Advances in Continuous and Discrete Models

RESEARCH Open Access

A pair of centro-symmetric heteroclinic orbits coined

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Abstract

Although the axis-symmetric heteroclinic orbits of Lorenz-like systems have been intensively studied in the past decades, scholars seem to pay scant attention to the centro-symmetric ones. To achieve this target, the present paper introduces a new subquadratic centro-symmetric three-dimensional Lorenz-like system: $\dot{x} = a(y - x)$, $\dot{y} = cx - \sqrt[3]{x^2}z$, $\dot{z} = -bz + \sqrt[3]{x^2}y$, and proves the existence of a pair of centro-symmetric to E_0 and $E_\pm$ combining the definitions of α-limit and ω-limit set, Lyapunov functions. The effectiveness and correctness of the theoretical conclusions are verified via a few numerical examples. Not only does the study provide new ideas for finding heteroclinic orbits, but also it poses an interesting question that the existence of heteroclinic orbits may depend on the degrees of the considered models.

Mathematics Subject Classification: 65P30; 65P40; 37C27; 37C29

Keywords: Hilbert's 16th problem; Subquadratic centro-symmetric Lorenz-like system; Heteroclinic orbit; Lyapunov function

1 Introduction

Since the introduction of the Lorenz system and attractor, the study of chaos entered a new era [1]. Utilizing theoretical tools, numerical and circuit simulation, etc., researchers and engineers performed a systematical analysis for numerous chaotic systems [2–7]. Not only did they reveal distinctive properties, such as extremely sensitive dependence on initial conditions, deterministicness, unpredictability, existence of at least one positive Lyapunov exponent, boundedness, hidden attractors, conservative chaotic flow, multistability, and so on, but also they explained the forming mechanism of strange attractors to some degree, i.e., the bifurcation of singular orbits (including homoclinic and heteroclinic orbits, singularly degenerate heteroclinic cycles, etc.) and invariant algebraic surfaces, the loss of global stability, etc. [2, 4–9].

Recently, Zhang et al. posed the extension of the second part of the celebrated Hilbert's 16th problem [9, 10], i.e., the degree of polynomials in the studied models determines the number and mutual disposition of attractors and repellers (if they exist). Inspired by this, Wang et al. introduced two new subquadratic Lorenz-like systems of degree $\frac{4}{3}$ and $\frac{6}{5}$, and they found a multitude of two-wing hidden attractors in a broader range of parameters [11, 12]. Moreover, by aid of Lyapunov functions and the definitions of both α-limit set and ω-limit set [7, 13–22], Wang et al. proved the existence of a pair of heteroclinic orbits when $\frac{b}{a} \geq \frac{4}{3}$ and $\frac{b}{a} \geq \frac{6}{5}$, respectively. Another two Lorenz-like analogues of degree $\frac{4}{3}$ also exhibit two pairs of heteroclinic orbits when $\frac{b}{a} \geq \frac{4}{3}$ [13, 14]. Likewise, there exist heteroclinic orbits in the cubic and quadratic Lorenz-like systems when $\frac{b}{a} \geq 3$ and $\frac{b}{a} \geq 2$ [15–21]. On these grounds, the degree of some Lorenz-like systems may have some relativity with heteroclinic orbits. In addition, all of the aforementioned heteroclinic orbits are axis-symmetric or single ones. However, the scenario of centro-symmetric heteroclinic orbits of it is not considered at all to the best of our knowledge. Therefore, it is an urgent task to conduct the study.

The newly introduced Lorenz-like system has to satisfy at least three principles:

(1) This model has to be a centro-symmetric analogue.

(2) The degree of it should guarantee the generation of a pair of nontrivial centro-symmetric equilibria w.r.t. the origin.

(3) The method of combination of Lyapunov functions, the definitions of both α-limit set and ω-limit set should be applicable to it when proving the existence of heteroclinic orbits.

Based on the above three tips and trial-and-error, we try to search for a new subquadratic centro-symmetric Lorenz-like system with the targeted heteroclinic orbits.

To our knowledge, little seems to be known about the Lorenz-like system yet with cross products $\sqrt[3]{x^2}z$ and $\sqrt[3]{x^2}y$. Innovations of this paper are:

(1) Proposing a new three-dimensional subquadratic centro-symmetric Lorenz-like system.

(2) Proving the existence of a pair of centro-symmetric heteroclinic orbits.

(3) Confirming the correlation between the degree and heteroclinic orbits to some extent.

As a result, it is theoretically and practically important to analyze such a Lorenz-like system, motivating the follow-up research of this work.

