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A q-analogue for partial-fraction decomposition of a rational function and its application

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Abstract

In this paper, by using the residue method of complex analysis, we obtain a q-analogue for partial-fraction decomposition of the rational function $\frac{x^M}{(x+1)_n^\lambda}$. As applications, we deduce the corresponding q-algebraic and q-combinatorial identities which are the q-extensions of Chu' results.

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1 Introduction and main result

Throughout this paper, we always make use of the following notation: $\mathbb{N} = \{1, 2, 3, \ldots\}$ denotes the set of natural numbers, $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$ denotes the set of nonnegative integers, $\mathbb{C}$ denotes the set of complex numbers.

For $a \in \mathbb{C}$, the shifted factorial are defined by

$$(a)_0 = 1 \quad \text{and} \quad (a)_n = a(a+1)\cdots(a+n-1) \quad \text{for } n = 1, 2, \ldots$$

The q-shifted factorial are defined by

$$(a;q)_0 = 1, \qquad (a;q)_n = (1-a)(1-aq)\cdots\left(1-aq^{n-1}\right) = \prod_{k=0}^{n-1}\left(1-aq^k\right) \quad (n = 1, 2, \ldots),$$

$$(a;q)_\infty = (1-a)(1-aq)\cdots\left(1-aq^n\right)\cdots = \prod_{k=0}^{\infty}\left(1-aq^k\right) \quad \left(|q| < 1; a, q \in \mathbb{C}\right).$$

The q-numbers and q-numbers factorial are defined by

$$[a]_q = \frac{1-q^a}{1-q} \quad (q \neq 1); \qquad [0]_q! = 1, \qquad [n]_q! = [1]_q[2]_q\cdots[n]_q \quad (n \in \mathbb{N}, a \in \mathbb{C}),$$

respectively. Clearly,

$$\lim_{q\to 1}[a]_q = a, \qquad \lim_{q\to 1}[n]_q! = n!.$$

The q-shifted factorial of q-numbers defined by

$$\big([a]_q\big)_n := [a]_q[a+1]_q\cdots[a+n-1]_q \quad (n\in\mathbb{N}, a\in\mathbb{C}).$$

Clearly,

$$\lim_{q\to 1}\big([a]_q\big)_n = (a)_n,$$

i.e., q-numbers shifted factorial $([a]_q)_n$ is an q-analogue of the shifted factorial $(a)_n$.

The q-shifted factorial $(a;q)_n$ is not a q-analogue of the shifted factorial $(a)_n$. Let $q\mapsto q^a$ and then divide $(1-q)^n$. Therefore $(a;q)_n$ becomes $([a]_q)_n = \frac{(q^a;q)_n}{(1-q)^n}$ which is a q-analogue of the shifted factorial $(a)_n$.

The q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_{n-k}(q;q)_k},$$

which satisfies the following relationships:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ n-k \end{bmatrix}_q \quad (0\le k\le n), \qquad \begin{bmatrix} n \\ k \end{bmatrix}_q = 0 \quad (n<k),$$
$$\begin{bmatrix} x \\ k \end{bmatrix}_q = \begin{bmatrix} x-1 \\ k-1 \end{bmatrix}_q + q^k\begin{bmatrix} x-1 \\ k \end{bmatrix}_q \quad (n,k\in\mathbb{N}; x\in\mathbb{C}).$$

The above q-standard notation can be found in [1] and [11].

The generalized harmonic numbers are defined by

$$H_0^{(r)} = 0 \quad \text{and} \quad H_n^{(r)} = \sum_{k=1}^{n}\frac{1}{k^r} \quad \text{for } n,r = 1,2,\ldots,$$

when $r=1$, they reduce to the classical harmonic numbers as $H_n = H_n^{(1)}$.

Two q-generalized harmonic numbers are respectively defined by

$$H_{0;q}^{(r)} = 0 \quad \text{and} \quad H_{n;q}^{(r)} = \sum_{k=1}^{n}\frac{1}{(1-q^k)^r} \quad \text{for } n,r = 1,2,\ldots,$$

and

$$\tilde{H}_{0;q}^{(r)} = 0 \quad \text{and} \quad \tilde{H}_{n;q}^{(r)} = \sum_{k=1}^{n}\frac{q^{kr}}{(1-q^k)^r} \quad \text{for } n,r = 1,2,\ldots,$$

when $r=1$, they reduce to the q-harmonic numbers $H_{n;q} = H_{n;q}^{(1)}$ and $\tilde{H}_{n;q} = \tilde{H}_{n;q}^{(1)}$, respectively.

