markov chain and NSDE (4.5) with control (4.4) and $\tau = 0.02$

5 Conclusion

In this paper, we studied exponential stability of highly nonlinear hybrid NSDEs. Our significant contribution in this paper is that the variable delay feedback controls are designed to stabilize highly nonlinear hybrid NSDEs. The key technique used in this paper is the method of Lyapunov functional. A significant amount of mathematics has been developed to deal with the difficulties due to the neutral term. An example with computer simulations has been used to illustrate our theory. Finally, following the work of Fei et al. [29], we can investigate the stabilisation of G-neutral stochastic differential equations with delay by the feedback control.

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Declarations

Consent for publication
Not applicable.

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