2 New subquadratic centro-symmetric Lorenz-like system and basic dynamics

Combining the axis-symmetric quadratic/subquadratic Lorenz system family [11–14] and trial-and-error, we firstly introduce the new subquadratic centro-symmetric analogue as follows:

$$\begin{cases} \dot{x} = a(y - x), \\ \dot{y} = cx - \sqrt[3]{x^2}z, \quad a \neq 0, (c, b) \in \mathbb{R}^2, \\ \dot{z} = -bz + \sqrt[3]{x^2}y, \end{cases} \tag{2.1}$$

which is invariant under transformation $(x, y, z) \to (-x, -y, -z)$. Next, aiming at revealing the heteroclinic orbits, we present some basic dynamics of system (2.1) in the following theorems, i.e., the distribution of equilibrium points, stability, Hopf bifurcation, inequality, etc.

Theorem 2.1 (1) *When* $b = 0$, *system* (2.1) *has nonisolated equilibria* $E_z = \{(0, 0, z)|z \in \mathbb{R}\}$.

(2) *When* $b \neq 0$ *and* $bc < 0$, *system* (2.1) *has a single equilibrium point* $E_0 = (0, 0, 0)$.

(3) *When* $bc > 0$, *system* (2.1) *has two equilibria* $E_\pm = \pm(\sqrt[4]{(bc)^3}, \sqrt[4]{(bc)^3}, c\sqrt[4]{bc})$ *except for* E_0.

Table 1 Local dynamics of E_0

b	a	c	Property of E_0
<0	<0	<0	A 1D W^s_{loc} and a 2D W^u_{loc}
		>0	A 3D W^u_{loc}
	>0	<0	A 2D W^s_{loc} and a 1D W^u_{loc}
		>0	A 1D W^s_{loc} and a 2D W^u_{loc}
>0	<0	<0	A 2D W^s_{loc} and a 1D W^u_{loc}
		>0	A 1D W^s_{loc} and a 2D W^u_{loc}
	>0	<0	A 3D W^s_{loc}
		>0	A 2D W^s_{loc} and a 1D W^u_{loc}

Theorem 2.2 *When $abc \neq 0$, the local dynamical behaviors of E_0 are totally summarized in Table 1. While $a \neq 0$ and $b = 0$, all of E_z are unstable.*

Set $W = \{(a,c,b)|a \neq 0, bc > 0\}$, $W_1 = \{(a,c,b) \in W : a+b > 0, ab+bc-\frac{ac}{3} > 0, \frac{4abc}{3} > 0\}$, $\Delta = ab(a+b) - c[\frac{(a-b)(3b+a)}{3}]$, $W_2 = W \backslash W_1$, $W_1^1 = \{(a,c,b) \in W_1 : \Delta < 0\}$, $W_1^2 = \{(a,c,b) \in W_1 : \Delta = 0\}$, and $W_1^3 = \{(a,c,b) \in W_1 : \Delta > 0\}$.

Theorem 2.3 (1) *When $(a,c,b) \in W_1^1$ (resp. W_1^3), $E_\pm$ are unstable (resp. asymptotically stable).*

(2) *When $(a,c,b) \in W_1^2$, Hopf bifurcation happens at $E_\pm$.*

Theorem 2.4 *If $5a > 3b > 0$ and $t \to \infty$, then the inequality $z \geq \frac{3}{5a}\sqrt[3]{x^5}$ holds.*

The rest content is arranged as follows. The proofs of Theorems 2.2–2.4 are outlined in Sect. 3. Section 4 studies the existence of centro-symmetric heteroclinic orbits. Lastly, some conclusions are drawn, and the correlation between power of the polynomials and dynamics is also discussed.

3 Basic dynamics and proofs of Theorems 2.2–2.4

In this section, proofs of Theorems 2.2–2.4 are sketched as follows.

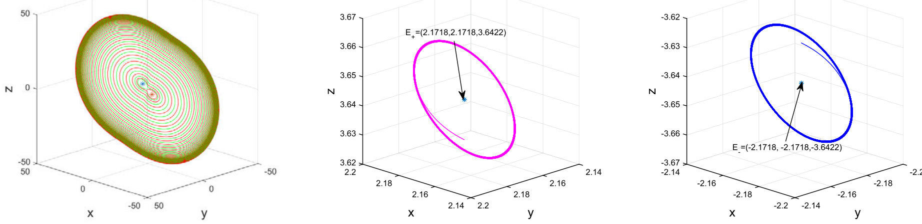
Proof of Theorem 2.2 Based on linear analysis, Theorem 2.2 easily follows, and the proof is omitted here. □

Proof of Theorem 2.3 The characteristic equation of matrix associated with the vector field of system (2.1) at $E_\pm$ is calculated as follows:

$$\lambda^3 + (a+b)\lambda^2 + \left[ab + bc - \frac{ac}{3}\right]\lambda + \frac{4abc}{3} = 0. \tag{3.1}$$

On the basis of Eq. (3.1) and Routh–Hurwitz criterion, $E_\pm$ are unstable (resp. asymptotically stable) when $(a,c,b) \in W_1^1$ (resp. W_1^3).