The complete Bell polynomials $\mathbf{B}_n(x_1,x_2,\ldots,x_n)$ are defined by (see [9, p. 133–137] or [16, p. 173–174])

$$\exp\left(\sum_{k=1}^{\infty} x_k \frac{z^k}{k!}\right) = \sum_{n=0}^{\infty} \mathbf{B}_n(x_1,x_2,\ldots,x_n)\frac{z^n}{n!}, \quad \mathbf{B}_0 := 1, \tag{1}$$

with an exact expression being

$$\mathbf{B}_n(x_1,x_2,\ldots,x_n) = \sum_{\pi(n)} \frac{n!}{k_1!k_2!\cdots k_n!}\left(\frac{x_1}{1!}\right)^{k_1}\left(\frac{x_2}{2!}\right)^{k_2}\cdots\left(\frac{x_n}{n!}\right)^{k_n}, \tag{2}$$

where $\pi(n)$ denotes a partition of n, usually denoted $1^{k_1}2^{k_2}\cdots n^{k_n}$, with $k_1+2k_2+\cdots+nk_n = n$. The complete Bell polynomials $\mathbf{B}_n(x_1,x_2,\ldots,x_n)$ for $n=0$ to 5 are respectively given by:

$$\begin{aligned}
&\mathbf{B}_0 = 1,\\
&\mathbf{B}_1(x_1) = x_1,\\
&\mathbf{B}_2(x_1,x_2) = x_1^2 + x_2,\\
&\mathbf{B}_3(x_1,x_2,x_3) = x_1^3 + 3x_1x_2 + x_3,\\
&\mathbf{B}_4(x_1,x_2,x_3,x_4) = x_1^4 + 6x_1^2x_2 + 4x_1x_3 + 3x_2^2 + x_4,\\
&\mathbf{B}_5(x_1,x_2,x_3,x_4,x_5) = x_1^5 + 10x_1^3x_2 + 15x_1x_2^2 + 10x_1^2x_3 + 10x_2x_3 + 5x_1x_4 + x_5.
\end{aligned}$$

From (2) we easily obtain

$$\mathbf{B}_n\bigl(-x_1,x_2,\ldots,(-1)^n x_n\bigr) = (-1)^n\mathbf{B}_n(x_1,x_2,\ldots,x_n). \tag{3}$$

For convenience, we define the above sum as equal to zero for $n<0$, i.e., $\mathbf{B}_n(x_1,x_2,\ldots,x_n) = 0$ when $n<0$.

Chu [3] established the partial fraction decompositions of two rational functions $\frac{1}{(z)_{n+1}^{\lambda}}$ and $\frac{z^M}{(z)_{n+1}^{\lambda}}$ based on the induction principle and famous Faà di Bruno formula and obtained several striking harmonic number identities from two partial fraction decompositions, therefore resolved completely the open problem of Driver et al. [10]. It is not difficult, by using (2), to reformulate two main results of Chu as follows:

Theorem 1 ([3, Theorem 2]) *Suppose that* λ *and* n *are positive integers,* $z \in \mathbb{C} \setminus \{0,-1,\ldots,-n\}$. *Then the following partial fraction decomposition holds*:

$$\frac{(n!)^{\lambda}}{(z)_{n+1}^{\lambda}} = \sum_{k=0}^{n}(-1)^{k\lambda}\binom{n}{k}^{\lambda}\sum_{j=0}^{\lambda-1}\frac{\mathbf{B}_j(x_1,x_2,\ldots,x_j)}{j!(z+k)^{\lambda-j}}, \tag{4}$$

where

$$x_i = \lambda(i-1)!\bigl(H_k^{(i)} + (-1)^i H_{n-k}^{(i)}\bigr), \quad i = 1,2,\ldots,\lambda-1.$$

