Wang *Advances in Continuous and Discrete Models* (2024) 2024:17
https://doi.org/10.1186/s13662-024-03813-8

Advances in Continuous and Discrete Models

RESEARCH **Open Access**

Mild solution for (ρ, k, Ψ)-proportional Hilfer fractional Cauchy problem

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Abstract

Hilfer fractional derivative is an important and interesting operator in fractional calculus, and it can be applicable in pure theories and other fields. It yields to other notable definitions, Ψ-Hilfer, (k, Ψ)-Hilfer derivatives, etc. Motivated by the concepts of the proportional fractional derivative and (k, Ψ)-Hilfer fractional derivative, we first introduce new definitions of integral and derivative, termed the (ρ, k, Ψ)-proportional integral and (ρ, k, Ψ)-proportional Hilfer fractional derivative. This type of fractional derivative is advantageous as it aligns with earlier studies on fractional differential equations. Additionally, we present a more generalized version of the $(\rho, \alpha, \beta, k, r)$-resolvent family, followed by an exploration of its properties. By analyzing the generalized resolvent family, we examine the existence of mild solutions to the (ρ, k, Ψ)-proportional Hilfer fractional Cauchy problem, supported by an illustrative example to show the main result.

Mathematics Subject Classification: 26A33; 34A12

Keywords: $(\rho, \alpha, \beta, k, r)$-resolvent family; (ρ, k, Ψ)-proportional Hilfer derivative; Mild solution; Existence

1 Introduction

Currently, the application of fractional differential equations in physics, engineering, medicine, and other fields is garnering increasing attention, as evidenced by the references [6, 15, 18, 28, 33, 34]. R. Hilfer [20] proposed the Hilfer (H) fractional derivative, which has technical properties that make it significantly more general than previous fractional derivatives. Due to this reason, the H fractional derivative serves as a more comprehensive derivative for analyzing real-world events and the resulting technological advancements [19]. In 2018, a new fractional derivative was proposed by Sousa et al. [10], called the Ψ-Hilfer (Ψ-H) fractional derivative. It generalizes earlier and important fractional derivatives, such as the H, Caputo (C), and Riemann-Liouville (RL) fractional derivatives. This Ψ-H derivative type has the advantage of being flexible in choosing the kernel Ψ, allowing the unification and recovery of most earlier studies to fractional differential equations. Furthermore, the Leibniz-type rule for the Ψ-H derivative was considered in [11]. Recently, K.A. Aldwoah et al. [1] considered the existence and Ulam-Hyers stability of solutions for a terminal value w-Hilfer fractional differential system in different weighted

spaces. However, the H- and Ψ-H fractional derivatives are relevant to the gamma function. In 2007, Díaz et al. [14] proposed the generalized gamma function $\Gamma_k(\cdot)$, the fractional derivatives have been updated and expanded since then. For example, Kucche et al. gave the (k,Ψ)-Hilfer ((k,Ψ)-H) fractional derivative [25], which is more general than the Ψ-H fractional derivative. Some exceptional cases, such as the k-Hilfer-Hadamard fractional derivative, (k,Ψ)-C, (k,Ψ)-RL, and others can be obtained by choosing the corresponding kernel Ψ, $\nu \in [0,1]$ and $k > 0$. In 2015, Anderson et al. [4] defined the proportional fractional derivative, which is more general than the fractional derivative given in [23]. The advantage of this fractional derivative type is its flexibility to choose the proportion ρ. Furthermore, in addition to the (k,Ψ)-H fractional derivative, the proportional fractional derivative also extends certain earlier fractional derivatives from another aspect.

The fractional evolution equation is more general than the fractional differential equation. An important method in studying the fractional evolution equation is the resolvent operator. The research publications [5, 32] of the fractional resolvent operator theory of abstract integral equations can be reviewed by the readers. Remarkably, Zhou et al. [37, 38] considered the abstract Cauchy problem by using the property of the C_0-semigroup and probability density function when the order $\alpha \in (0,1)$. In 2021, Sousa et al. [13] presented a new Gronwall inequality in the H derivative sense and then used it to consider the existence of mild solution for H impulsive differential equations in Pδ-normed space, the generalized δ-Ulam-Hyers-Rassias stability of the equations was also discussed. Li et al. [27] introduced the fractional resolvent operator $S_\alpha(t)$ and dealt with a class of fractional abstract problems when the order $\alpha \in (1,2)$. The existence of a mild solution is the fundamental concept of evolution equation theory, and it is interesting for pure mathematicians. Various technical tools are used to arrive at this purpose, including the fixed point theorem, iterative technique, stability, etc. In [31], R. Ponce introduced the concept of (α,β)-resolvent family and outlined its properties. The existence of mild solutions to two types of nonlocal problems was also obtained by using the properties of $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$. In 2021, Sousa et al. [12] considered the Ulam-Hyers stability of mild solutions of the H fractional abstract Cauchy problem on both finite and infinite intervals, respectively. Based on $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$, Chang et al. [8] introduced the more general resolvent family and examined the fractional evolution equations of the Sobolev type. We also emphasize the paper by Sousa et al. [9] on the ε-regular mild solution for the H fractional abstract equation. However, studies of more general resolvent families of operators have received limited attention. How to define the proper resolvent family $\{S_{\rho,\alpha,\beta,k,r}\}_{t\geq 0}$ corresponding to the (ρ,k,Ψ)-proportional H fractional derivative from the purely algebraic conditions? Moreover, this resolvent family should maintain the properties of $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$. The problems mentioned above are the main challenges in this work. Our objective is not only to propose a more general version for resolvent families of operators but also to discuss the norm continuous, compactness, and other properties of it. To the best of our knowledge, no research on the (ρ,k,Ψ)-proportional H fractional derivative and the resolvent family $\{S_{\rho,\alpha,\beta,k,r}\}_{t\geq 0}$ has been published. We highlight the main contributions in this work, namely:

1. We propose new (ρ,k,Ψ)-proportional integral and (ρ,k,Ψ)-proportional H fractional derivative. Some particular cases are illustrated to point out their generality.
2. We present a new definition of the resolvent family in the sense of $\{S_{\rho,\alpha,\beta,k,r}\}_{t\geq 0}$, which extends some previous standard resolvent operator functions.

3. The existence of mild solutions of (ρ, k, Ψ)-proportional H fractional Cauchy problem is considered. An example was given to show the corresponding Theorems.
4. All Results can be applied for special cases, see Remarks 2.2, 2.3, and 2.4.

Our results can be regarded as a generalization of corresponding conclusions from some authors' papers. In Sect. 2, a listing of preliminary information would be beneficial. Section 3 examines into the properties of this general resolvent family, including but not limited to its continuous and compactness. In Sect. 4, we consider the existence of mild solutions for the (ρ, k, Ψ)-proportional H fractional Cauchy problem, and two theorems are obtained through diverse methods. Finally, we give an example to illustrate the main results.

2 Preliminaries

Let J be a finite closed interval of $\mathbb{R}$, and assume that E is a Banach space endowed with the norm $\|\cdot\|$. Let $C(J,E)$ be a Banach space of $\{f \mid f : J \to E \text{ is continuous}\}$ and endowed with the usual norm $\|x\|_C = \max_{t\in J} \|x(t)\|$. Without confusion, the norm in $C(J,\mathbb{R})$ is defined by $\|x\|_\infty = \max_{t\in J} |x(t)|$. Moreover, let $\mathfrak{B}(X,Y)$ denote the space of all bounded linear operators mapping from a Banach space X into another Banach space Y, and we abbreviate $\mathfrak{B}(X,X)$ to $\mathfrak{B}(X)$.

Further, we let $\Psi : \mathbb{R} \to \mathbb{R}$ be an odd function with $\Psi'(t) > 0$.

Definition 2.1 The operator family $\{S(t)\}_{t\geq 0} \subset \mathfrak{B}(E)$ is said to be general exponentially bounded (GEB) if there exist $M, k, r > 0$, $\omega \in \mathbb{R}$ such that the following inequality is satisfied:

$$\|S(t)\| \leq M e^{\omega k^{1-\frac{r}{k}} \Psi(t)}, \quad t \geq 0.$$

We say that a type of $S(t)$ is (M, ω, k, r). Set $\omega_0(S) = \inf_{\omega\in\mathbb{R}}\{\exists M \geq 0 \text{ such as } \|S(t)\| \leq M e^{\omega k^{1-\frac{r}{k}} \Psi(t)}, t \geq 0\}$.

Definition 2.2 [21] If f and h are two functions, the more general convolution of f and h is given by

$$(f *_\Psi h)(t) = \int_0^t f(\tau)\overline{h}(t,\tau)\Psi'(\tau)\,\mathrm{d}\tau,$$

where we abbreviate this notation $h(\Psi^{-1}(\Psi(t) - \Psi(\tau)))$ to $\overline{h}(t,\tau)$.

Definition 2.3 If f and h are two functions, we define another convolution by

$$\big(f *^\Psi h\big)(t) = \int_{-\infty}^{\infty} f(\tau)\overline{h}(t,\tau)\Psi'(\tau)\,\mathrm{d}\tau.$$

Let $\rho, k, \eta > 0$, for brevity, denote

$$g_{\rho,k,\eta}(t) = \begin{cases} 0, & t \leq 0, \\ \frac{\Psi^{\frac{\eta}{k}-1}(t)}{\rho^{\frac{\eta}{k}} k\Gamma_k(\eta)} e^{\frac{(\rho-1)\Psi(t)}{\rho k}}, & t > 0, \end{cases}$$

and $\hat{g}_{\rho,k,\eta}(t,s,\tau) = \frac{[\Psi(t)-\Psi(\tau)]^{\frac{\eta}{k}-1}}{\rho^{\frac{\eta}{k}}k\Gamma_k(\eta)}\mathrm{e}^{\frac{(\rho-1)[\Psi(s)-\Psi(\tau)]}{\rho k}}\Psi'(\tau)$, $t,s,\tau \geq 0$, where $\Gamma_k(z)$ denotes the k-gamma function

$$\Gamma_k(z) = \int_0^\infty s^{z-1}\mathrm{e}^{-\frac{s^k}{k}}\,\mathrm{d}s, \quad \mathrm{Re}(z) > 0.$$

Definition 2.4 If $\rho \in (0,1]$, $k > 0$, then the first-order (ρ,k,Ψ)-proportional derivative ${}^kD^{1,\rho;\Psi}h$ is defined by

$${}^kD^{1,\rho;\Psi}h(t) = \rho\left(\frac{k}{\Psi'(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right)h(t) + (1-\rho)h(t).$$

Remark 2.1 For $k = 1$, $\Psi(t) = t$, the first-order (ρ,k,Ψ)-proportional derivative ${}^kD^{1,\rho;\Psi}h(t)$ reduces to $D^{1,\rho}h(t)$ [22].

Definition 2.5 Let $k > 0$, $\rho \in (0,1]$, and h is integrable on $[0,b]$, then

$${}^kI_{0+}^{1,\rho;\Psi}h(t) = \frac{1}{\rho k}\int_0^t \mathrm{e}^{\frac{\rho-1}{\rho k}[\Psi(t)-\Psi(s)]}\Psi'(s)h(s)\,\mathrm{d}s$$

is said to be the first-order (ρ,k,Ψ)-proportional integral of the function h.

Definition 2.6 [7] Let $r,k > 0$ and $h : [0,\infty) \to \mathbb{R}$. The more general (k,Ψ)-Laplace transform to h is given by

$$\mathcal{L}_k^{r;\Psi}\big(h(t)\big)(\lambda) = \int_0^\infty \mathrm{e}^{-\lambda k^{1-\frac{r}{k}}\Psi(t)}h(t)\Psi'(t)\,\mathrm{d}t.$$

Definition 2.7 Let $r,k > 0$ and $h : \mathbb{R} \to \mathbb{R}$. We define the more general transform of h as follows:

$$\mathcal{F}_k^{r;\Psi}\big(h(t)\big)(\lambda) = \int_{-\infty}^\infty \mathrm{e}^{-i\lambda k^{1-\frac{r}{k}}\Psi(t)}h(t)\Psi'(t)\,\mathrm{d}t.$$

Definition 2.8 If $k > 0$, $\eta > 0$, $\rho \in (0,1]$, h is integrable on $[0,b]$, then we say

$${}^kI_{0+}^{\eta,\rho;\Psi}h(t) = \int_0^t \hat{g}_{\rho,k,\eta}(t,t,s)h(s)\,\mathrm{d}s$$

is the (ρ,k,Ψ)-proportional integral of h, and η is the order of h.

Remark 2.2 For $\rho = 1$, (ρ,k,Ψ)-proportional integral reduces to (k,Ψ)-RL integral [26]. For $\rho = 1$, $\Psi(t) = t$, (ρ,k,Ψ)-proportional integral reduces to k-RL integral [29]. For $\rho = 1$, $k = 1$, (ρ,k,Ψ)-proportional integral reduces to Ψ-RL integral [24]. For $k = 1$, $\Psi(t) = t$, (ρ,k,Ψ)-proportional integral reduces to GPF integral [22].

The use of integrals in the aforementioned definitions implies that the function is Bochner integrable if h is an abstract-valued function.