While $(a,c,b) \in W_1^2$, Eq. (3.1) has the negative real root $\lambda_1 = -(a+b) < 0$ and a pair of conjugate purely imaginary roots $\lambda_{2,3} = \pm\omega i = \pm 2ab\sqrt{\frac{1}{(a-b)(3b+a)}}i$. Furthermore, the transversal



(a) (b) (c)

Figure 1 When $(a,c,b) = (5, 2.8125, 1)$, (a) $(x_0^{1,2}, y_0^{1,2}, z_0^{1,2}) = (3.14, 1.618, 2.718) \times 10^{-6}$, $(x_0^{3,4}, y_0^{3,4}, z_0^{3,4}) = \pm(2.1618, 2.1818, 3.6322)$, (b) $(x_0^3, y_0^3, z_0^3) = (2.1618, 2.1818, 3.6322)$, (c) $(x_0^4, y_0^4, z_0^4) = -(2.1618, 2.1818, 3.6322)$, phase portraits of system (2.1). These figures illustrate that there exist three limit cycles when undergoing Hopf bifurcation at $E_\pm$

condition $\frac{d\,Re(\lambda_2)}{dc}|_{c=c^*} = \frac{(a-b)(a+3b)}{6[\omega^2+(a+b)^2]} \neq 0$ holds, where $c_* = \frac{3ab(a+b)}{(a-b)(3b+a)}$. Therefore, Hopf bifurcation occurs at $E_\pm$, as shown in Fig. 1. The proof is completed. □

Remark 3.1 When $(a,c,b) = (5, 2.8125, 1)$, the eigenvalues of $E_\pm$ are $\lambda_1 = -6$, $\lambda_{2,3} = \pm 1.7678i$.

Proof of Theorem 2.4 Set $Q(x,z) = z - \frac{3}{5a}\sqrt[3]{x^5}$ and compute the derivative of it along any one orbit of system (2.1): $\frac{dQ(x,z)}{dt}|_{(2.1)} = -bz + \frac{5a}{3}\sqrt[3]{x^5}$, i.e., $\dot{Q} + bQ = -(b - \frac{5a}{3})\sqrt[3]{x^5}$.

Based on the comparison principle, if $b - \frac{5a}{3} < 0$, then $\dot{Q} + bQ \geq 0$ leads to

$$Q(t) \geq Q_0 e^{-b(t-t_0)} \to 0, \quad (t \to \infty), \forall Q(t_0) = Q_0.$$

Namely, for $b - \frac{5a}{3} < 0$, we arrive at the inequality $\lim_{t\to\infty} Q(t) = \lim_{t\to\infty}[z - \frac{3}{5a}\sqrt[3]{x^5}] \geq 0$. The proof is finished. □

For discussion purposes, the following denotations have to be introduced:

(1) $p(t;q_0) = (x(t;x_0), y(t;y_0), z(t;z_0))$: each solution of system (2.1) through the initial value $q_0 = (x_0, y_0, z_0)$.

(2) $\gamma^\pm = \{p_\pm(t;q_0) | p_\pm(t;q_0) = \pm(x_+(t;x_0), y_+(t;y_0), z_+(t;z_0)) \in W^u_\pm, t \in \mathbb{R}\}$: the branch of the unstable manifold $W^u(E_0)$ corresponding to $x_+ > 0$ and $-x_+ < 0$ when $t \to -\infty$.

4 Existence of heteroclinic orbit

Combining the concepts of both α-limit and ω-limit set and Lyapunov functions, we prove the existence of centro-symmetric heteroclinic orbits of system (2.1) and arrive at the following result.

Theorem 4.1 *If* $c > 0$ *and* $3b \geq 5a > 0$, *then there exist no homoclinic orbits but a pair of centro-symmetric heteroclinic orbits*: $\gamma^\pm$ *to* E_0 *and* $E_\pm$.

Next, we prove Theorem 4.1 in two steps: (1) $3b - 5a > 0$, (2) $3b - 5a = 0$.

4.1 The case $3b - 5a > 0$

In this subsection, we first construct the following Lyapunov function:

$$\begin{aligned} V_1\bigl(p(t;q_0)\bigr) = \frac{1}{2}\biggl[& b(3b-5a)(y-x)^2 \\ & + 3\bigl(-bz + \sqrt[3]{x^5}\bigr)^2 + \frac{4\sqrt{bc}(3b-5a)}{5a}\bigl(-\sqrt{bc}\sqrt[3]{x^2} + \sqrt[3]{x^4}\bigr)^2 \\ & + \frac{2\sqrt{bc}(3b-5a)}{5a}\bigl(-bc + \sqrt[3]{x^4}\bigr)^2 + \frac{3(3b-5a)}{5a}\bigl(-\sqrt{bc}x + \sqrt[3]{x^5}\bigr)^2\biggr] \end{aligned}$$

and derive the following statements.