Theorem 2 ([3, Theorem 5]) *Suppose that n, M, and λ are three natural numbers with $\lambda \le M < \lambda(n+1)$, $z \in \mathbb{C} \setminus \{0, -1, \dots, -n\}$. Then the following partial fraction decomposition holds*:

$$\frac{z^M}{(z)_{n+1}^{\lambda}} = \sum_{k=0}^{n} (-1)^{\lambda k + M} \frac{k^M}{(n!)^{\lambda}} \binom{n}{k}^{\lambda} \sum_{j=0}^{\lambda-1} \frac{\mathbf{B}_j(x_1, x_2, \dots, x_j)}{j!(z+k)^{\lambda-j}}, \tag{5}$$

where

$$x_i = (i-1)! \left[\lambda \left(H_k^{(i)} + (-1)^i H_{n-k}^{(i)} \right) - \frac{M}{k^i} \right], \quad i = 1, 2, \dots, \lambda - 1.$$

The partial fraction decomposition plays an important role in the study of the combinatorial identities and related questions (for example, see [2–8, 12, 15, 17–23] and the references therein). But if a rational fraction is improper, the method of the partial fraction decomposition is invalid. So how do we decompose an improper rational fraction into partial fractions? Zhu and Luo answered this question using the contour integral and Cauchy's residue theorem and gave an explicit decomposition for the *general* rational function $\frac{x^M}{(x+1)_n^{\lambda}}$. We rewrite the main result of Zhu and Luo as follows:

Theorem 3 ([24, Theorem 1]) *Suppose that M is any nonnegative integer, λ and n are any positive integers such that $N = \lambda n$, and z is a complex number such that $z \in \mathbb{C} \setminus \{-1, -2, \dots, -n\}$. Then the following partial fraction decomposition holds*:

$$\begin{aligned} \frac{z^M}{(z+1)_n^{\lambda}} = {} & \sum_{j=0}^{M-N} \frac{\mathbf{B}_j(x_1, \dots, x_j)}{j!} z^{M-N-j} \\ & + \sum_{k=1}^{n} \frac{(-1)^{\lambda k}}{(n!)^{\lambda}} \binom{n}{k}^{\lambda} (-k)^{\lambda+M} \sum_{j=0}^{\lambda-1} \frac{\mathbf{B}_j(y_1, y_2, \dots, y_j)}{j!(z+k)^{\lambda-j}}, \end{aligned} \tag{6}$$

where

$$x_i = \lambda(-1)^i (i-1)! \sum_{j=1}^{n} j^i, \quad i = 1, 2, \dots, M - N,$$

$$y_i = (i-1)! \left[\lambda \left(H_k^{(i)} + (-1)^i H_{n-k}^{(i)} \right) - \frac{\lambda + M}{k^i} \right], \quad i = 1, 2, \dots, \lambda - 1.$$

When $M - N \ge 0$ in Theorem 3, i.e., the rational function $\frac{x^M}{(x+1)_n^{\lambda}}$ is improper, putting $a_j = \frac{\mathbf{B}_j(x_1,\dots,x_j)}{j!}$ and $a_{k,j} = \frac{(-1)^{\lambda k}}{j!(n!)^{\lambda}} \binom{n}{k}^{\lambda} (-k)^{\lambda+M} \mathbf{B}_j(y_1, y_2, \dots, y_j)$, (6) becomes the following explicit form:

$$\frac{x^M}{(x+1)_n^{\lambda}} = \sum_{j=0}^{M-N} a_{M-N-j} x^j + \sum_{k=1}^{n} \sum_{j=0}^{\lambda-1} \frac{a_{k,j}(x+k)^j}{(x+k)^{\lambda}},$$

which is an explicit result of the polynomial x^M divided by the polynomial $(x+1)_n^{\lambda}$. Therefore we say that Theorem 3 provides a new idea and method for the division of two *general* polynomials.

When $M - N < 0$, i.e., the degree of the numerator polynomial M is smaller than the degree of the denominator polynomial $N = \lambda n$, we deduce Chu's results (4) and (5). Therefore we say that Theorem 3 is an interesting extension of Chu's results.

In the present paper, we will provide a q-analogue of Theorem 3 using the contour integral and Cauchy's residue theorem. As some applications, we obtain the corresponding q-algebraic and q-combinatorial identities.

We state our main result in the following theorem.