Definition 2.9 Let $\rho \in (0,1]$, $k > 0$, $m \in \mathbb{Z}^+$ with $m - 1 < \frac{\eta}{k} \leq m$, function h has m times continuously differentiable on $[0,b]$, then we say

$$ {}^{k,H}D_{0+}^{\eta,\rho,\nu;\Psi}h(t) = {}^{k}I_{0+}^{\nu(mk-\eta),\rho;\Psi}\,{}^{k}D^{m,\rho;\Psi}\,{}^{k}I_{0+}^{(1-\nu)(mk-\eta),\rho;\Psi}h(t), \quad \nu \in [0,1] $$

is the (ρ,k,Ψ)-proportional H fractional derivative of h. Moreover, η and ν are the order and type of h, receptively.

Remark 2.3 For $\Psi(t) = t$, $\rho = 1$, $\nu = 0$, (ρ,k,Ψ)-proportional H derivative reduces to k-RL derivative [16], and RL derivative for $k = 1$ [24]. For $\Psi(t) = t$, $\rho = 1$, $\nu = 1$, $k = 1$, (ρ,k,Ψ)-proportional H derivative reduces to C derivative [24]. For $\Psi(t) = t$, $\rho = 1$, (ρ,k,Ψ)-proportional H derivative reduces to k-H derivative [17], and H derivative for $k = 1$ [20]. For $\Psi(t) = t$, $\nu = 0$, $k = 1$, (ρ,k,Ψ)-proportional H derivative reduces to GPF RL derivative [22]. For $\Psi(t) = t$, $\nu = 1$, $k = 1$, (ρ,k,Ψ)-proportional H derivative reduces to GPF C derivative [22]. For $\rho = 1$, $\nu = 0$, (ρ,k,Ψ)-proportional H derivative reduces to (k,Ψ)-RL derivative [26], and Ψ-RL derivative for $k = 1$ [24]. For $\rho = 1$, $\nu = 1$, (ρ,k,Ψ)-proportional H derivative reduces to (k,Ψ)-C derivative [24], and Ψ-C derivative for $k = 1$ [3]. For $\rho = 1$, (ρ,k,Ψ)-proportional H derivative reduces to (k,Ψ)-H derivative [25], and Ψ-H derivative for $k = 1$ [2].

Lemma 2.1 *If $\rho \in (0,1]$, $r,k,\eta > 0$ and h is piecewise continuous and GEB on $[0,b]$, then*

$$ \mathcal{L}_k^{r;\Psi}\big({}^{k}I_{0+}^{\eta,\rho;\Psi}h(t)\big)(\lambda) = \left(\frac{1}{\rho k^{2-\frac{r}{k}}\lambda + 1 - \rho}\right)^{\frac{\eta}{k}} \mathcal{L}_k^{r;\Psi}\big(h(t)\big)(\lambda), \tag{2.1} $$

for $\lambda > \frac{\rho-1}{\rho k^{2-\frac{r}{k}}}$.

Proof If we apply the transform $\mathcal{L}_k^{r;\Psi}$ to ${}^{k}I_{0+}^{\eta,\rho;\Psi}h(t)$, then we attain

$$ \mathcal{L}_k^{r;\Psi}\big({}^{k}I_{0+}^{\eta,\rho;\Psi}h(t)\big)(\lambda) = \mathcal{L}_k^{r;\Psi}\big(g_{\rho,k,\eta}(t) *_{\Psi} h(t)\big)(\lambda). $$

By choosing u such that $\Psi(u) = \frac{\rho k^{2-\frac{r}{k}}\lambda+1-\rho}{\rho k}\Psi(t)$, we can write

$$ \begin{aligned} \mathcal{L}_k^{r;\Psi}\big(g_{\rho,k,\eta}(t)\big)(\lambda) &= \int_0^\infty \mathrm{e}^{-\lambda k^{1-\frac{r}{k}}\Psi(t)} \frac{\Psi^{\frac{\eta}{k}-1}(t)}{\rho^{\frac{\eta}{k}} k\Gamma_k(\eta)} \mathrm{e}^{\frac{(\rho-1)\Psi(t)}{\rho k}} \Psi'(t)\,\mathrm{d}t \\ &= \frac{1}{\rho^{\frac{\eta}{k}} k\Gamma_k(\eta)} \int_0^\infty \mathrm{e}^{(\frac{\rho-1}{\rho k} - \lambda k^{1-\frac{r}{k}})\Psi(t)} \Psi^{\frac{\eta}{k}-1}(t)\Psi'(t)\,\mathrm{d}t \\ &= \frac{1}{\rho^{\frac{\eta}{k}} k\Gamma_k(\eta)} \left(\frac{\rho k}{\rho k^{2-\frac{r}{k}}\lambda + 1 - \rho}\right)^{\frac{\eta}{k}} \int_0^\infty \mathrm{e}^{-\Psi(u)} \Psi^{\frac{\eta}{k}-1}(u)\Psi'(u)\,\mathrm{d}u \\ &= \left(\frac{1}{\rho k^{2-\frac{r}{k}}\lambda + 1 - \rho}\right)^{\frac{\eta}{k}}. \end{aligned} \tag{2.2} $$

From the convolution, it is evident that (2.1) holds. □

Theorem 2.1 *If $\rho \in (0,1]$, $r,k,\eta > 0$ with $m = \lceil \frac{\eta}{k} \rceil$ and h is a piecewise continuous and GEB function, then*

$$\begin{aligned}
&\mathcal{L}_k^{r;\Psi}\big({}^{k,H}D_{0+}^{\eta,\rho,\nu;\Psi}h(t)\big)(\lambda) \\
&\quad = \big(\rho k^{2-\frac{r}{k}}\lambda + 1 - \rho\big)^{\frac{\eta}{k}}\mathcal{L}_k^{r;\Psi}\big(h(t)\big)(\lambda) \\
&\qquad - \rho k \sum_{i=0}^{m-1}\big(\rho k^{2-\frac{r}{k}}\lambda + 1 - \rho\big)^{m-i-1+\frac{\nu(\eta-mk)}{k}}\big[{}^kD^{i,\rho;\Psi}\,{}^kI_{0+}^{(1-\nu)(mk-\eta),\rho;\Psi}h(t)\big]_{t=0},
\end{aligned}$$

for $\lambda > 0$.

Proof Initially, using the Definition 2.6, an integration by parts leads to

$$\begin{aligned}
\mathcal{L}_k^{r;\Psi}\left[\left(\frac{k}{\Psi'(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right)h(t)\right](\lambda) &= \int_0^\infty \mathrm{e}^{-\lambda k^{1-\frac{r}{k}}\Psi(t)}k\frac{\mathrm{d}}{\mathrm{d}t}h(t)\,\mathrm{d}t \\
&= -kh(0) + k^{2-\frac{r}{k}}\lambda\mathcal{L}_k^{r;\Psi}\big(h(t)\big)(\lambda).
\end{aligned} \tag{2.3}$$

Second, we prove that

$$\begin{aligned}
\mathcal{L}_k^{r;\Psi}\big[{}^kD^{m,\rho;\Psi}h(t)\big](\lambda) &= \big(\rho k^{2-\frac{r}{k}}\lambda + 1 - \rho\big)^m\mathcal{L}_k^{r;\Psi}\big(h(t)\big)(\lambda) \\
&\quad - \rho k\sum_{i=0}^{m-1}\big(\rho k^{2-\frac{r}{k}}\lambda + 1 - \rho\big)^{m-i-1}\big[{}^kD^{i,\rho;\Psi}h(t)\big]_{t=0}
\end{aligned} \tag{2.4}$$

is true, where $m \in \mathbb{Z}^+$.

For $m = 1$, from Definition 2.4 and (2.3),

$$\begin{aligned}
\mathcal{L}_k^{r;\Psi}\big[{}^kD^{1,\rho;\Psi}h(t)\big](\lambda) &= \mathcal{L}_k^{r;\Psi}\left[\rho\left(\frac{k}{\Psi'(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right)h(t) + (1-\rho)h(t)\right] \\
&= \rho\big[-kh(0) + k^{2-\frac{r}{k}}\lambda\mathcal{L}_k^{r;\Psi}\big(h(t)\big)(\lambda)\big] + (1-\rho)\mathcal{L}_k^{r;\Psi}\big(h(t)\big)(\lambda) \\
&= -\rho kh(0) + \big(\rho k^{2-\frac{r}{k}}\lambda + 1 - \rho\big)\mathcal{L}_k^{r;\Psi}\big(h(t)\big)(\lambda).
\end{aligned}$$

Assume that (2.4) is satisfied for $m = l$. Next, we verify that (2.4) is also satisfied for $m = l + 1$,

$$\begin{aligned}
\mathcal{L}_k^{r;\Psi}\big[{}^kD^{l+1,\rho;\Psi}h(t)\big](\lambda) &= \mathcal{L}_k^{r;\Psi}\left[\rho\left(\frac{k}{\Psi'(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right){}^kD^{l,\rho;\Psi}h(t) + (1-\rho){}^kD^{l,\rho;\Psi}h(t)\right](\lambda) \\
&= \big(\rho k^{2-\frac{r}{k}}\lambda + 1 - \rho\big)\mathcal{L}_k^{r;\Psi}\big[{}^kD^{l,\rho;\Psi}h(t)\big](\lambda) - \rho k\big[{}^kD^{l,\rho;\Psi}h(t)\big]_{t=0} \\
&= \big(\rho k^{2-\frac{r}{k}}\lambda + 1 - \rho\big)^{l+1}\mathcal{L}_k^{r;\Psi}(h(t)(\lambda) - \rho k\big[{}^kD^{l,\rho;\Psi}h(t)\big]_{t=0} \\
&\quad - \rho k\sum_{i=0}^{l-1}\big(\rho k^{2-\frac{r}{k}}\lambda + 1 - \rho\big)^{l-i}\big[{}^kD^{i,\rho;\Psi}h(t)\big]_{t=0} \\
&= \big(\rho k^{2-\frac{r}{k}}\lambda + 1 - \rho\big)^{l+1}\mathcal{L}_k^{r;\Psi}\big(h(t)\big)(\lambda) \\
&\quad - \rho k\sum_{i=0}^{l}\big(\rho k^{2-\frac{r}{k}}\lambda + 1 - \rho\big)^{l-i}\big[{}^kD^{i,\rho;\Psi}h(t)\big]_{t=0}.
\end{aligned}$$

The relation (2.4) is proved.

In view of Definition 2.9 and (2.1), one has

$$\begin{aligned}
&\mathcal{L}_k^{r;\Psi}\big({}^{k,H}D_{0+}^{\eta,\rho,\nu;\Psi}h(t)\big)(\lambda)\\
&\quad=\mathcal{L}_k^{r;\Psi}\big({}^{k}I_{0+}^{\nu(mk-\eta),\rho;\Psi}\,{}^{k}D^{m,\rho;\Psi}\,{}^{k}I_{0+}^{(1-\nu)(mk-\eta),\rho;\Psi}h(t)\big)(\lambda)\\
&\quad=\left(\frac{1}{\rho k^{2-\frac{r}{k}}\lambda+1-\rho}\right)^{\frac{\nu(mk-\eta)}{k}}\mathcal{L}_k^{r;\Psi}\big({}^{k}D^{m,\rho;\Psi}\,{}^{k}I_{0+}^{(1-\nu)(mk-\eta),\rho;\Psi}h(t)\big)(\lambda).
\end{aligned}$$

Notice also that (2.1) and (2.4), we find

$$\begin{aligned}
&\mathcal{L}_k^{r;\Psi}\big({}^{k,H}D_{0+}^{\eta,\rho,\nu;\Psi}h(t)\big)(\lambda)\\
&\quad=\left(\frac{1}{\rho k^{2-\frac{r}{k}}\lambda+1-\rho}\right)^{\frac{\nu(mk-\eta)}{k}}\Bigg\{\big(\rho k^{2-\frac{r}{k}}\lambda+1-\rho\big)^{m}\mathcal{L}_k^{r;\Psi}\big({}^{k}I_{0+}^{(1-\nu)(mk-\eta),\rho;\Psi}h(t)\big)(\lambda)\\
&\qquad-\rho k\sum_{i=0}^{m-1}\big(\rho k^{2-\frac{r}{k}}\lambda+1-\rho\big)^{m-1-i}\big[{}^{k}D^{i,\rho;\Psi}\,{}^{k}I_{0+}^{(mk-\eta)(1-\nu),\rho;\Psi}h(t)\big]_{t=0}\Bigg\}\\
&\quad=\big(\rho k^{2-\frac{r}{k}}\lambda+1-\rho\big)^{\frac{\eta}{k}}\mathcal{L}_k^{r;\Psi}\big(h(t)\big)(\lambda)\\
&\qquad-\rho k\sum_{i=0}^{m-1}\big(\rho k^{2-\frac{r}{k}}\lambda+1-\rho\big)^{m-i-1+\frac{\nu(\eta-mk)}{k}}\big[{}^{k}D^{i,\rho;\Psi}\,{}^{k}I_{0+}^{(1-\nu)(mk-\eta),\rho;\Psi}h(t)\big]_{t=0}.
\end{aligned}$$