Lemma 4.2 *When $c > 0$ and $3b - 5a > 0$, we arrive at the following statements*:

1. *If $\exists t_{1,2}$, $t_1 < t_2$ and $V_1(p(t_1;q_0)) = V_1(p(t_2;q_0))$, then q_0 is one of the equilibrium points.*
2. *If $\lim_{t\to-\infty} p(t;q_0) = E_0$ and $x(t_3;q_0) > 0$, $\exists t_3 \in \mathbb{R}$, then $V_1(E_0) > V_1(p(t;q_0))$ and $x(t;x_0) > 0$, $\forall t \in \mathbb{R}$. As a result, $q_0 \in \gamma^+$.*

Proof (1) Taking the derivative of V_1 along $p(t;q_0)$ results in

$$\frac{dV_1(p(t;q_0))}{dt}\bigg|_{(2.1)} = -ab(3b-5a)(y-x)^2 - 3b\bigl(-bz + \sqrt[3]{x^5}\bigr)^2, \tag{4.1}$$

and thus leads to

$$y(t;y_0) - x(t;x_0) \equiv 0, \qquad -bz(t;z_0) + \sqrt[3]{x^5(t;x_0)} \equiv 0, \tag{4.2}$$

under the condition of (1), $\forall t \in (t_1, t_2)$.

On the basis of Eq. (4.2) and system (2.1), we obtain the identities $\dot{x}(t;x_0) \equiv \dot{y}(t;y_0) \equiv \dot{z}(t;z_0) \equiv 0$, $\forall t \in (t_1, t_2)$. Therefore, system (2.1) has the stationary point q_0.

(2) Let us first show $V_1(E_0) > V_1(p(t;q_0))$, $\forall t \in \mathbb{R}$. Otherwise, $V_1(E_0) \le V_1(p(t;q_0))$, $\exists t \in \mathbb{R}$. Then, the first assertion yields that q_0 is just one of the equilibria of system (2.1), which contradicts the assumed condition $\lim_{t\to-\infty} p(t;q_0) = E_0$ and $x(t_3;x_0) > 0$. Namely, $V_1(E_0) > V_1(p(t;q_0))$, $\forall t \in \mathbb{R}$.

Next, we prove $x(t;x_0) > 0$, $\forall t \in \mathbb{R}$. If not, $x(t_4;x_0) \le 0$, $\exists t_4 \in \mathbb{R}$. Due to $x(t_3;x_0) > 0$, $t_3 \in \mathbb{R}$, we obtain $x(t_5;x_0) = 0$, $\exists t_5 \in \mathbb{R}$. Since $V_1(E_0) > V_1(p(t;q_0))$, $\forall t \in \mathbb{R}$, we arrive at

$$p(t_5;q_0) \in \bigl\{(x,y,z) | V_1(x,y,z) < V_1(E_0)\bigr\} \cap \bigl\{(x,y,z) | x = 0\bigr\}.$$

In addition, the fact holds:

$$\begin{aligned} & \bigl\{(x,y,z) | V_1(x,y,z) < V_1(E_0)\bigr\} \cap \bigl\{(x,y,z) | x = 0\bigr\} \\ & = \left\{(0,y,z) \Big| \frac{1}{2}\left[b(3b-5a)y^2 + 3b^2z^2 + \frac{2\sqrt{bc}b^2c^2(3b-5a)}{5a}\right] < \frac{\sqrt{bc}b^2c^2(3b-5a)}{5a}\right\} \\ & = \emptyset. \end{aligned}$$

A contradiction happens. Consequently, the fact $x(t;x_0) > 0$, $\forall t \in \mathbb{R}$ holds, and the proof is completed. □

Lemma 4.3 *When $c > 0$, $3b > 5a > 0$, and $t \to \infty$, each solution of system (2.1) approaches one of its equilibrium points. In a word, closed orbits are nonexistent in system (2.1).*

Proof From Eq. (4.1), we deduce $\lim_{t\to+\infty} V_1(p(t;q_0)) = \Phi(q_0)$ and $0 \le V_1(p(t;q_0)) \le V_1(p(0;q_0)) = V_1(q_0)$, $\forall t \ge 0$, implying the boundedness of $x(t;x_0)$, $y(t;y_0)$ and $z(t;z_0)$, $t \in [0,+\infty)$. Namely, the set $\{p(t;q_0)|t \ge 0\}$ is bounded.

Let $\Omega(q_0) \ne \emptyset$ be the ω-limit set of $p(t;q_0)$. For $\forall q \in \Omega(q_0)$, i.e., $\exists\{t_n\}$, such that

$$\lim_{n\to+\infty} t_n = +\infty, \qquad \lim_{n\to+\infty} p(t_n,q_0) = q.$$

Next, $\forall t \in \mathbb{R}$, $p(t;q) = \lim_{n\to+\infty} p(t;p(t_n;q_0)) = \lim_{n\to+\infty} p(t+t_n;q_0)$ yields $V_1(p(t;q)) = V_1[\lim_{n\to+\infty} p(t;p(t_n;q_0))] = \lim_{n\to+\infty} V_1(p(t+t_n;q_0)) = \Phi(q_0)$. Therefore, $q \in \{E_-, E_0, E_+\}$. Since $\Omega(q_0)$ is connected, we only obtain $\Omega(q_0) = \{E_-\}$ or $\Omega(q_0) = \{E_0\}$, or $\Omega(q_0) = \{E_+\}$, which suggests that $p(t;q_0)$ converges to one of the equilibria when $t \to +\infty$. Therefore, the proof is finished. □

From Lemmas 4.2–4.3, we prove the existence of heteroclinic orbits.