Theorem 4 *If z is a complex variable, M is a nonnegative integer, n and λ are two positive integers such that $N = n\lambda$, then the following q-algebraic identity holds*:

$$\begin{aligned}\frac{z^M}{((z+1)q;q)_n^\lambda} &= (-1)^N q^{-\lambda\binom{n+1}{2}} \sum_{j=0}^{M-N} \frac{\mathbf{B}_{M-N-j}(x_1,\ldots,x_{M-N-j})}{(M-N-j)!} z^j \\ &\quad + \sum_{k=1}^{n} (-1)^{\lambda(k-1)} q^{\lambda\binom{k}{2}-kM} \frac{(1-q^k)^{\lambda+M}}{(q;q)_n^\lambda} \begin{bmatrix} n \\ k \end{bmatrix}_q^\lambda \sum_{j=0}^{\lambda-1} q^{-kj} \frac{\mathbf{B}_j(y_1,\ldots,y_j)}{j!(1-(z+1)q^k)^{\lambda-j}}, \end{aligned} \tag{7}$$

where

$$x_i = \lambda(i-1)!(1-q)^i \sum_{j=1}^{n} \left(q^{-j}[j]\right)^i, \quad i = 1,2,\ldots,M-N, \tag{8}$$

$$y_i = (i-1)! q^{ki} \left[\lambda\left(H_{k;q}^{(i)} + (-1)^i \tilde{H}_{n-k;q}^{(i)}\right) - \frac{\lambda+M}{(1-q^k)^i} \right], \quad i = 1,2,\ldots,\lambda-1. \tag{9}$$

Remark 5 Formula (7) is a q-analogue of formula (6).

2 *q*-Algebraic identities

In this section, we will provide the q-analogues of some algebraic identities.

When $M < N$, we obtain the following q-algebraic identity:

Corollary 6 *If z is a complex variable, M is a nonnegative integer, n and λ are two positive integers such that $N = n\lambda$, then the following q-algebraic identity holds*:

$$\begin{aligned}\frac{z^M}{((z+1)q;q)_n^\lambda} &= \sum_{k=1}^{n} (-1)^{\lambda(k-1)} q^{\lambda\binom{k}{2}-kM} \frac{(1-q^k)^{\lambda+M}}{(q;q)_n^\lambda} \begin{bmatrix} n \\ k \end{bmatrix}_q^\lambda \sum_{j=0}^{\lambda-1} q^{-kj} \frac{\mathbf{B}_j(y_1,\ldots,y_j)}{j!(1-(z+1)q^k)^{\lambda-j}}, \end{aligned} \tag{10}$$

where

$$y_i = (i-1)! q^{ki} \left[\lambda\left(H_{k;q}^{(i)} + (-1)^i \tilde{H}_{n-k;q}^{(i)}\right) - \frac{\lambda+M}{(1-q^k)^i} \right], \quad i = 1,2,\ldots,\lambda-1. \tag{11}$$

When $M \geq N$, we give the following new and interesting q-algebraic identities: for $M = N$, we have

Corollary 7 *Suppose that z is a complex variable, n and λ are two positive integers. Then the following q-algebraic identity holds*:

$$\begin{aligned}&\frac{z^{n\lambda}}{((z+1)q;q)_n^{\lambda}}\\&\quad=\frac{(-1)^{n\lambda}}{q^{\lambda\binom{n+1}{2}}}+\sum_{k=1}^{n}(-1)^{\lambda(k-1)}q^{\lambda\binom{k}{2}-kM}\frac{(1-q^k)^{\lambda+n\lambda}}{(q;q)_n^{\lambda}}\begin{bmatrix}n\\k\end{bmatrix}_q^{\lambda}\sum_{j=0}^{\lambda-1}q^{-kj}\frac{\mathbf{B}_j(y_1,\dots,y_j)}{j!(1-(z+1)q^k)^{\lambda-j}},\end{aligned}\tag{12}$$

where

$$y_i=\lambda(i-1)!q^{ki}\left[\left(H_{k;q}^{(i)}+(-1)^i\tilde{H}_{n-k;q}^{(i)}\right)-\frac{n+1}{(1-q^k)^i}\right],\quad i=1,2,\dots,\lambda-1.$$