The proof is now complete. □

For $k,r,n>0$, we denote by $D_{k,r,n}(t)=\frac{\sin k^{1-\frac{r}{k}}\Psi(n)\Psi(t)}{\pi k^{1-\frac{r}{k}}\Psi(t)}$, $t\in\mathbb{R}$, then

$$\begin{aligned}
\big(D_{k,r,n}*^{\Psi}h\big)(t)&=\int_{-\infty}^{\infty}h(\tau)\frac{1}{2\pi}\int_{-\Psi(n)}^{\Psi(n)}\mathrm{e}^{isk^{1-\frac{r}{k}}(\Psi(t)-\Psi(\tau))}\,\mathrm{d}s\Psi'(\tau)\,\mathrm{d}\tau\\
&=\frac{1}{2\pi}\int_{-\Psi(n)}^{\Psi(n)}\mathrm{e}^{isk^{1-\frac{r}{k}}\Psi(t)}\left(\int_{-\infty}^{\infty}\mathrm{e}^{-isk^{1-\frac{r}{k}}\Psi(\tau)}h(\tau)\Psi'(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s\\
&=\frac{1}{2\pi}\int_{-\Psi(n)}^{\Psi(n)}\mathrm{e}^{isk^{1-\frac{r}{k}}\Psi(t)}\mathcal{F}_k^{r;\Psi}(h)(s)\,\mathrm{d}s.
\end{aligned}$$

Let $S:[0,\infty)\to\mathfrak{B}(X,Y)$, we denote the shift $S_{k,r,\omega}$ by $S_{k,r,\omega}(t)=\mathrm{e}^{-\omega k^{1-\frac{r}{k}}\Psi(t)}S(t)$, $t\geq 0$. Now, if the operator S is strongly continuous, then

$$\begin{aligned}
\mathcal{L}_k^{r;\Psi}(S)(\omega+is)&=\int_0^{\infty}\mathrm{e}^{-(\omega+is)k^{1-\frac{r}{k}}\Psi(t)}S(t)\Psi'(t)\,\mathrm{d}t\\
&=\int_0^{\infty}\mathrm{e}^{-isk^{1-\frac{r}{k}}\Psi(t)}S_{k,r,\omega}(t)\Psi'(t)\,\mathrm{d}t\\
&=\mathcal{F}_k^{r;\Psi}(S_{k,r,\omega})(s),
\end{aligned}$$

which leads to

$$\begin{aligned}
K_{k,r,n}(t) &= \frac{1}{2\pi i}\int_{\omega-i\Psi(n)}^{\omega+i\Psi(n)} \mathrm{e}^{k^{1-\frac{r}{k}}\Psi(t)z}\mathcal{L}_k^{r;\Psi}(S)(z)\,\mathrm{d}z \\
&= \frac{1}{2\pi}\int_{-\Psi(n)}^{\Psi(n)} \mathrm{e}^{k^{1-\frac{r}{k}}\Psi(t)\omega}\mathrm{e}^{ik^{1-\frac{r}{k}}\Psi(t)s}\mathcal{L}_k^{r;\Psi}(S)(\omega+is)\,\mathrm{d}s \\
&= \mathrm{e}^{k^{1-\frac{r}{k}}\Psi(t)\omega}\frac{1}{2\pi}\int_{-\Psi(n)}^{\Psi(n)} \mathrm{e}^{ik^{1-\frac{r}{k}}\Psi(t)s}\mathcal{L}_k^{r;\Psi}(S)(\omega+is)\,\mathrm{d}s \\
&= \mathrm{e}^{k^{1-\frac{r}{k}}\Psi(t)\omega}\frac{1}{2\pi}\int_{-\Psi(n)}^{\Psi(n)} \mathrm{e}^{isk^{1-\frac{r}{k}}\Psi(t)}\mathcal{F}_k^{r;\Psi}(S_{k,r,\omega})(s)\,\mathrm{d}s \\
&= \mathrm{e}^{k^{1-\frac{r}{k}}\Psi(t)\omega}\bigl(D_{k,r,n} *^{\Psi} S_{k,r,\omega}\bigr)(t),
\end{aligned}$$

namely $(K_{k,r,n})_\omega = D_{k,r,n} *^{\Psi} S_{k,r,\omega}$.

Theorem 2.2 *If $S:[0,\infty)\to\mathfrak{B}(X,Y)$ is strongly continuous, $b\in L^1_{\mathrm{loc}}([0,\infty),\mathbb{R})$, functions b and S are finite GBE. Then, one has*

$$\lim_{n\to\infty}\frac{1}{2\pi i}\int_{\omega-i\Psi(n)}^{\omega+i\Psi(n)} \mathrm{e}^{k^{1-\frac{r}{k}}\Psi(t)z}\mathcal{L}_k^{r;\Psi}\left(\frac{b}{\sqrt{\Psi'}} *^{\Psi} S\right)(z)\,\mathrm{d}z = k^{\frac{r}{k}-1}b *^{\Psi} S,$$

$\omega>\omega_0(S),\omega(b)$ in $\mathfrak{B}(X,Y)$.

Proof If we replace the operator S by $\frac{b}{\sqrt{\Psi'}} *^{\Psi} S$ in the expression $(K_{k,r,n})_\omega = D_{k,r,n} *^{\Psi} S_{k,r,\omega}$, then we conclude that $\mathrm{e}^{-\omega k^{1-\frac{r}{k}}\Psi(t)}(K_{k,r,n})(t) = D_{k,r,n} *^{\Psi} (\frac{b}{\sqrt{\Psi'}} *^{\Psi} S)_{k,r,\omega}$, where

$$\begin{aligned}
\left(\frac{b}{\sqrt{\Psi'}} *^{\Psi} S\right)_{k,r,\omega} &= \mathrm{e}^{-\omega k^{1-\frac{r}{k}}\Psi(t)}\int_{-\infty}^{\infty}\frac{b(\tau)}{\sqrt{\Psi'(\tau)}}\overline{S}(t,\tau)\Psi'(\tau)\,\mathrm{d}\tau \\
&= \int_{-\infty}^{\infty}\mathrm{e}^{-\omega k^{1-\frac{r}{k}}[\Psi(\tau)+\Psi(t)-\Psi(\tau)]}\frac{b(\tau)}{\sqrt{\Psi'(\tau)}}\overline{S}(t,\tau)\Psi'(\tau)\,\mathrm{d}\tau \\
&= \frac{b}{\sqrt{\Psi'}}_{k,r,\omega} *^{\Psi} S_{k,r,\omega}.
\end{aligned}$$

Hence, $\mathrm{e}^{-\omega k^{1-\frac{r}{k}}\Psi(t)}(K_{k,r,n})(t) = D_{k,r,n} *^{\Psi} (\frac{b}{\sqrt{\Psi'}}_{k,r,\omega} *^{\Psi} S_{k,r,\omega}) = D_{k,r,n} *^{\Psi} \frac{b}{\sqrt{\Psi'}}_{k,r,\omega} *^{\Psi} S_{k,r,\omega}$. Using the Plancherel theorem, one has $\sqrt{\Psi'}(D_{k,r,n} *^{\Psi} \frac{h}{\sqrt{\Psi'}}) \to k^{\frac{r}{k}-1}h\sqrt{\Psi'}$ in $L^2(\mathbb{R})$ as $n\to\infty$ for $h\sqrt{\Psi'}\in L^2(\mathbb{R})$. Furthermore, using the Young inequality, we can get $(D_{k,r,n} *^{\Psi} \frac{b}{\sqrt{\Psi'}}_{k,r,\omega} *^{\Psi} S_{k,r,\omega}) \to k^{\frac{r}{k}-1}b_{k,r,\omega} *^{\Psi} S_{k,r,\omega}$ in $\mathfrak{B}(X,Y)$ as $n\to\infty$ uniformly in $t\geq 0$. The proof is completed. □

Definition 2.10 Let $\rho\in(0,1],\alpha,\beta,k,r>0$, A is a closed linear operator mapping from $D(A)\subset E$ into the Banach space E. Operator A is said to be the generator of (ρ,α,β,k,r)-resolvent family, if there exist $\omega\geq 0$ and a strongly continuous GEB operator $S_{\rho,\alpha,\beta,k,r}(t):[0,\infty)\to\mathfrak{B}(E)$, $\{(\rho k^{2-\frac{r}{k}}\lambda+1-\rho)^{\frac{\alpha}{k}}:((\rho k^{2-\frac{r}{k}}\lambda+1-\rho)^{\frac{\alpha}{k}}I-A)$ is invertible in $\mathfrak{B}(E)$, and

$\operatorname{Re}\lambda > \omega\}$,

$$\begin{aligned}&\left(\rho k^{2-\frac{r}{k}}\lambda + 1 - \rho\right)^{\frac{\alpha-\beta}{k}}\left(\left(\rho k^{2-\frac{r}{k}}\lambda + 1 - \rho\right)^{\frac{\alpha}{k}} I - A\right)^{-1} x \\ &\quad = \int_0^\infty \mathrm{e}^{-\lambda k^{1-\frac{r}{k}}\Psi(t)} S_{\rho,\alpha,\beta,k,r}(t) x \Psi'(t)\,\mathrm{d}t, \quad x \in E.\end{aligned} \tag{2.5}$$

Then, we say that A generates $\{S_{\rho,\alpha,\beta,k,r}(t)\}_{t\geq 0}$.

Remark 2.4 For $\rho = k = r = \alpha = \beta = 1$, $S_{\rho,\alpha,\beta,k,r}(t)$ reduces to C_0-semigroup [30]. For $\rho = k = r = 1$, $\alpha = \beta = 2$, $S_{\rho,\alpha,\beta,k,r}(t)$ reduces to sine family [35]. For $\rho = k = r = \beta = 1$, $S_{\rho,\alpha,\beta,k,r}(t)$ reduces to α-resolvent family [32], which contains cosine family. For $\rho = k = r = 1$, $S_{\rho,\alpha,\beta,k,r}(t)$ reduces to (α,β)-resolvent family [31].

Theorem 2.3 *For $\rho \in (0,1]$, $\alpha,\beta,k,r > 0$, the following functional equation holds*:

$$\begin{aligned}&S_{\rho,\alpha,\beta,k,r}(s)(g_{\rho,k,\alpha} *_\Psi S_{\rho,\alpha,\beta,k,r})(t) - (g_{\rho,k,\alpha} *_\Psi S_{\rho,\alpha,\beta,k,r})(s) S_{\rho,\alpha,\beta,k,r}(t) \\ &\quad = g_{\rho,k,\beta}(s)(g_{\rho,k,\alpha} *_\Psi S_{\rho,\alpha,\beta,k,r})(t) - g_{\rho,k,\beta}(t)(g_{\rho,k,\alpha} *_\Psi S_{\rho,\alpha,\beta,k,r})(s), \quad t,s \geq 0.\end{aligned} \tag{2.6}$$

Proof For $\operatorname{Re}\lambda, \operatorname{Re}\mu > \omega$ and $\lambda \neq \mu$, applying $\mathcal{L}_k^{r;\Psi}$ to both sides of (2.6), using (2.2) and Definition 2.10, we have

$$\begin{aligned}&\int_0^\infty \int_0^\infty \mathrm{e}^{-\lambda k^{1-\frac{r}{k}}\Psi(t)} \mathrm{e}^{-\mu k^{1-\frac{r}{k}}\Psi(s)} \big[S_{\rho,\alpha,\beta,k,r}(s)(g_{\rho,k,\alpha} *_\Psi S_{\rho,\alpha,\beta,k,r})(t) \\ &\quad - (g_{\rho,k,\alpha} *_\Psi S_{\rho,\alpha,\beta,k,r})(s) S_{\rho,\alpha,\beta,k,r}(t)\big] \Psi'(s)\Psi'(t)\,\mathrm{d}s\,\mathrm{d}t \\ &= \left(\rho k^{2-\frac{r}{k}}\lambda + 1 - \rho\right)^{-\frac{\beta}{k}} \left(\rho k^{2-\frac{r}{k}}\mu + 1 - \rho\right)^{-\frac{\beta}{k}} \left[\left(\rho k^{2-\frac{r}{k}}\lambda + 1 - \rho\right)^{\frac{\alpha}{k}} I - A\right]^{-1} \\ &\quad \times \left[\left(\rho k^{2-\frac{r}{k}}\mu + 1 - \rho\right)^{\frac{\alpha}{k}} I - A\right]^{-1} \left[\left(\rho k^{2-\frac{r}{k}}\mu + 1 - \rho\right)^{\frac{\alpha}{k}} - \left(\rho k^{2-\frac{r}{k}}\lambda + 1 - \rho\right)^{\frac{\alpha}{k}}\right] \\ &= \left(\rho k^{2-\frac{r}{k}}\lambda + 1 - \rho\right)^{-\frac{\beta}{k}} \left(\rho k^{2-\frac{r}{k}}\mu + 1 - \rho\right)^{-\frac{\beta}{k}} \left[\left(\rho k^{2-\frac{r}{k}}\lambda + 1 - \rho\right)^{\frac{\alpha}{k}} I - A\right]^{-1} \\ &\quad - \left(\rho k^{2-\frac{r}{k}}\lambda + 1 - \rho\right)^{-\frac{\beta}{k}} \left(\rho k^{2-\frac{r}{k}}\mu + 1 - \rho\right)^{-\frac{\beta}{k}} \left[\left(\rho k^{2-\frac{r}{k}}\mu + 1 - \rho\right)^{\frac{\alpha}{k}} I - A\right]^{-1}.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\int_0^\infty \int_0^\infty \mathrm{e}^{-\lambda k^{1-\frac{r}{k}}\Psi(t)} \mathrm{e}^{-\mu k^{1-\frac{r}{k}}\Psi(s)} \big[g_{\rho,k,\beta}(s)(g_{\rho,k,\alpha} *_\Psi S_{\rho,\alpha,\beta,k,r})(t) \\ &\quad - g_{\rho,k,\beta}(t)(g_{\rho,k,\alpha} *_\Psi S_{\rho,\alpha,\beta,k,r})(s)\big] \Psi'(t)\Psi'(s)\,\mathrm{d}t\,\mathrm{d}s \\ &= \left(\rho k^{2-\frac{r}{k}}\mu + 1 - \rho\right)^{-\frac{\beta}{k}} \left(\rho k^{2-\frac{r}{k}}\lambda + 1 - \rho\right)^{-\frac{\alpha}{k}} \left(\rho k^{2-\frac{r}{k}}\lambda + 1 - \rho\right)^{\frac{\alpha-\beta}{k}} \\ &\quad \times \left[\left(\rho k^{2-\frac{r}{k}}\lambda + 1 - \rho\right)^{\frac{\alpha}{k}} I - A\right]^{-1} \\ &\quad - \left(\rho k^{2-\frac{r}{k}}\lambda + 1 - \rho\right)^{-\frac{\beta}{k}} \left(\rho k^{2-\frac{r}{k}}\mu + 1 - \rho\right)^{-\frac{\alpha}{k}} \left(\rho k^{2-\frac{r}{k}}\mu + 1 - \rho\right)^{\frac{\alpha-\beta}{k}} \\ &\quad \times \left[\left(\rho k^{2-\frac{r}{k}}\mu + 1 - \rho\right)^{\frac{\alpha}{k}} I - A\right]^{-1} \\ &= \left(\rho k^{2-\frac{r}{k}}\lambda + 1 - \rho\right)^{-\frac{\beta}{k}} \left(\rho k^{2-\frac{r}{k}}\mu + 1 - \rho\right)^{-\frac{\beta}{k}} \left[\left(\rho k^{2-\frac{r}{k}}\lambda + 1 - \rho\right)^{\frac{\alpha}{k}} I - A\right]^{-1} \\ &\quad - \left(\rho k^{2-\frac{r}{k}}\lambda + 1 - \rho\right)^{-\frac{\beta}{k}} \left(\rho k^{2-\frac{r}{k}}\mu + 1 - \rho\right)^{-\frac{\beta}{k}} \left[\left(\rho k^{2-\frac{r}{k}}\mu + 1 - \rho\right)^{\frac{\alpha}{k}} I - A\right]^{-1}.\end{aligned}$$