Theorem 4.4 *If $c > 0$ and $3b > 5a > 0$, then*

1. *Homoclinic orbits are nonexistent in system (2.1);*
2. *System (2.1) has a pair of centro-symmetric heteroclinic orbits: γ^+ joining E_+ and E_0, and γ^- joining and E_- and E_0.*

Proof Let us prove that both homoclinic orbits and heteroclinic orbits to $E_\pm$ are nonexistent in system (2.1) when $c > 0$ and $3b > 5a > 0$. Otherwise, let $p(t) = (x(t),y(t),z(t))$ be a homoclinic (resp. heteroclinic) orbit to E_0 or E_+, or E_- (resp. E_+ and E_-), i.e., $\lim_{t\to-\infty} p(t) = e^-$, $\lim_{t\to+\infty} p(t) = e^+$, where $e^- = e^+ \in \{E_-, E_0, E_+\}$ or $\{e^-, e^+\} = \{E_-, E_+\}$.

It follows from Eq. (4.1) that

$$V_1(e^-) \ge V_1(p(t)) \ge V_1(e^+) \tag{4.3}$$

holds. In either case, we only obtain the relation $V_1(e^-) = V_1(e^+)$, which thus results in $V_1(p(t)) \equiv V_1(e^+)$. In virtue of the first assertion of Lemma 4.2, $p(t)$ is just one of the fixed points. As a result, there exist neither homoclinic orbits to E_0 or E_+, or E_-, nor heteroclinic orbits to E_- and E_+.

Next, we show that γ^+ is a heteroclinic orbit to E_0 and E_+, i.e., $\lim_{t\to+\infty} p(t) = E_+$. On the basis of the definition of γ^+ and the second assertion of Lemma 4.2, we arrive at $x_+(t) > 0$, $\forall t \in \mathbb{R}$, which also yields $\lim_{t\to+\infty} p(t) \ne E_-$. Meanwhile, the definition of γ^+ leads to $\lim_{t\to+\infty} p(t) \ne E_0$. Thus, $\lim_{t\to+\infty} p_+(t) = E_+$ holds.

At last, we prove that, if there exists a heteroclinic orbit to E_0 and E_+ in system (2.1), then it is nothing but γ^+.

Define the $p_1(t) = (x_1(t),y_1(t),z_1(t))$ to be any one solution of system (2.1) such that

$$\lim_{t\to-\infty} p_1(t) = e_1^-, \qquad \lim_{t\to+\infty} p_1(t) = e_1^+,$$

where $\{e_1^-, e_1^+\} = \{E_0, E_+\}$. Similar to Eq. (4.3), we arrive at $V_1(e_1^-) \ge V_1(p_1(t)) \ge V_1(e_1^+)$, $\forall t \in \mathbb{R}$, based on Eq. (4.2). Since $V_1(E_0) > V_1(E_+)$, we deduce that $e_1^- = E_0$ and $e_1^+ = E_+$, i.e.,

$$\lim_{t\to-\infty} p_1(t) = E_0, \qquad \lim_{t\to+\infty} p_1(t) = E_+,$$

which results in $p_1(t) \in \gamma^+$ from the second assertion of Lemma 4.2. Since system (2.1) is centro-symmetrical w.r.t. the origin E_0, there is a unique heteroclinic orbit γ^- centro-symmetrical to γ^+. The proof is completed. □

4.2 The case $3b - 5a = 0$

This subsection first introduces the second Lyapunov function

$$\begin{aligned} V_2\big(p(t;q_0)\big) = \frac{1}{2}\Bigg[&\frac{25a^2}{9}(y-x)^2 + \frac{4}{3}\sqrt{\frac{5ac}{3}}\left(-\sqrt{\frac{5ac}{3}}\sqrt[3]{x^2} + \sqrt[3]{x^4}\right)^2 \\ &+ \frac{2}{3}\sqrt{\frac{5ac}{3}}\left(-\frac{5ac}{3} + \sqrt[3]{x^4}\right)^2 + \left(-\frac{5ac}{3}x + \sqrt[3]{x^5}\right)^2\Bigg] \end{aligned}$$

and the following statements.