For $M=N+1$, we have

Corollary 8 *Suppose that z is a complex variable, n and λ are two positive integers. Then the following q-algebraic identity holds*:

$$\begin{aligned}&\frac{z^{n\lambda+1}}{((z+1)q;q)_n^{\lambda}}\\&\quad=\frac{(-1)^{n\lambda}}{q^{\lambda\binom{n+1}{2}}}\left(z+\lambda\left(q^{-n}[n]-n\right)\right)\\&\qquad+\sum_{k=1}^{n}(-1)^{\lambda(k-1)}q^{\lambda\binom{k}{2}-kM}\frac{(1-q^k)^{\lambda+n\lambda+1}}{(q;q)_n^{\lambda}}\begin{bmatrix}n\\k\end{bmatrix}_q^{\lambda}\sum_{j=0}^{\lambda-1}q^{-kj}\frac{\mathbf{B}_j(y_1,\dots,y_j)}{j!(1-(z+1)q^k)^{\lambda-j}},\end{aligned}\tag{13}$$

where

$$y_i=(i-1)!q^{ki}\left[\lambda\left(H_{k;q}^{(i)}+(-1)^i\tilde{H}_{n-k;q}^{(i)}\right)-\frac{\lambda(n+1)+1}{(1-q^k)^i}\right],\quad i=1,2,\dots,\lambda-1.\tag{14}$$

For $M=N+2$, we have

Corollary 9 *Suppose that z is a complex variable, n and λ are two positive integers. Then the following q-algebraic identity holds*:

$$\begin{aligned}\frac{z^{n\lambda+2}}{((z+1)q;q)_n^{\lambda}}&=\frac{(-1)^{n\lambda}}{q^{\lambda\binom{n+1}{2}}}\Bigg[z^2+\lambda\left(q^{-n}[n]-n\right)z+\frac{1}{2}\left(\lambda\left(q^{-n}[n]-n\right)\right)^2\\&\quad+\lambda\left(\frac{[2n]}{(1+q)q^{2n}}-2q^{-n}[n]+3n\right)\Bigg]\\&\quad+\sum_{k=1}^{n}(-1)^{\lambda(k-1)}q^{\lambda\binom{k}{2}-kM}\frac{(1-q^k)^{\lambda(n+1)+2}}{(q;q)_n^{\lambda}}\begin{bmatrix}n\\k\end{bmatrix}_q^{\lambda}\\&\quad\times\sum_{j=0}^{\lambda-1}q^{-kj}\frac{\mathbf{B}_j(y_1,\dots,y_j)}{j!(1-(z+1)q^k)^{\lambda-j}},\end{aligned}\tag{15}$$

where

$$y_i=(i-1)!q^{ki}\left[\lambda\left(H_{k;q}^{(i)}+(-1)^i\tilde{H}_{n-k;q}^{(i)}\right)-\frac{\lambda(n+1)+2}{(1-q^k)^i}\right],\quad i=1,2,\dots,\lambda-1.\tag{16}$$

Taking $M = 0$ in (7), we deduce

Corollary 10 *Suppose that z is a complex variable, n and λ are two positive integers. Then the following q-algebraic identity holds*:

$$\frac{(q;q)_n^\lambda}{((z+1)q;q)_n^\lambda} = \sum_{k=1}^{n} (-1)^{\lambda(k-1)} q^{\lambda\binom{k}{2}} \left(1-q^k\right)^\lambda \begin{bmatrix} n \\ k \end{bmatrix}_q^\lambda \sum_{j=0}^{\lambda-1} q^{-kj} \frac{\mathbf{B}_j(y_1,\ldots,y_j)}{j!(1-(z+1)q^k)^{\lambda-j}}, \tag{17}$$

where

$$y_i = (i-1)!q^{ki}\left[\lambda\left(H_{k;q}^{(i)} + (-1)^i \tilde{H}_{n-k;q}^{(i)}\right) - \frac{\lambda}{(1-q^k)^i}\right], \quad i = 1,2,\ldots,\lambda-1. \tag{18}$$

Theorem 11 *If z is a complex variable, M is a nonnegative integer, n and λ are two positive integers such that $N = n\lambda$. Then the following q-algebraic identity holds*:

$$\begin{aligned} \frac{(q^z-1)^M}{(q^{z+1};q)_n^\lambda} &= (-1)^N q^{-\lambda\binom{n+1}{2}} \sum_{j=0}^{M-N} \frac{\mathbf{B}_{M-N-j}(x_1,\ldots,x_{M-N-j})}{(M-N-j)!} \left(q^z-1\right)^j \\ &\quad + \sum_{k=1}^{n} (-1)^{\lambda(k-1)} q^{\lambda\binom{k}{2}-kM} \frac{(1-q^k)^{\lambda+M}}{(q;q)_n^\lambda} \begin{bmatrix} n \\ k \end{bmatrix}_q^\lambda \sum_{j=0}^{\lambda-1} q^{-kj} \frac{\mathbf{B}_j(y_1,\ldots,y_j)}{j!(1-q^{z+k})^{\lambda-j}}, \end{aligned} \tag{19}$$