From the uniqueness, we get that (2.6) is true. □

If $A : D(A) \to E$ generates $S_{\rho,\alpha,\beta,k,r}(t)$, we define

$$Ax = \lim_{t\to 0+} \frac{S_{\rho,\alpha,\beta,k,r}(t)x - g_{\rho,k,\beta}(t)x}{g_{\rho,k,\alpha+\beta}(t)}.$$

Theorem 2.4 *If $\rho \in (0,1], \alpha, \beta, k, r > 0$ and $\{S_{\rho,\alpha,\beta,k,r}(t)\}_{t\geq 0}$ is generated by A, then $S_{\rho,\alpha,\beta,k,r}(t)$ has the following properties*:

(i) *$S_{\rho,\alpha,\beta,k,r}(t)$ is a mapping from $D(A)$ into itself, besides, $S_{\rho,\alpha,\beta,k,r}(t)Ax = AS_{\rho,\alpha,\beta,k,r}(t)x$;*

(ii) *If $\beta \leq k$, then operator $(g_{\rho,k,\alpha} *_\Psi S_{\rho,\alpha,\beta,k,r})(t)$ is a mapping from E into $D(A)$, and*

$$S_{\rho,\alpha,\beta,k,r}(t)x = g_{\rho,k,\beta}(t)x + A(g_{\rho,k,\alpha} *_\Psi S_{\rho,\alpha,\beta,k,r})(t)x, \quad x \in E. \tag{2.7}$$

If $\beta > k$ and $x \in D(A)$, then (2.7) also holds. Moreover, if $x \in D(A)$, then

$$S_{\rho,\alpha,\beta,k,r}(t)x = g_{\rho,k,\beta}(t)x + (g_{\rho,k,\alpha} *_\Psi AS_{\rho,\alpha,\beta,k,r})(t)x.$$

Proof (i) For $x \in D(A)$, one can see that

$$\begin{aligned} S_{\rho,\alpha,\beta,k,r}(t)Ax &= \lim_{s\to 0+} \frac{S_{\rho,\alpha,\beta,k,r}(t)[S_{\rho,\alpha,\beta,k,r}(s)x - g_{\rho,k,\beta}(s)x]}{g_{\rho,k,\alpha+\beta}(s)} \\ &= \lim_{s\to 0+} \frac{S_{\rho,\alpha,\beta,k,r}(s)S_{\rho,\alpha,\beta,k,r}(t)x - g_{\rho,k,\beta}(s)S_{\rho,\alpha,\beta,k,r}(t)x}{g_{\rho,k,\alpha+\beta}(s)} \\ &= \lim_{s\to 0+} \frac{S_{\rho,\alpha,\beta,k,r}(t)[S_{\rho,\alpha,\beta,k,r}(s)x - g_{\rho,k,\beta}(s)x]}{g_{\rho,k,\alpha+\beta}(s)} \\ &= AS_{\rho,\alpha,\beta,k,r}(t)x, \end{aligned}$$

which implies that $S_{\rho,\alpha,\beta,k,r}(t)$ is a mapping from $D(A)$ into itself, and $S_{\rho,\alpha,\beta,k,r}(t)Ax = AS_{\rho,\alpha,\beta,k,r}(t)x$.

(ii) First, by Theorem 2.3,

$$\begin{aligned} &\lim_{s\to 0+} \frac{S_{\rho,\alpha,\beta,k,r}(s)(g_{\rho,k,\alpha} *_\Psi S_{\rho,\alpha,\beta,k,r})(t)x - g_{\rho,k,\beta}(s)(g_{\rho,k,\alpha} *_\Psi S_{\rho,\alpha,\beta,k,r})(t)x}{g_{\rho,k,\alpha+\beta}(s)} \\ &\quad = \lim_{s\to 0+} \frac{(g_{\rho,k,\alpha} *_\Psi S_{\rho,\alpha,\beta,k,r})(s)S_{\rho,\alpha,\beta,k,r}(t)x - g_{\rho,k,\beta}(t)(g_{\rho,k,\alpha} *_\Psi S_{\rho,\alpha,\beta,k,r})(s)x}{g_{\rho,k,\alpha+\beta}(s)} \\ &\quad = \lim_{s\to 0+} \frac{(g_{\rho,k,\alpha} *_\Psi S_{\rho,\alpha,\beta,k,r})(s)[S_{\rho,\alpha,\beta,k,r}(t)x - g_{\rho,k,\beta}(t)x]}{g_{\rho,k,\alpha+\beta}(s)}. \end{aligned} \tag{2.8}$$

Furthermore, using those facts

$$(g_{\rho,k,\alpha} *_\Psi S_{\rho,\alpha,\beta,k,r})(s) = \int_0^s \hat{g}_{\rho,k,\alpha}(s,s,\tau)S_{\rho,\alpha,\beta,k,r}(\tau)\,\mathrm{d}\tau$$

and

$$\begin{aligned} g_{\rho,k,\alpha+\beta}(s) &= \frac{1}{\rho^{\frac{\beta}{k}} k\Gamma_k(\beta)} \int_0^s \hat{g}_{\rho,k,\alpha}(s,s,\tau)\mathrm{e}^{\frac{(\rho-1)\Psi(\tau)}{\rho k}} \Psi^{\frac{\beta}{k}-1}(\tau)\,\mathrm{d}\tau \\ &= \int_0^s \hat{g}_{\rho,k,\alpha}(s,s,\tau)g_{\rho,k,\beta}(\tau)\,\mathrm{d}\tau, \end{aligned}$$

we get

$$\begin{aligned}&\left\|\frac{(g_{\rho,k,\alpha}*_\Psi S_{\rho,\alpha,\beta,k,r})(s)u}{g_{\rho,k,\alpha+\beta}(s)}-u\right\|\\&\quad=\frac{1}{g_{\rho,k,\alpha+\beta}(s)}\left\|\int_0^s\hat{g}_{\rho,k,\alpha}(s,s,\tau)\big[S_{\rho,\alpha,\beta,k,r}(\tau)-g_{\rho,k,\beta}(\tau)\big]u\,\mathrm{d}\tau\right\|.\end{aligned}$$

If $\beta\le k$, then

$$\begin{aligned}&\left\|\frac{(g_{\rho,k,\alpha}*_\Psi S_{\rho,\alpha,\beta,k,r})(s)u}{g_{\rho,k,\alpha+\beta}(s)}-u\right\|\\&\quad\le\frac{\sup_{\tau\in[0,s]}\|S_{\rho,\alpha,\beta,k,r}(\tau)u-g_{\rho,k,\beta}(\tau)u\|}{g_{\rho,k,\alpha+\beta}(s)}\int_0^s\hat{g}_{\rho,k,\alpha}(s,s,\tau)\,\mathrm{d}\tau\\&\quad\le\frac{\rho^{\frac{\beta}{k}}k\Gamma_k(\alpha+\beta)}{\alpha\Gamma_k(\alpha)\mathrm{e}^{\frac{(\rho-1)\Psi(s)}{\rho k}}}\Psi^{1-\frac{\beta}{k}}(s)\sup_{\tau\in[0,s]}\big\|S_{\rho,\alpha,\beta,k,r}(\tau)u-g_{\rho,k,\beta}(\tau)u\big\|\\&\quad\to0,\quad s\to0+.\end{aligned}\tag{2.9}$$

From (2.8) and (2.9), we have that $(g_{\rho,k,\alpha}*_\Psi S_{\rho,\alpha,\beta,k,r})(t)x$ is a mapping from E into $D(A)$, $t\ge0$, and (2.7) is true.

If $\beta>k$, in view of $g'_{\rho,k,\alpha+\beta}(s)=\frac{1}{\rho^{\frac{\alpha+\beta}{k}}k\Gamma_k(\alpha+\beta)}\Psi^{\frac{\alpha+\beta}{k}-2}(s)\mathrm{e}^{\frac{(\rho-1)\Psi(s)}{\rho k}}\Psi'(s)[(\frac{\alpha+\beta}{k}-1)+\Psi(s)\frac{\rho-1}{\rho k}]$, $s>0$. Then, $g_{\rho,k,\alpha+\beta}(s)$ is monotone increasing on $[0,\delta]$, where $\delta>0$ is sufficiently small.

It follows that

$$\begin{aligned}&\left\|\frac{(g_{\rho,k,\alpha}*_\Psi S_{\rho,\alpha,\beta,k,r})(s)u}{g_{\rho,k,\alpha+\beta}(s)}-u\right\|\\&\quad\le\frac{\sup_{\tau\in[0,s]}\|S_{\rho,\alpha,\beta,k,r}(\tau)u-g_{\rho,k,\beta}(\tau)u\|}{g_{\rho,k,\alpha+\beta}(s)}\int_0^s\hat{g}_{\rho,k,\alpha}(s,s,\tau)\,\mathrm{d}\tau\\&\quad\le\sup_{\tau\in[0,s]}\left\|\frac{S_{\rho,\alpha,\beta,k,r}(\tau)u-g_{\rho,k,\beta}(\tau)u}{g_{\rho,k,\alpha+\beta}(\tau)}\right\|\int_0^s\hat{g}_{\rho,k,\alpha}(s,s,\tau)\,\mathrm{d}\tau\\&\quad\to0,\quad s\to0+,\end{aligned}\tag{2.10}$$

for $u\in D(A)$. From (i), we know that $(g_{\rho,k,\alpha}*_\Psi S_{\rho,\alpha,\beta,k,r})(t)x\in D(A)$, $t\ge0$. Furthermore, $S_{\rho,\alpha,\beta,k,r}(t)x=g_{\rho,k,\beta}(t)x+A(g_{\rho,k,\alpha}*_\Psi S_{\rho,\alpha,\beta,k,r})(t)x$ for $x\in D(A)$ can be obtained by (2.10). The rest part is obviously follows from (i). □

3 Properties of $S_{\rho,\alpha,\beta,k,r}(t)$

Theorem 3.1 *Let $\rho\in(0,1],\alpha,\beta,\gamma,k,r>0$. If $\{S_{\rho,\alpha,\beta,k,r}(t)\}_{t\ge0}$ is generated by A, and its type is (M,ω,k,r), then A generates $\{S_{\rho,\alpha,\beta+\gamma,k,r}(t)\}_{t\ge0}$, which type is $((\rho k^{2-\frac{r}{k}}\omega+1-\rho)^{-\frac{\gamma}{k}}M,\omega,k,r)$.*

Proof For $t \geq 0$, using the variable $\Psi(u) = \Psi(t) - \Psi(\tau)$, it follows that

$$\begin{aligned}
\left\|(g_{\rho,k,\gamma} *_\Psi S_{\rho,\alpha,\beta,k,r})(t)\right\| &\leq M \int_0^t \hat{g}_{\rho,k,\gamma}(t,t,\tau) \mathrm{e}^{\omega k^{1-\frac{r}{k}} \Psi(\tau)} \,\mathrm{d}\tau \\
&= M \int_0^t \frac{\Psi^{\frac{\gamma}{k}-1}(u)}{\rho^{\frac{\gamma}{k}} k \Gamma_k(\gamma)} \mathrm{e}^{\frac{(\rho-1)\Psi(u)}{\rho k}} \Psi'(u) \mathrm{e}^{\omega k^{1-\frac{r}{k}} [\Psi(t)-\Psi(u)]} \,\mathrm{d}u \\
&= M \mathrm{e}^{\omega k^{1-\frac{r}{k}} \Psi(t)} \int_0^t \frac{\Psi^{\frac{\gamma}{k}-1}(u)}{\rho^{\frac{\gamma}{k}} k \Gamma_k(\gamma)} \mathrm{e}^{\frac{(\rho-1)\Psi(u)}{\rho k}} \Psi'(u) \mathrm{e}^{-\omega k^{1-\frac{r}{k}} \Psi(u)} \,\mathrm{d}u \\
&\leq M \mathrm{e}^{\omega k^{1-\frac{r}{k}} \Psi(t)} \mathcal{L}_k^{r;\Psi}\big(g_{\rho,k,\gamma}(t)\big)(\omega) \\
&= \left(\frac{1}{\rho k^{2-\frac{r}{k}} \omega + 1 - \rho}\right)^{\frac{\gamma}{k}} M \mathrm{e}^{\omega k^{1-\frac{r}{k}} \Psi(t)},
\end{aligned}$$

which implies that the generalized Laplace transform of $S_{\rho,\alpha,\beta+\gamma,k,r}(t)$ exists.