Lemma 4.5 *For $c > 0$ and $3b = 5a > 0$, the following assertions hold*:

(i) If $\lim_{t\to-\infty} p(t;q_0)$ *is bounded, then* $z(t;z_0) = \frac{3}{5a}\sqrt[3]{x^5(t;x_0)}$;

(ii) If $z(t;z_0) = \frac{3}{5a}\sqrt[3]{x^5(t;x_0)}$, *then* $\frac{dV_2(p(t;q_0))}{dt}|_{(2.1)} = -\frac{25a^3}{9}(y-x)^2 \le 0$;

(iii) If $z(t;z_0) = \frac{3}{5a}\sqrt[3]{x^5(t;x_0)}$ *and* $V_2(p(t_1;q_0)) = V_2(p(t_2;q_0))$, $\exists t_{1,2}$, $t_1 < t_2$, *then q_0 is one of the equilibrium points*;

(iv) If $\lim_{t\to-\infty} p(t;q_0) = E_0$ *and* $x(t_3;q_0) > 0$, $\exists t_3 \in \mathbb{R}$, *then* $V_2(E_0) > V_2(p(t;q_0))$ *and* $x(t;q_0) > 0$, $\forall t \in \mathbb{R}$. *In a word,* $q_0 \in \gamma^+$.

Proof (i) From Proof of Theorem 2.4, we arrive at $\frac{dQ(p(t;q_0))}{dt}|_{(2.1)} = -\frac{5a}{3}Q(p(t;q_0))$, i.e.,

$$Q\big(p(t;q_0)\big) = Q\big(p(\tau;q_0)\big)e^{-\frac{5a}{3}(t-\tau)}, \quad \forall \tau, t \in \mathbb{R}. \tag{4.4}$$

Because $\lim_{\tau\to-\infty} p(\tau;q_0)$ is bounded, Eq. (4.4) suggests $Q(p(t;q_0)) \equiv 0$, i.e., $z(t;z_0) \equiv \frac{3}{5a}\sqrt[3]{x^5(t;x_0)}$.

(ii) The result easily follows from the first assertion $z(t;z_0) \equiv \frac{3}{5a}\sqrt[3]{x^5(t;x_0)}$ and system (2.1).

(iii) The second assertion yields $\frac{dV_2(p(t,q_0))}{dt}|_{(2.1)} = 0$, $\forall t \in (t_1,t_2)$, i.e.,

$$y(t;y_0) - x(t;x_0) \equiv 0. \tag{4.5}$$

Combining $\dot{x} = a(y-x)$, Eq. (4.5), and $z(t;z_0) \equiv \frac{3}{5a}\sqrt[3]{x^5(t;x_0)}$, we derive

$$\dot{x}(t;x_0) \equiv \dot{y}(t;y_0) \equiv \dot{z}(t;z_0) \equiv 0, \quad \forall t \in (t_1,t_2).$$

Hence, q_0 is just one of the equilibria.

(iv) Let us prove $V_2(E_0) > V_2(p(t;q_0))$, $\forall t \in \mathbb{R}$. Otherwise, $V_2(E_0) \le V_2(p(t_0;q_0))$, $\exists t_0 \in \mathbb{R}$. On the other hand, assertions (i)–(iii) yield that q_0 is one of the equilibria, which contradicts $\lim_{t\to-\infty} p(t;q_0) = 0$ and $x(t_3;x_0) > 0$. Thus, we arrive at $V_2(E_0) > V_2(p(t;q_0))$, $\forall t \in \mathbb{R}$.

Next, one shows $x(t;x_0) > 0$, $\forall t \in \mathbb{R}$. Otherwise, $\exists t_4 \in \mathbb{R}$ such that $x(t_4;x_0) \le 0$. Because of $x(t_3;x_0) > 0$, $\exists t_5 \in \mathbb{R}$ such that $x(t_5;x_0) = 0$. Since $V_2(E_0) > V_2(p(t;q_0))$, $\forall t \in \mathbb{R}$, we arrive at

$$p(t_5;q_0) \in \big\{(x,y,z)|V_2(x,y,z) < V_2(E_0)\big\} \cap \big\{(x,y,z)|x = 0\big\}.$$

Instead, $\{(x,y,z)|V_2(x,y,z) < V_2(E_0)\} \cap \{(x,y,z)|x = 0\} = \{(0,y,z)|\frac{1}{2}[\frac{25a^2}{9}y^2 + \frac{50a^2c^2}{27}\sqrt{\frac{5ac}{3}}] < \frac{25a^2c^2}{27}\sqrt{\frac{5ac}{3}}\} = \emptyset$. A contradiction occurs. Consequently, $x(t;x_0) > 0$ is true, $\forall t \in \mathbb{R}$. □

Lemma 4.6 *Consider $c > 0$ and $3b = 5a > 0$. If any one negative orbit with initial point q_0 is bounded, then $\lim_{t\to-\infty} p(t,q_0)$ converges to one of the equilibria of system (2.1). Therefore, closed orbits are nonexistent in system (2.1).*

Proof It follows from assertions (i)–(ii) that $\lim_{t\to-\infty} V_2(p(t;q_0)) = \Psi(q_0)$ exists. Suppose $q \in \alpha(q_0)$, i.e., $\exists\{t_n\}$ such that

$$\lim_{n\to+\infty} t_n = -\infty \quad \text{and} \quad \lim_{n\to+\infty} p(t_n;q_0) = q.$$

$\forall t \in \mathbb{R}$, the fact

$$p(t;q) = \lim_{n\to+\infty} p\big(t;p(t_n;q_0)\big) = \lim_{n\to+\infty} p(t + t_n;q_0)$$

results in

$$\begin{cases} p(t;q) \quad \text{is bounded on } \mathbb{R}, \\ V_2(p(t;q)) = \lim_{n\to+\infty} V_2(p(t + t_n;q_0)) = \Psi(q_0). \end{cases} \tag{4.6}$$

According to Lemma 4.5, we arrive at $q \in \{E_-, E_0, E_+\}$. Thus,

$$\alpha(q_0) \subseteq \{E_-, E_0, E_+\}.$$

Since $\alpha(q_0)$ is connected, we obtain $\alpha(q_0) = \{E_-\}$ or $\alpha(q_0) = \{E_0\}$, or $\alpha(q_0) = \{E_+\}$, which yields that $\lim_{n\to+\infty} p(t;q_0)$ converges to one of the equilibria. The proof is over. □

Theorem 4.7 *If $c > 0$ and $3b = 5a > 0$, then*

(*i*) *system (2.1) has no homoclinic orbits;*

(*ii*) *system (2.1) has only a pair of centro-symmetric heteroclinic orbits: γ^+ joining E_0 and E_+, and γ^- joining E_0 and E_-.*

Proof (i) Let us prove that neither homoclinic orbits nor heteroclinic orbits joining E_- and E_+ exist in system (2.1) when $c > 0$ and $3b = 5a > 0$. If not, suppose that $p(t) = (x(t), y(t), z(t))$ is a homoclinic or heteroclinic orbit to E_- and E_+, i.e.,

$$\lim_{t\to-\infty} p(t) = e^- \quad \text{and} \quad \lim_{t\to+\infty} p(t) = e^+,$$

where e^- and e^+ satisfy either

$$e^- = e^+ \in \{E_-, E_0, E_+\} \quad \text{or} \quad \big\{e^-, e^+\big\} = \{E_-, E_+\}.$$

It follows from Lemma 4.5 and $V_2(e^-) = V_2(e^+)$ that $p(t)$ is just one of the stationary points.

As a result, both homoclinic orbits and heteroclinic orbits to $E_\pm$ are nonexistent in system (2.1).

(ii) Let us prove that if system (2.1) has a heteroclinic orbit to E_0 and E_+, then it is just γ^+. Assume $p_1(t) = (x_1(t), y_1(t), z_1(t))$ is any one solution of system (2.1) such that

$$\lim_{t\to-\infty} p_1(t) = e_1^- \quad \text{and} \quad \lim_{t\to+\infty} p_1(t) = e_1^+,$$

where $\{e_1^-, e_1^+\} = \{E_0, E_+\}$. For $\forall t \in \mathbb{R}$, assertions (i)–(ii) of Lemma 4.5 suggest

$$V_2(e_1^-) \geq V_2(p_1(t)) \geq V_2(e_1^+).$$

Since $V_2(E_0) > V_2(E_+)$, we arrive at $e_1^- = E_0$ and $e_1^+ = E_+$, i.e.,

$$\lim_{t\to-\infty} p_1(t) = E_0 \quad \text{and} \quad \lim_{t\to+\infty} p_1(t) = E_+, \tag{4.7}$$

yielding $p_1(t) \in \gamma^+$ based on the fourth assertion of Lemma 4.5.

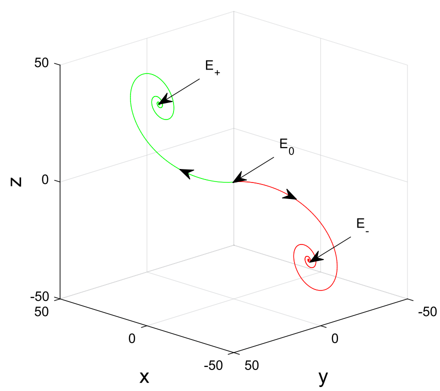
At last, let us prove that γ^+ is a heteroclinic orbit to E_0 and E_+, i.e., $\lim_{t\to+\infty} p_+(t) = E_+$. From Lemma 4.5, we arrive at

$$\begin{cases} z_+(t; z_0) \equiv \frac{3}{5a}\sqrt[3]{x_+^5(t; x_0)}, \\ \frac{dV_2(p_+(t))}{dt}\big|_{(2.1)} = -\frac{25a^3}{9}(y_+(t; y_0) - x_+(t; x_0))^2, \\ V_2(p_+(t)) < V_2(E_0), \quad \forall t \in \mathbb{R}, \\ x_+(t) > 0, \quad \forall t \in \mathbb{R}. \end{cases} \tag{4.8}$$