Furthermore, if $\operatorname{Re} \lambda > \omega$, then we have

$$\begin{aligned}
\mathcal{L}_k^{r;\Psi}(g_{\rho,k,\gamma} *_\Psi S_{\rho,\alpha,\beta,k,r})(\lambda) &= \left(\frac{1}{\rho k^{2-\frac{r}{k}} \lambda + 1 - \rho}\right)^{\frac{\gamma}{k}} \big(\rho k^{2-\frac{r}{k}} \lambda + 1 - \rho\big)^{\frac{\alpha-\beta}{k}} \\
&\quad \times \big(\big(\rho k^{2-\frac{r}{k}} \lambda + 1 - \rho\big)^{\frac{\alpha}{k}} I - A\big)^{-1} \\
&= \big(\rho k^{2-\frac{r}{k}} \lambda + 1 - \rho\big)^{\frac{\alpha-\beta-\gamma}{k}} \big(\big(\rho k^{2-\frac{r}{k}} \lambda + 1 - \rho\big)^{\frac{\alpha}{k}} I - A\big)^{-1} \\
&= \mathcal{L}_k^{r;\Psi}(S_{\rho,\alpha,\beta+\gamma,k,r})(\lambda).
\end{aligned}$$

Therefore, we see that the $(\rho,\alpha,\beta+\gamma,k,r)$-resolvent family is generated by A, and its type is $((\rho k^{2-\frac{r}{k}} \omega + 1 - \rho)^{-\frac{\gamma}{k}} M, \omega, k, r)$. □

Theorem 3.2 *Let $\rho \in (0,1]$, $\beta > \beta_1 > 0$, $\alpha, k, r > 0$. If $\{S_{\rho,\alpha,\beta_1,k,r}(t)\}_{t\geq 0}$ is generated by A, and its type is (M,ω,k,r), then $S_{\rho,\alpha,\beta,k,r}(t)$ is norm continuous in $\mathfrak{B}(E)$ for $t > 0$.*

Proof We conclude that $\{S_{\rho,\alpha,\beta,k,r}(t)\}_{t\geq 0}$ is generated by A, and its type is $((\rho k^{2-\frac{r}{k}} \omega + 1 - \rho)^{-\frac{\beta-\beta_1}{k}} M, \omega, k, r)$ from Theorem 3.1. To prove the conclusion, there are two cases that need to be discussed.

Case 1: $\beta \neq k + \beta_1$.

Let $0 \leq t_1 < t_2$, from $S_{\rho,\alpha,\beta,k,r}(t) = (g_{\rho,k,\beta-\beta_1} *_\Psi S_{\rho,\alpha,\beta_1,k,r})(t)$, $t \geq 0$, it follows that

$$\begin{aligned}
&S_{\rho,\alpha,\beta,k,r}(t_2) - S_{\rho,\alpha,\beta,k,r}(t_1) \\
&\quad = (g_{\rho,k,\beta-\beta_1} *_\Psi S_{\rho,\alpha,\beta_1,k,r})(t_2) - (g_{\rho,k,\beta-\beta_1} *_\Psi S_{\rho,\alpha,\beta_1,k,r})(t_1) \\
&\quad = \int_{t_1}^{t_2} \hat{g}_{\rho,k,\beta-\beta_1}(t_2,t_2,\tau) S_{\rho,\alpha,\beta_1,k,r}(\tau) \,\mathrm{d}\tau \\
&\qquad + \int_0^{t_1} \big(\hat{g}_{\rho,k,\beta-\beta_1}(t_2,t_2,\tau) - \hat{g}_{\rho,k,\beta-\beta_1}(t_1,t_1,\tau)\big) S_{\rho,\alpha,\beta_1,k,r}(\tau) \,\mathrm{d}\tau \\
&\quad = I_1 + I_2.
\end{aligned} \tag{3.1}$$

For I_1, we have

$$\begin{aligned}
\|I_1\| &\le \int_{t_1}^{t_2} \hat{g}_{\rho,k,\beta-\beta_1}(t_2,t_2,\tau)\left\|S_{\rho,\alpha,\beta_1,k,r}(\tau)\right\| \mathrm{d}\tau \\
&\le M\int_{t_1}^{t_2} \hat{g}_{\rho,k,\beta-\beta_1}(t_2,t_2,\tau)\mathrm{e}^{\omega k^{1-\frac{r}{k}}\Psi(\tau)}\,\mathrm{d}\tau \\
&\le \frac{M\mathrm{e}^{\omega k^{1-\frac{r}{k}}\Psi(t_2)}}{\rho^{\frac{\beta-\beta_1}{k}}k\Gamma_k(\beta-\beta_1)}\int_{t_1}^{t_2}\left[\Psi(t_2)-\Psi(\tau)\right]^{\frac{\beta-\beta_1}{k}-1}\Psi'(\tau)\,\mathrm{d}\tau \\
&= \frac{M\mathrm{e}^{\omega k^{1-\frac{r}{k}}\Psi(t_2)}}{\rho^{\frac{\beta-\beta_1}{k}}\Gamma_k(\beta+k-\beta_1)}\left(\Psi(t_2)-\Psi(t_1)\right)^{\frac{\beta-\beta_1}{k}}.
\end{aligned} \tag{3.2}$$

From the continuous of $\Psi(t)$, we can see that $\lim_{t_1\to t_2}\|I_1\| = 0$.

For I_2, we have

$$\|I_2\| \le M\mathrm{e}^{\omega k^{1-\frac{r}{k}}\Psi(t_1)}\int_0^{t_1}\left|\hat{g}_{1,k,\beta-\beta_1}(t_2,\tau,\tau)-\hat{g}_{1,k,\beta-\beta_1}(t_1,\tau,\tau)\right|\mathrm{d}\tau, \tag{3.3}$$

when $\rho = 1$ and

$$\begin{aligned}
\|I_2\| &\le M\mathrm{e}^{\omega k^{1-\frac{r}{k}}\Psi(t_1)}\int_0^{t_1}\left|\hat{g}_{\rho,k,\beta-\beta_1}(t_2,t_2,\tau)-\hat{g}_{\rho,k,\beta-\beta_1}(t_2,t_1,\tau)\right|\mathrm{d}\tau \\
&\quad + M\mathrm{e}^{\omega k^{1-\frac{r}{k}}\Psi(t_1)}\int_0^{t_1}\left|\hat{g}_{\rho,k,\beta-\beta_1}(t_2,t_1,\tau)-\hat{g}_{\rho,k,\beta-\beta_1}(t_1,t_1,\tau)\right|\mathrm{d}\tau,
\end{aligned} \tag{3.4}$$

when $\rho \ne 1$.

Subcase 1: $\rho = 1$ and $\beta_1 < \beta < k + \beta_1$.

We can see that

$$\begin{aligned}
\|I_2\| &\le M\mathrm{e}^{\omega k^{1-\frac{r}{k}}\Psi(t_1)}\int_0^{t_1}\left(\hat{g}_{1,k,\beta-\beta_1}(t_1,\tau,\tau)-\hat{g}_{1,k,\beta-\beta_1}(t_2,\tau,\tau)\right)\mathrm{d}\tau \\
&= kM\mathrm{e}^{\omega k^{1-\frac{r}{k}}\Psi(t_1)}\left[\hat{g}_{1,k,\beta-\beta_1+k}(t_1,0,0)-\hat{g}_{1,k,\beta-\beta_1+k}(t_2,0,0)\right. \\
&\quad \left.+\hat{g}_{1,k,\beta-\beta_1+k}(t_2,t_1,t_1)\right].
\end{aligned} \tag{3.5}$$

Subcase 2: $\rho = 1$ and $\beta > k + \beta_1$.

Similar to Subcase 1, we can get

$$\begin{aligned}
\|I_2\| &\le M\mathrm{e}^{\omega k^{1-\frac{r}{k}}\Psi(t_1)}\int_0^{t_1}\left(\hat{g}_{1,k,\beta-\beta_1}(t_2,\tau,\tau)-\hat{g}_{1,k,\beta-\beta_1}(t_1,\tau,\tau)\right)\mathrm{d}\tau \\
&= kM\mathrm{e}^{\omega k^{1-\frac{r}{k}}\Psi(t_1)}\left[\hat{g}_{1,k,\beta-\beta_1+k}(t_2,0,0)-\hat{g}_{1,k,\beta-\beta_1+k}(t_1,0,0)\right. \\
&\quad \left.-\hat{g}_{1,k,\beta-\beta_1+k}(t_2,t_1,t_1)\right].
\end{aligned} \tag{3.6}$$

Subcase 3: $\rho \ne 1$ and $\beta_1 < \beta < k + \beta_1$.

Using the Lagrange mean value theorem, we have

$$\begin{aligned}
\|I_2\| &\le M\mathrm{e}^{\omega k^{1-\frac{r}{k}}\Psi(t_1)}\Big\{\big[\Psi(t_2)-\Psi(t_1)\big]^{\frac{\beta-\beta_1}{k}-1} \\
&\quad\times\int_0^{t_1}\big[\hat{g}_{\rho,k,\beta-\beta_1}(\tau,t_1,\tau)-\hat{g}_{\rho,k,\beta-\beta_1}(\tau,t_2,\tau)\big]\,\mathrm{d}\tau \\
&\quad+\int_0^{t_1}\big[\hat{g}_{\rho,k,\beta-\beta_1}(t_1,\tau,\tau)-\hat{g}_{\rho,k,\beta-\beta_1}(t_2,\tau,\tau)\big]\,\mathrm{d}\tau\Big\} \\
&= \frac{M\mathrm{e}^{\omega k^{1-\frac{r}{k}}\Psi(t_1)}}{\rho^{\frac{\beta-\beta_1}{k}}k\Gamma_k(\beta-\beta_1)}\big[\Psi(t_2)-\Psi(t_1)\big]^{\frac{\beta-\beta_1}{k}}\frac{1-\rho}{\rho k}\int_0^{t_1}\mathrm{e}^{\frac{(\rho-1)\xi}{\rho k}}\Psi'(\tau)\,\mathrm{d}\tau \\
&\quad+kM\mathrm{e}^{\omega k^{1-\frac{r}{k}}\Psi(t_1)}\big[\hat{g}_{1,k,\beta-\beta_1+k}(t_1,0,0)-\hat{g}_{1,k,\beta-\beta_1+k}(t_2,0,0) \\
&\quad+\hat{g}_{1,k,\beta-\beta_1+k}(t_2,t_1,t_1)\big] \\
&\le \frac{M\mathrm{e}^{\omega k^{1-\frac{r}{k}}\Psi(t_1)}}{\rho^{\frac{\beta-\beta_1}{k}}k\Gamma_k(\beta-\beta_1)}\big[\Psi(t_2)-\Psi(t_1)\big]^{\frac{\beta-\beta_1}{k}}\frac{1-\rho}{\rho k}\Psi(t_1) \\
&\quad+kM\mathrm{e}^{\omega k^{1-\frac{r}{k}}\Psi(t_1)}\big[\hat{g}_{1,k,\beta-\beta_1+k}(t_1,0,0)-\hat{g}_{1,k,\beta-\beta_1+k}(t_2,0,0) \\
&\quad+\hat{g}_{1,k,\beta-\beta_1+k}(t_2,t_1,t_1)\big],
\end{aligned}\tag{3.7}$$

where $\xi\in[\Psi(t_1)-\Psi(\tau),\Psi(t_2)-\Psi(\tau)]$.

Subcase 4: $\rho\neq 1$ and $\beta>k+\beta_1$.

Similar to Subcase 3, we have

$$\begin{aligned}
\|I_2\| &\le M\mathrm{e}^{\omega k^{1-\frac{r}{k}}\Psi(t_1)}\Big\{\Psi^{\frac{\beta-\beta_1}{k}-1}(t_2)\int_0^{t_1}\big[\hat{g}_{\rho,k,\beta-\beta_1}(\tau,t_1,\tau)-\hat{g}_{\rho,k,\beta-\beta_1}(\tau,t_2,\tau)\big]\,\mathrm{d}\tau \\
&\quad+\int_0^{t_1}\big[\hat{g}_{\rho,k,\beta-\beta_1}(t_2,\tau,\tau)-\hat{g}_{\rho,k,\beta-\beta_1}(t_1,\tau,\tau)\big]\,\mathrm{d}\tau\Big\} \\
&\le \frac{M\mathrm{e}^{\omega k^{1-\frac{r}{k}}\Psi(t_1)}}{\rho^{\frac{\beta-\beta_1}{k}}k\Gamma_k(\beta-\beta_1)}\Psi^{\frac{\beta-\beta_1}{k}}(t_2)\frac{1-\rho}{\rho k}\Psi(t_1) \\
&\quad+kM\mathrm{e}^{\omega k^{1-\frac{r}{k}}\Psi(t_1)}\big[\hat{g}_{1,k,\beta-\beta_1+k}(t_2,0,0)-\hat{g}_{1,k,\beta-\beta_1+k}(t_1,0,0) \\
&\quad-\hat{g}_{1,k,\beta-\beta_1+k}(t_2,t_1,t_1)\big].
\end{aligned}\tag{3.8}$$

Now, we can state that $\lim_{t_1\to t_2}\|I_2\|=0$ holds in all Subcases from (3.5)–(3.8).

Case 2: $\beta=k+\beta_1$.

Let $0\le t_1<t_2$, from

$$\begin{aligned}
S_{\rho,\alpha,k+\beta_1,k,r}(t) &= (g_{\rho,k,k}*_\Psi S_{\rho,\alpha,\beta_1,k,r})(t) \\
&= \int_0^t \hat{g}_{\rho,k,k}(\tau,t,\tau)S_{\rho,\alpha,\beta_1,k,r}(\tau)\,\mathrm{d}\tau,
\end{aligned}$$

we conclude that

$$\begin{aligned}
&\left\|S_{\rho,\alpha,k+\beta_1,k,r}(t_2)-S_{\rho,\alpha,k+\beta_1,k,r}(t_1)\right\| \\
&\quad\le \int_0^{t_1}\left[\hat{g}_{\rho,k,k}(\tau,t_1,\tau)-\hat{g}_{\rho,k,k}(\tau,t_2,\tau)\right]\left\|S_{\rho,\alpha,\beta_1,k,r}(\tau)\right\|\,\mathrm{d}\tau \\
&\qquad+\frac{1}{\rho\Gamma_k(2k)}\int_{t_1}^{t_2}\Psi'(\tau)\left\|S_{\rho,\alpha,\beta_1,k,r}(\tau)\right\|\,\mathrm{d}\tau \\
&\quad\le \frac{M\mathrm{e}^{\omega k^{1-\frac{r}{k}}\Psi(t_2)}}{\rho\Gamma_k(2k)}\left[\Psi(t_2)-\Psi(t_1)\right]\left[\frac{1-\rho}{\rho k}\Psi(t_1)+1\right]\to 0,\quad \text{as } t_2\to t_1.
\end{aligned}$$

The proof is complete. □

Definition 3.1 $\{S_{\rho,\alpha,\beta,k,r}(t)\}_{t\ge 0}$ is said to be compact if $S_{\rho,\alpha,\beta,k,r}(t)$ is a compact when $t>0$.

Theorem 3.3 *If $\rho\in(0,1]$, $\alpha>0$, $\beta>\beta_1>0$ and $k,r>0$, A generates $\{S_{\rho,\alpha,\beta_1,k,r}(t)\}_{t\ge 0}$, which type is (M,ω,k,r), then the following assertions are equivalent*:

(i) *$S_{\rho,\alpha,\beta,k,r}(t)$ is a compact operator for $t>0$;*

(ii) *$((\rho k^{2-\frac{r}{k}}\lambda+1-\rho)^{\frac{\alpha}{k}}I-A)^{-1}$ is a compact operator for* $\operatorname{Re}\lambda>\omega$.

Proof Assume that $S_{\rho,\alpha,\beta,k,r}(t)$ is compact. Let $\operatorname{Re}\lambda>\omega$, then

$$\begin{aligned}
&\left(\rho k^{2-\frac{r}{k}}\lambda+1-\rho\right)^{\frac{\alpha-\beta}{k}}\left(\left(\rho k^{2-\frac{r}{k}}\lambda+1-\rho\right)^{\frac{\alpha}{k}}I-A\right)^{-1}x \\
&\quad=\int_0^{\infty}\mathrm{e}^{-\lambda k^{1-\frac{r}{k}}\Psi(t)}S_{\rho,\alpha,\beta,k,r}(t)x\Psi'(t)\,\mathrm{d}t
\end{aligned}\tag{3.9}$$

from Definition 2.10. However, noting that the uniform continuous of $\{S_{\rho,\alpha,\beta,k,r}(t)\}_{t>0}$ by Theorem 3.2, hence $((\rho k^{2-\frac{r}{k}}\lambda+1-\rho)^{\frac{\alpha}{k}}I-A)^{-1}$ is compact from Corollary 2.3 in [36].

Conversely, in view of $g_{\rho,k,\beta-\beta_1}\in L^1_{\mathrm{loc}}[0,\infty)$ and Theorem 2.2, we have

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1}{2\pi i}\int_{\omega-i\Psi(n)}^{\omega+i\Psi(n)}\mathrm{e}^{k^{1-\frac{r}{k}}\Psi(t)z}\mathcal{L}_k^{r;\Psi}\left(\frac{g_{\rho,k,\beta-\beta_1}}{\sqrt{\Psi'}}*^{\Psi}S_{\rho,\alpha,\beta_1,k,r}\right)(z)\,\mathrm{d}z \\
&\quad=k^{\frac{r}{k}-1}g_{\rho,k,\beta-\beta_1}*^{\Psi}S_{\rho,\alpha,\beta_1,k,r}.
\end{aligned}$$

Notice that $g_{\rho,k,\beta-\beta_1}*^{\Psi}S_{\rho,\alpha,\beta_1,k,r}=g_{\rho,k,\beta-\beta_1}*_{\Psi}S_{\rho,\alpha,\beta_1,k,r}$, then one derives the following:

$$\begin{aligned}
&\frac{1}{2\pi i}\int_L\mathrm{e}^{k^{1-\frac{r}{k}}\Psi(t)z}\mathcal{L}_k^{r;\Psi}\left(\frac{g_{\rho,k,\beta-\beta_1}}{\sqrt{\Psi'}}\right)(\lambda)\left(\rho k^{2-\frac{r}{k}}\lambda+1-\rho\right)^{\frac{\alpha-\beta}{k}} \\
&\quad\times\left(\left(\rho k^{2-\frac{r}{k}}\lambda+1-\rho\right)^{\frac{\alpha}{k}}I-A\right)^{-1}\mathrm{d}z=k^{\frac{r}{k}-1}S_{\rho,\alpha,\beta,k,r},\quad t>0,
\end{aligned}\tag{3.10}$$

where L is the line $\operatorname{Re}(z)=\omega$. $S_{\rho,\alpha,\beta,k,r}(t)$ is compact for $\rho\in(0,1]$, $\alpha>0$, $\beta>\beta_1>0$, $k,r>0$ can be found using Corollary 2.3 in [36] again. □

4 Mild solution for (ρ, k, Ψ)-proportional Hilfer Cauchy problem

In the section, we investigate the following (ρ, k, Ψ)-proportional Hilfer Cauchy problem:

$$\begin{cases} {}^{k,H}D_{0+}^{\alpha,\rho,\nu;\Psi}x(t) = Ax(t) + f(t, x(t)), \quad t \in [0, T], \\ [{}^{k}I_{0+}^{2k-\alpha_k,\rho;\Psi}x(t)]_{t=0} = x_0, \\ [{}^{k}D^{1,\rho;\Psi}\,{}^{k}I_{0+}^{2k-\alpha_k,\rho;\Psi}x(t)]_{t=0} = x_1, \end{cases} \tag{4.1}$$

where $x_0, x_1 \in E$, $1 < \frac{\alpha}{k} < 2$, $0 \le \nu \le 1$, $\alpha_k = \alpha + \nu(2k - \alpha)$, the closed linear operator A generates $\{S_{\rho,\alpha,\alpha_k-k,k,r}(t)\}_{t\ge 0}$.

Let us apply $\mathcal{L}_k^{r;\Psi}$ to the first equality in (4.1), together with the initial value conditions in (4.1) and Lemma 2.1, we get

$$x(t) = \int_0^t \overline{S}_{\rho,\alpha,\alpha,k,r}(t,s) f\big(s, x(s)\big)\Psi'(s)\,\mathrm{d}s + \rho k S_{\rho,\alpha,\alpha_k-k,k,r}(t)x_0 + \rho k S_{\rho,\alpha,\alpha_k,k,r}(t)x_1, \tag{4.2}$$

where the symbol $\overline{S}_{\rho,\alpha,\alpha,k,r}(t,s)$ is defined in Definition 2.2.

Definition 4.1 Let $1 < \frac{\alpha}{k} < 2$, $0 \le \nu \le 1$. A generates $\{S_{\rho,\alpha,\alpha_k-k,k,r}(t)\}_{t\ge 0}$. We say that $x(t)$ is a mild solution of problem (4.1) if x satisfies (4.2).

For convenience, set

$$\Lambda_1 = \left(\frac{1}{\rho k^{2-\frac{r}{k}}\omega + 1 - \rho}\right)^{\frac{k-(\alpha_k-\alpha)}{k}},$$

and

$$\Lambda_2(t,s,\tau) = M\mathrm{e}^{\omega k^{1-\frac{r}{k}}\Psi(t)}\frac{\mathrm{e}^{-\omega k^{1-\frac{r}{k}}\Psi(s)} - \mathrm{e}^{-\omega k^{1-\frac{r}{k}}\Psi(\tau)}}{\omega k^{1-\frac{r}{k}}}, \quad t, s, \tau \in [0, T].$$

Theorem 4.1 *Suppose that A generates $\{S_{\alpha,\alpha_k-k,k,r}(t)\}_{t\ge 0}$, which type is (M, ω, k, r). Moreover, $S_{\alpha,\alpha_k-k,k,r}(t)$ is continuous for $t > 0$, and the following conditions hold*:

(H1) *$f \in C([0, T] \times E, E)$. Furthermore, there exist a scalar nondecreasing function $\Phi \in C(\mathbb{R}^+, \mathbb{R}^+)$ and two scalar functions $a, b \in C([0, T], \mathbb{R}^+)$ such that*

$$\big\|f(t,x)\big\| \le a(t) + b(t)\Phi\big(\|x\|\big),$$

where $x \in E$, $t \in [0, T]$;

(H2) *There exists a scalar function $c \in C([0, T], \mathbb{R}^+)$ such that*

$$\alpha\big(f(t,D)\big) \le c(t)\alpha(D)$$

for any bounded set D and $t \in [0, T]$.

If

$$\Lambda_1\Lambda_2(T, 0, T)\max\left\{\|b\|_\infty \liminf_{l\to\infty}\frac{\Phi(l)}{l}, 2\|c\|_\infty\right\} < 1, \tag{4.3}$$

then problem (4.1) possesses one mild solution on $[0, T]$.

Proof Set $E_l = \{x \in C([0,T],E) : \|x\| \le l\}$, $l > 0$, and $\Omega = \{x^n : n \ge 1\}$, F on $C([0,T],E)$ and $x^n(t)$ on Ω are given below.

$$\begin{aligned} Fx(t) &= \int_0^t \overline{S}_{\rho,\alpha,\alpha,k,r}(t,s) f\big(s,x(s)\big)\Psi'(s)\,\mathrm{d}s \\ &\quad + \rho k S_{\rho,\alpha,\alpha_k-k,k,r}(t)x_0 + \rho k S_{\rho,\alpha,\alpha_k,k,r}(t)x_1, \end{aligned}$$

$$x^n(t) = \begin{cases} \rho k S_{\rho,\alpha,\alpha_k-k,k,r}(t)x_0 + \rho k S_{\rho,\alpha,\alpha_k,k,r}(t)x_1, & t \in [0,\frac{T}{n}], \\ \int_0^{t-\frac{T}{n}} \overline{S}_{\rho,\alpha,\alpha,k,r}(t,s) f(s,x^n(s))\Psi'(s)\,\mathrm{d}s \\ \quad + \rho k S_{\rho,\alpha,\alpha_k-k,k,r}(t)x_0 + \rho k S_{\rho,\alpha,\alpha_k,k,r}(t)x_1, & t \in [\frac{T}{n},T], \end{cases}$$

respectively.

We subdivide the proof into several steps.

Step 1: $FE_{l_0} \subseteq E_{l_0}$ for some $l_0 > 0$.