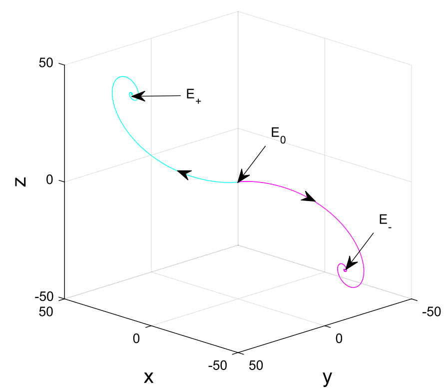
The second equation of Eq. (4.8) suggests that $\lim_{t\to\infty} V_2(p_+(t)) = \nu$ exists. Again, Eq. (4.8) indicates the boundedness of $x_+(t)$, $y_+(t)$ and $z_+(t)$, $\forall t \in [0, +\infty)$, i.e., the set $\{p_+(t) | t \geq 0\}$ is bounded. Define Ω to be the ω-limit set of solution $p_+(t)$. Suppose $q \in \Omega$, i.e., $\exists \{t_n\}$ such that $\lim_{n\to+\infty} t_n = +\infty$ and $\lim_{n\to+\infty} p_+(t_n) = q$. Therefore, $\forall t \in \mathbb{R}$, the relation

$$p(t; q) = \lim_{n\to+\infty} p(t; p_+(t_n)) = \lim_{n\to+\infty} p_+(t + t_n) = q,$$
$$\begin{cases} p(t; q) \quad \text{is bounded on } \mathbb{R}, \\ V_2(p(t; q)) = \lim_{n\to+\infty} V_2(p_+(t + t_n)) = \nu, \end{cases} \tag{4.9}$$

and together with assertions (i)–(iii) of Lemma 4.5 results in $q \in \{E_-, E_0, E_+\}$. Consequently, $\Omega \subseteq \{E_-, E_0, E_+\}$. Because Ω is connected, we obtain $\Omega = E_-$ or $\Omega = E_0$, or $\Omega = E_+$. On the basis of assertion (ii) of Lemma 4.5 and the fourth equality of Eq. (4.8), we arrive at $\Omega \neq E_0$ and $\Omega \neq E_-$. Thus, $\Omega = E_+$, i.e., $\lim_{n\to+\infty} p_+(t) = E_+$. Due to the central symmetry of system (2.1), there is a unique heteroclinic orbit γ^- to E_- and E_0, as illustrated in Fig. 2. The proof is completed. □



(a) $b = 5$



(b) $b = 8$

Figure 2 Phase portraits of system (2.1) with $(a,c) = (3,12)$, $(x_0^{1,2}, y_0^{1,2}, z_0^{1,2}) = \pm(3.14, 1.618, 2.718) \times 10^{-6}$, (a) $b = 5$, (b) $b = 8$. Both figures suggest that system (2.1) has a pair of centro-symmetric heteroclinic orbits to $E_\pm$ and E_0 when $c > 0$ and $3b \geq 5a > 0$

5 Conclusions

As an effective method to study the existence of heteroclinic orbits, the combination of Lyapunov functions, the definitions of ω-limit set and α-limit set has been widely applied in many axis-symmetric Lorenz-like systems. Whether or not it is applicable to the centro-symmetric ones. In this effort, based on the extension of the second part of the celebrated Hilbert's 16th problem and a trial and error process, this paper reports another new subquadratic three-dimensional Lorenz-like system and proves the existence of a pair of centro-symmetric heteroclinic orbits by aid of the aforementioned method. Moreover, for $(a, c) = (3, 12)$ and $b = 5, 8$, Fig. 2 validates the correctness of theoretical results.

In future work, some interesting issues deserve consideration. First, whether or not strange attractors and pseudo singularly degenerate heteroclinic cycles exist. Second, the existence of some other global dynamics, i.e., homoclinic orbits, boundedness, and so on. Finally, the relationship between the degree and heteroclinic orbits, and real world applications.

Acknowledgements
This work is supported in part by the National Natural Science Foundation of China under Grant 12001489, in part by Zhejiang Public Welfare Technology Application Research Project of China Grant LGN21F020003, in part by the Natural Science Foundation of Zhejiang Guangsha Vocational and Technical University of Construction under Grant 2022KYQD-KGY, in part by the Natural Science Foundation of Taizhou University under Grant T20210906033. Meanwhile, the authors would like to express their sincere thanks to the anonymous editors and reviewers for their conscientious reading and numerous constructive comments, which improved the manuscript substantially.

Author contributions
Haijun Wang: conceptualization, software, writing—original draft, Investigation. Jun Pan: supervision, visualization, validation, writing—review & editing. Guiyao Ke: software, methodology, investigation, visualization. Feiyu Hu: software, validation.
All authors read and approved the final manuscript.

Data availability
There is no data because the results obtained in this paper can be reproduced based on the information given in this paper.

Declarations

Competing interests
The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Received: 24 April 2024 Accepted: 2 May 2024 Published online: 22 May 2024

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