In fact, if not, then for every $l > 0$, $\exists x_l \in E_l$, $t_l \in [0,T]$, one has $\|(Fx_l)(t_l)\| > l$. First, we notice that for $x \in E$,

$$\begin{aligned} \big\|Fx(t)\big\| &\le \int_0^t \big\|\overline{S}_{\rho,\alpha,\alpha,k,r}(t,s)\big\| \big\|f\big(s,x(s)\big)\big\| \Psi'(s)\,\mathrm{d}s \\ &\quad + \rho k \big\|S_{\rho,\alpha,\alpha_k-k,k,r}(t)\big\| \|x_0\| + \rho k \big\|S_{\rho,\alpha,\alpha_k,k,r}(t)\big\| \|x_1\| \\ &\le \Lambda_1 M \mathrm{e}^{\omega k^{1-\frac{r}{k}}\Psi(t)} \int_0^t \mathrm{e}^{-\omega k^{1-\frac{r}{k}}\Psi(s)} \big[a(s) + b(s)\Phi\big(\|x\|\big)\big]\Psi'(s)\,\mathrm{d}s \\ &\quad + \rho k M \mathrm{e}^{\omega k^{1-\frac{r}{k}}\Psi(t)} \|x_0\| + \rho \frac{k}{\rho k^{2-\frac{r}{k}}\omega + 1 - \rho} M \mathrm{e}^{\omega\Psi(t)} \|x_1\| \\ &\le \Lambda_1\Lambda_2(T,0,T)\big[\|a\|_\infty + \|b\|_\infty \Phi\big(\|x\|\big)\big] + \rho k M \mathrm{e}^{\omega k^{1-\frac{r}{k}}\Psi(T)} \|x_0\| \\ &\quad + \rho \frac{k}{\rho k^{2-\frac{r}{k}}\omega + 1 - \rho} M \mathrm{e}^{\omega\Psi(T)} \|x_1\|, \quad t \in [0,T]. \end{aligned}$$

Consequently,

$$\begin{aligned} l &< \big\|Fx_l(t_l)\big\| \\ &\le \Lambda_1\Lambda_2(T,0,T)\big[\|a\|_\infty + \|b\|_\infty \Phi(l)\big] + \rho k M \mathrm{e}^{\omega k^{1-\frac{r}{k}}\Psi(T)} \|x_0\| \\ &\quad + \rho \frac{k}{\rho k^{2-\frac{r}{k}}\omega + 1 - \rho} M \mathrm{e}^{\omega\Psi(T)} \|x_1\|. \end{aligned}$$

By multiplying above by $\frac{1}{l}$, and then taking $\liminf_{l\to\infty}$, we obtain the following result:

$$1 \le \Lambda_1\Lambda_2(T,0,T)\|b\|_\infty \liminf_{l\to\infty} \frac{\Phi(l)}{l},$$

and it is a contradiction. Thus, $FE_{l_0} \subseteq E_{l_0}$.

Step 2: The operator F is continuous on E_{r_0}.

If $\{x_n\}, x \in E_{l_0}$ with $\lim_{n\to\infty} x_n = x$. Noting that $f(s,x_n(s)) \to f(s,x(s))$, $t \in [0,T]$ as $n \to \infty$, we have $\sup_{t\in[0,T]} \|(Fx_n)(t) - (Fx)(t)\| \to 0$ when $n \to \infty$, namely F is continuous.

Step 3: Ω is equicontinuous on $[0,T]$.

Case 1: $0 \le t_1 < t_2 \le \frac{T}{n}$ and $x^n \in \Omega$.

Noting that

$$\begin{aligned}\left\|x^n(t_2)-x^n(t_1)\right\| &\le \rho k\left\|S_{\rho,\alpha,\alpha_k-k,k,r}(t_2)-S_{\rho,\alpha,\alpha_k-k,k,r}(t_1)\right\|\|x_0\| \\ &\quad+\rho k\left\|S_{\rho,\alpha,\alpha_k,k,r}(t_2)-S_{\rho,\alpha,\alpha_k,k,r}(t_1)\right\|\|x_1\|.\end{aligned}$$

By hypothesis and Theorem 3.2, we conclude that $\|x^n(t_2)-x^n(t_1)\| \to 0$ if $t_1 \to t_2$.

Case 2: $0 \le t_1 \le \frac{T}{n} < t_2 \le T$ and $x^n \in \Omega$.

Then

$$\begin{aligned}\left\|x^n(t_2)-x^n(t_1)\right\| &\le \left\|\int_0^{t_2-\frac{T}{n}} \overline{S}_{\rho,\alpha,\alpha,k,r}(t_2,s)f\bigl(s,x^n(s)\bigr)\Psi'(s)\,\mathrm{d}s\right\| \\ &\quad+\rho k\left\|S_{\rho,\alpha,\alpha_k-k,k,r}(t_2)-S_{\rho,\alpha,\alpha_k-k,k,r}(t_1)\right\|\|x_0\| \\ &\quad+\rho k\left\|S_{\rho,\alpha,\alpha_k,k,r}(t_2)-S_{\rho,\alpha,\alpha_k,k,r}(t_1)\right\|\|x_1\| \\ &\le \Lambda_1 M\mathrm{e}^{\omega k^{1-\frac{r}{k}}\Psi(t_2)}\int_0^{t_2-\frac{T}{n}} \mathrm{e}^{-\omega k^{1-\frac{r}{k}}\Psi(s)}\bigl[a(s)+b(s)\Phi\bigl(\|x\|\bigr)\bigr]\Psi'(s)\,\mathrm{d}s \\ &\quad+\rho k\left\|S_{\rho,\alpha,\alpha_k-k,k,r}(t_2)-S_{\rho,\alpha,\alpha_k-k,k,r}(t_1)\right\|\|x_0\| \\ &\quad+\rho k\left\|S_{\rho,\alpha,\alpha_k,k,r}(t_2)-S_{\rho,\alpha,\alpha_k,k,r}(t_1)\right\|\|x_1\| \\ &\le \Lambda_1\Lambda_2\left(t_2,0,t_2-\frac{T}{n}\right)\bigl[\|a\|_\infty+\|b\|_\infty\Phi(l_0)\bigr] \\ &\quad+\rho k\left\|S_{\rho,\alpha,\alpha_k-k,k,r}(t_2)-S_{\rho,\alpha,\alpha_k-k,k,r}(t_1)\right\|\|x_0\| \\ &\quad+\rho k\left\|S_{\rho,\alpha,\alpha_k,k,r}(t_2)-S_{\rho,\alpha,\alpha_k,k,r}(t_1)\right\|\|x_1\|.\end{aligned}$$

We can also obtain that $\|x^n(t_2)-x^n(t_1)\| \to 0$ if $t_1 \to t_2$.

Case 3: $\frac{T}{n} \le t_1 < t_2 \le T$ and $x^n \in \Omega$.

We can see that

$$\begin{aligned}\left\|x^n(t_1)-x^n(t_2)\right\| &\le \left\|\int_{t_1-\frac{T}{n}}^{t_2-\frac{T}{n}} \overline{S}_{\rho,\alpha,\alpha,k,r}(t_2,s)f\bigl(s,x^n(s)\bigr)\Psi'(s)\,\mathrm{d}s\right\| \\ &\quad+\left\|\int_0^{t_1-\frac{T}{n}} \bigl[\overline{S}_{\rho,\alpha,\alpha,k,r}(t_2,s)-\overline{S}_{\rho,\alpha,\alpha,k,r}(t_1,s)\bigr]\Psi'(s)f\bigl(s,x^n(s)\bigr)\,\mathrm{d}s\right\| \\ &\quad+\rho k\left\|S_{\rho,\alpha,\alpha_k-k,k,r}(t_2)-S_{\rho,\alpha,\alpha_k-k,k,r}(t_1)\right\|\|x_0\| \\ &\quad+\rho k\left\|S_{\rho,\alpha,\alpha_k,k,r}(t_2)-S_{\rho,\alpha,\alpha_k,k,r}(t_1)\right\|\|x_1\| \\ &\le I_1+I_2+I_3+I_4.\end{aligned}$$

Observe that

$$I_1 \le \Lambda_1\Lambda_2\left(t_2,t_1-\frac{T}{n},t_2-\frac{T}{n}\right)\bigl[\|a\|_\infty+\|b\|_\infty\Phi(l_0)\bigr],$$

hence $\lim_{t_1\to t_2} I_1 = 0$ independently of $x^n \in \Omega$. Moreover, the inequality

$$I_2 \le \big[\|a\|_\infty + \|b\|_\infty \Phi(l_0)\big] \int_0^{t_1-\frac{T}{n}} \big\|\overline{S}_{\rho,\alpha,\alpha,k,r}(t_2,s) - \overline{S}_{\rho,\alpha,\alpha,k,r}(t_1,s)\big\| \Psi'(s)\,\mathrm{d}s,$$

together with

$$I_2 \le 2\big[\|a\|_\infty + \|b\|_\infty \Phi(l_0)\big]\Lambda_1 M \mathrm{e}^{\omega k^{1-\frac{r}{k}}\Psi(T)} \frac{1}{\omega k^{1-\frac{r}{k}}},$$

and $\overline{S}_{\rho,\alpha,\alpha,k,r}$ is norm continuous by Theorem 3.2, which leads to $\lim_{t_1\to t_2} I_2 = 0$ independently of $x^n \in \Omega$. Moreover, we derive that $\lim_{t_1\to t_2} I_3 = 0$ and $\lim_{t_1\to t_2} I_4 = 0$ independently of $x^n \in \Omega$ using the hypothesis and Theorem 3.2, respectively.

Step 4: Ω is compact in E_{r_0}.

Let $\Omega(t) = \{x^n(t) : n \ge 1, t \in [0,T]\}$. In view of $\mathrm{e}^{-\omega k^{1-\frac{r}{k}}\Psi(t)}$ is continuous on any closed interval, then for arbitrary $\varepsilon > 0$, there exists a $\delta > 0$, which satisfies

$$2\Lambda_1\Lambda_2(T,t_1,t_2)\big[\|a\|_\infty + \|b\|_\infty \Phi(l_0)\big] < \varepsilon, \quad |t_1 - t_2| < \delta.$$

Now, choose a $N \in \mathbb{Z}^+$ so that $\frac{T}{n} < \delta$ for $n > N$, we conclude that

$$\begin{aligned}
&\alpha\left(\left\{\int_{t-\frac{T}{n}}^{t} \overline{S}_{\rho,\alpha,\alpha,k,r}(t,s) f\big(s,x^n(s)\big)\Psi'(s)\,\mathrm{d}s : n > N\right\}\right) \\
&\quad \le 2\Lambda_1\Lambda_2\left(T, t-\frac{T}{n}, t\right)[\|a\|_\infty + \|b\|_\infty \Phi(l_0) < \varepsilon, \quad t \in [0,T].
\end{aligned}$$

Furthermore,

$$\alpha\left(\left\{\int_{t-\frac{T}{n}}^{t} \overline{S}_{\rho,\alpha,\alpha,k,r}(t,s) f\big(s,x^n(s)\big)\Psi'(s)\,\mathrm{d}s : n \ge 1\right\}\right) < \varepsilon.$$

It leads to

$$\begin{aligned}
\alpha\big(\Omega(t)\big) &\le \alpha\left(\left\{\int_0^t \overline{S}_{\rho,\alpha,\alpha,k,r}(t,s) f\big(s,x^n(s)\big)\Psi'(s)\,\mathrm{d}s : n \ge 1\right\}\right) \\
&\quad + \alpha\left(\left\{\int_{t-\frac{T}{n}}^{t} \overline{S}_{\rho,\alpha,\alpha,k,r}(t,s) f\big(s,x^n(s)\big)\Psi'(s)\,\mathrm{d}s : n \ge 1\right\}\right) \\
&\le 2\int_0^t \overline{S}_{\rho,\alpha,\alpha,k,r}(t,s) c(s) \alpha\big(\Omega(s)\big)\Psi'(s)\,\mathrm{d}s + \varepsilon \\
&\le 2\Lambda_1\Lambda_2(T,0,T)\|c\|_\infty \sup_{s\in[0,t]} \alpha\big(\Omega(s)\big) + \varepsilon,
\end{aligned}$$

which means that $\alpha(\Omega(t)) = 0$ for all $t \in [0,T]$ from (4.3). Hence, Ω is relatively compact in E_{l_0} by the fact that $\alpha(\Omega) = 0$.

Step 5: The last step is to verify that the mapping F has one fixed point.

By Step 4, without loss of generality, let $\{x^n : n \ge 1\} \subset \Omega$ converge to x on $[0,T]$. Suppose that

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \big\|x^n(t) - x(t)\big\| = 0, \tag{4.4}$$

one can easily derive

$$\lim_{n\to\infty}\sup_{t\in[0,T]}\left\|Fx(t)-Fx^n(t)\right\|=0 \tag{4.5}$$

by the continuous of operator F.

Case 1: $t\in[0,\frac{T}{n}]$.

Then

$$\begin{aligned}\left\|Fx^n(t)-x^n(t)\right\| &\le \int_0^t \overline{S}_{\rho,\alpha,\alpha,k,r}(t,s)\left\|f\left(s,x^n(s)\right)\right\|\Psi'(s)\,\mathrm{d}s\\ &\le \Lambda_1\Lambda_2\left(\frac{T}{n},0,\frac{T}{n}\right)\left[\|a\|_\infty+\|b\|_\infty\Phi(l_0)\right].\end{aligned}$$

Case 2: $t\in[\frac{T}{n},T]$.

Then

$$\begin{aligned}\left\|Fx^n(t)-x^n(t)\right\| &\le \int_{t-\frac{T}{n}}^t \overline{S}_{\rho,\alpha,\alpha,k,r}(t,s)\left\|f\left(s,x^n(s)\right)\right\|\Psi'(s)\,\mathrm{d}s\\ &\le \Lambda_1\Lambda_2\left(t,t-\frac{T}{n},t\right)\left[\|a\|_\infty+\|b\|_\infty\Phi(l_0)\right].\end{aligned}$$

In any case, we can obtain

$$\sup_{t\in[0,T]}\left\|Fx^n(t)-x^n(t)\right\|\to 0,\quad n\to\infty. \tag{4.6}$$

From the triangle inequality

$$\left\|Fx(t)-x(t)\right\|\le\left\|x^n(t)-x(t)\right\|+\left\|Fx^n(t)-Fx(t)\right\|+\left\|Fx^n(t)-x^n(t)\right\|$$

and (4.4)–(4.6), we have $Fx=x$. It is equivalent that the Cauchy problem (4.1) has at least a mild solution x. □

Theorem 4.2 *Let A generate $\{S_{\rho,\alpha,\alpha_k-k,k,r}(t)\}_{t\ge0}$, which type is (M,ω,k,r). $S_{\rho,\alpha,\alpha_k-k,k,r}(t)$ is continuous for $t>0$, and $((\rho k^{2-\frac{r}{k}}\lambda+1-\rho)^{\frac{\alpha}{k}}I-A)^{-1}$ is compact for $\lambda>\omega$. Suppose that*

$$\Lambda_1\Lambda_2(T,0,T)\|b\|_\infty\liminf_{l\to\infty}\frac{\Phi(l)}{l}<1, \tag{4.7}$$

then (4.1) *has at least one mild solution under the assumption* (H1).

Proof We consider two operators F_1 and F_2 mapping from $C([0,T],E)$ into itself as follows:

$$\begin{aligned}F_1x(t)&=\int_0^t\overline{S}_{\rho,\alpha,\alpha,k,r}(t,s)f\left(s,x(s)\right)\Psi'(s)\,\mathrm{d}s,\\ F_2x(t)&=\rho kS_{\rho,\alpha,\alpha_k-k,k,r}(t)x_0+\rho kS_{\rho,\alpha,\alpha_k,k,r}(t)x_1,\quad t\in[0,T].\end{aligned}$$

According to the same process of Step 1 in Theorem 4.1, it is easy to verify that there exists a $l_0>0$ such that $F_1x+F_2y\in E_{l_0}$, $x,y\in E_{l_0}$. Obviously, F_2 is a contraction on E_{l_0}

with contraction constant 0. Moreover, similar to Step 2 in Theorem 4.1, F_1 is continuous on E_{l_0}.

Now we show that $\{F_1x : x \in E_{l_0}\}$ is equiconinuous. If $0 \le t_1 < t_2 \le T$ and $x \in E_{l_0}$, then

$$\begin{aligned}
&\left\|F_1x(t_2) - F_1(t_1)\right\| \\
&\quad \le \left\| \int_{t_1}^{t_2} \overline{S}_{\rho,\alpha,\alpha,k,r}(t_2,s) f\big(s,x^n(s)\big)\Psi'(s)\,\mathrm{d}s \right\| \\
&\qquad + \left\| \int_0^{t_1} \big[\overline{S}_{\rho,\alpha,\alpha,k,r}(t_2,s) - \overline{S}_{\rho,\alpha,\alpha,k,r}(t_1,s)\big]\Psi'(s) f\big(s,x^n(s)\big)\,\mathrm{d}s \right\| \\
&\quad \le I_1 + I_2.
\end{aligned}$$

Noting that

$$I_1 \le \Lambda_1\Lambda_2(T,t_1,t_2)\big[\|a\|_\infty + \|b\|_\infty \Phi(l_0)\big],$$

hence $\lim_{t_1\to t_2} I_1 = 0$.

$$I_2 \le \big[\|a\|_\infty + \|b\|_\infty \Phi(l_0)\big] \int_0^{t_1} \left\|\overline{S}_{\rho,\alpha,\alpha,k,r}(t_2,s) - \overline{S}_{\rho,\alpha,\alpha,k,r}(t_1,s)\right\| \Psi'(s)\,\mathrm{d}s,$$

observe that

$$I_2 \le 2\big[\|a\|_\infty + \|b\|_\infty \Phi(l_0)\big]\Lambda_1 M \mathrm{e}^{\omega k^{1-\frac{r}{k}}\Psi(T)} \frac{1}{\omega k^{1-\frac{r}{k}}},$$

and $S_{\rho,\alpha,\alpha,k,r}$ is norm continuous by Theorem 3.2, then we have $\lim_{t_1\to t_2} I_2 = 0$.

First, $\{(F_1x)(0) : x \in E_{l_0}\}$ is precompact. Second, let $0 < t \le T$, F_1^ϵ is defined by

$$\big(F_1^\epsilon x\big)(t) = \int_0^{t-\varepsilon} \overline{S}_{\rho,\alpha,\alpha,k,r}(t,s) f\big(s,x(s)\big)\Psi'(s)\,\mathrm{d}s, \quad \forall \varepsilon \in (0,t),$$

where $x \in E_{l_0}$. Hypothesis illustrates the compactness of $S_{\rho,\alpha,\alpha,k,r}(t)$ for $t > 0$, then $\{(F_1^\varepsilon x)(t) : x \in E_{l_0}\}$ is precompact in E_{l_0}. Furthermore, for every $x \in E_{l_0}$, one easily sees that

$$\begin{aligned}
&\left\|F_1x(t) - F_1^\varepsilon x(t)\right\| \\
&\quad \le \int_{t-\varepsilon}^{t} \left\|\overline{S}_{\rho,\alpha,\alpha,k,r}(t,s) f\big(s,x(s)\big)\Psi'(s)\right\| \mathrm{d}s \\
&\quad \le \Lambda_1\Lambda_2(T,t-\varepsilon,t)\big[\|a\|_\infty + \|b\|_\infty \Phi(l_0)\big] \to 0, \quad \varepsilon \to 0,
\end{aligned}$$

which implies that $\{F_1x(t) : x \in E_{l_0}, t \in (0,T]\}$ is precompact in E_{l_0}.

Hence, $\{F_1x : x \in E_{l_0}\}$ is precompact. Furthermore, F_1 is compact on E_{l_0}, and then $F_1 + F_2$ has a fixed point on E_{l_0} by the Kransnoselskii fixed point Theorem, namely (4.1) has a mild solution. □

5 An example

Example 5.1 Denote the space $E = L^2([0,\pi],\mathbb{R})$, and let $k < \alpha < 2k$, $\nu \in [0,1]$. We study the (ρ,k,Ψ)-proportional Hilfer fractional Cauchy problem as follows:

$$\begin{cases} {}^{k,H}D_{0+}^{\alpha,\rho,\nu;\Psi}x(t,\xi) = \frac{\partial^4 x(t,\xi)}{\partial\xi^4} + \frac{1}{1+t}\sqrt{x(t,\xi)}, \quad t \in [0,1], \\ ({}^{k}I_{0+}^{2k-\alpha_k,\rho;\Psi}x(t,\xi))_{t=0} = x_0(\xi), \quad \xi \in [0,\pi], \\ [{}^{k}D^{1,\rho;\Psi}\,{}^{k}I_{0+}^{2k-\alpha_k,\rho;\Psi}x(t,\xi)]_{t=0} = x_1(\xi), \quad \xi \in [0,\pi], \\ x(t,0) = 0 = x(t,\pi), \end{cases} \tag{5.1}$$

where ${}^{k,H}D_{0+}^{\alpha,\rho,\nu;\Psi}$ is the (k,Ψ)-proportional Hilfer derivative. Operator A is given by $Ax = x_{\xi\xi\xi\xi}$, let $D(A) = \{x \in E : x \in H^4([0,\pi]), x(t,0) = x(t,\pi) = 0\}$. Obviously, the eigenvalues and eigenvectors of A are $-n^4$ and $x_n(\xi) = \sqrt{\frac{2}{\pi}}\sin(n\xi)$, respectively.

Let $x \in E$ and $k \le \alpha \le 2k$, we have

$$\begin{aligned} &\left(\rho k^{2-\frac{r}{k}}\lambda + 1 - \rho\right)^{\frac{\alpha-\beta}{k}}\left(\left(\rho k^{2-\frac{r}{k}}\lambda + 1 - \rho\right)^{\frac{\alpha}{k}}I - A\right)^{-1}x \\ &\quad = \sum_{n=1}^{\infty}\langle x,x_n\rangle x_n \frac{(\rho k^{2-\frac{r}{k}}\lambda + 1 - \rho)^{\frac{\alpha-\beta}{k}}}{(\rho k^{2-\frac{r}{k}}\lambda + 1 - \rho)^{\frac{\alpha}{k}} + n^4} \\ &\quad = \sum_{n=1}^{\infty}\langle x,x_n\rangle x_n \mathcal{L}_k^{r;\Psi}\left(\rho^{-\frac{\beta}{k}}k^{-\frac{\beta}{k}}\Psi^{\frac{\beta}{k}-1}(t)\mathrm{e}^{\frac{(\rho-1)\Psi(t)}{\rho k}}E_{\frac{\alpha}{k},\frac{\beta}{k}}\left(-\rho^{-\frac{\alpha}{k}}k^{-\frac{\alpha}{k}}n^4\Psi^{\frac{\alpha}{k}}(t)\right)\right)(\lambda) \\ &\quad = \sum_{n=1}^{\infty}\langle x,x_n\rangle x_n \int_0^{\infty}\mathrm{e}^{-\lambda k^{1-\frac{r}{k}}\Psi(t)}\Psi'(t)\rho^{-\frac{\beta}{k}}k^{-\frac{\beta}{k}}\Psi^{\frac{\beta}{k}-1}(t)\mathrm{e}^{\frac{(\rho-1)\Psi(t)}{\rho k}} \\ &\qquad \times E_{\frac{\alpha}{k},\frac{\beta}{k}}\left(-\rho^{-\frac{\alpha}{k}}k^{-\frac{\alpha}{k}}n^4\Psi^{\frac{\alpha}{k}}(t)\right)\mathrm{d}t \\ &\quad = \int_0^{\infty}\mathrm{e}^{-\lambda k^{1-\frac{r}{k}}\Psi(t)}\Psi'(t)\sum_{n=1}^{\infty}\langle x,x_n\rangle x_n\rho^{-\frac{\beta}{k}}k^{-\frac{\beta}{k}}\Psi^{\frac{\beta}{k}-1}(t)\mathrm{e}^{\frac{(\rho-1)\Psi(t)}{\rho k}} \\ &\qquad \times E_{\frac{\alpha}{k},\frac{\beta}{k}}\left(-\rho^{-\frac{\alpha}{k}}k^{-\frac{\alpha}{k}}n^4\Psi^{\frac{\alpha}{k}}(t)\right)\mathrm{d}t. \end{aligned}$$

Hence, A generates $\{S_{\rho,\alpha,\beta,k,r}(t)\}_{t\ge 0}$,

$$S_{\rho,\alpha,\beta,k,r}(t)x = \sum_{n=1}^{\infty}\langle x,x_n\rangle x_n\rho^{-\frac{\beta}{k}}k^{-\frac{\beta}{k}}\Psi^{\frac{\beta}{k}-1}(t)\mathrm{e}^{\frac{(\rho-1)\Psi(t)}{\rho k}}E_{\frac{\alpha}{k},\frac{\beta}{k}}\left(-\rho^{-\frac{\alpha}{k}}k^{-\frac{\alpha}{k}}n^4\Psi^{\frac{\alpha}{k}}(t)\right),$$

$x \in E$. From the continuity of $E_{\frac{\alpha}{k},\frac{\beta}{k}}(\cdot)$, one can easily derive that $S_{\alpha,\beta,k}(t)$ is norm continuous. We also have the compactness of $(\rho k^{2-\frac{r}{k}}\lambda + 1 - \rho)^{\frac{\alpha-\beta}{k}}((\rho k^{2-\frac{r}{k}}\lambda + 1 - \rho)^{\frac{\alpha}{k}}I - A)^{-1}$ on E by $\lim_{n\to\infty}\frac{(\rho k^{2-\frac{r}{k}}\lambda+1-\rho)^{\frac{\alpha-\beta}{k}}}{(\rho k^{2-\frac{r}{k}}\lambda+1-\rho)^{\frac{\alpha}{k}}+n^4} = 0$, then we conclude the compactness of $((\rho k^{2-\frac{r}{k}}\lambda + 1 - \rho)^{\frac{\alpha}{k}}I - A)^{-1}$ for $\lambda > 0$.

Assume that $\|S_{\rho,\alpha,\beta,k,r}(t)x\| \le M\|x\|$, $x \in E$. Therefore, $S_{\rho,\alpha,\beta,k,r}(t)$ is type $(M,1,k,k)$.

We choose $a(t) = 0$, $b(t) = 1$ and $\Phi(r) = \sqrt{r}$, then (5.1) has one mild solution on $[0,1]$ by Theorem 4.2.

6 Conclusion

We propose new fractional integrals and derivatives, namely the (ρ, k, Ψ)-proportional integral and (ρ, k, Ψ)-proportional Hilfer derivative, respectively. We also introduce a new fractional resolvent operator. This study of the mild solution of generalized proportional (ρ, k, Ψ)-Hilfer fractional Cauchy problem yields results for numerous other distinct fractional derivatives as stated in Remark 2.3. Here, we have used the properties of the resolvent operator and the fixed point technique to get the existence of mild solutions. An example is also provided to illustrate the main result. All results are applicable to the more general fractional equation of the Sobolev type. In future work, we will consider the stability and controllability of mild solution and the ε-regular mild solution to (ρ, k, Ψ)-Hilfer fractional abstract differential equations.

Abbreviations
Not applicable.

Acknowledgements
The author is grateful to the editor and the anonymous reviewers for their constructive comments and suggestions which improved the quality of the paper.

Author contributions
The author contributed to the writing of this paper and approved the final manuscript.

Funding
This work is supported by the Hainan Provincial Natural Science Foundation of China (122MS088), Qiongtai Normal University (QTjg2022-4, QTjg2022-49).

Data availability
Not applicable.

Declarations

Ethics approval and consent to participate
Not applicable.

Competing interests
The author declares no competing interests.

Received: 21 January 2024 Accepted: 17 May 2024 Published online: 31 May 2024

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