where

$$x_i = \lambda(i-1)!(1-q)^i \sum_{j=1}^{n} \left(q^{-j}[j]\right)^i, \quad i = 1,2,\ldots,M-N, \tag{20}$$

$$y_i = (i-1)!q^{ki}\left[\lambda\left(H_{k;q}^{(i)} + (-1)^i \tilde{H}_{n-k;q}^{(i)}\right) - \frac{\lambda+M}{(1-q^k)^i}\right], \quad i = 1,2,\ldots,\lambda-1. \tag{21}$$

Proof Letting $z \mapsto q^z - 1$ in Theorem 4, we obtain Theorem 11 immediately. □

Remark 12 Formula (19) is another q-analogue of Luo and Zhu result (6).

Theorem 13 *If z is a complex variable, M is a nonnegative integer, n and λ are two positive integers such that $N = \lambda(n+1)$, then the following q-algebraic identity holds*:

$$\begin{aligned} \frac{(1-q^z)^M}{(q^z;q)_{n+1}^\lambda} &= (-1)^{M-N-\lambda} q^{-\lambda\binom{n+1}{2}} \sum_{j=0}^{M-N-\lambda} \frac{\mathbf{B}_{M-N-\lambda-j}(x_1,\ldots,x_{M-N-\lambda-j})}{(M-N-\lambda-j)!} \left(q^z-1\right)^j \\ &\quad + \sum_{k=0}^{n} (-1)^{\lambda k+M} q^{\lambda\binom{k+1}{2}-kM} \frac{(1-q^k)^M}{(q;q)_n^\lambda} \begin{bmatrix} n \\ k \end{bmatrix}_q^\lambda \sum_{j=0}^{\lambda-1} q^{-kj} \frac{\mathbf{B}_j(y_1,\ldots,y_j)}{j!(1-q^{z+k})^{\lambda-j}}, \end{aligned} \tag{22}$$

where

$$x_i = \lambda(i-1)!(1-q)^i \sum_{j=1}^{n} \left(q^{-j}[j]\right)^i, \quad i = 1,2,\ldots,M-N-\lambda, \tag{23}$$

$$y_i = (i-1)!q^{ki}\left[\lambda\left(H_{k;q}^{(i)} + (-1)^i \tilde{H}_{n-k;q}^{(i)}\right) - \frac{M}{(1-q^k)^i}\right], \quad i = 1,2,\ldots,\lambda-1. \tag{24}$$

Proof Letting $z \mapsto q^z - 1$ and $M \mapsto M - \lambda$, noting that $\sum_{k=1}^{n} = \sum_{k=0}^{n}$ in Theorem 4, and applying the relation

$$\frac{(q^z-1)^M}{(q^{z+1};q)_n^\lambda} \equiv (-1)^M \frac{(1-q^z)^{M+\lambda}}{(q^z;q)_{n+1}^\lambda},$$

we obtain Theorem 13 immediately. □

When $M < N$, and noting that $\sum_{k=1}^{n} = \sum_{k=0}^{n}$ in Theorem 13, we have

Corollary 14 *If z is a complex variable, n and M are nonnegative integers, λ is a positive integer such that $N = \lambda(n+1)$, then the following q-algebraic identity holds*:

$$\frac{(1-q^z)^M}{(q^z;q)_{n+1}^\lambda} = \sum_{k=0}^{n} (-1)^{\lambda k+M} q^{\lambda\binom{k+1}{2}-kM} \frac{(1-q^k)^M}{(q;q)_n^\lambda} \begin{bmatrix} n \\ k \end{bmatrix}_q^\lambda \sum_{j=0}^{\lambda-1} q^{-kj} \frac{\mathbf{B}_j(y_1,\dots,y_j)}{j!(1-q^{z+k})^{\lambda-j}}, \tag{25}$$

where

$$y_i = (i-1)! q^{ki} \left[\lambda \left(H_{k;q}^{(i)} + (-1)^i \tilde{H}_{n-k;q}^{(i)} \right) - \frac{M}{(1-q^k)^i} \right], \quad i = 1,2,\dots,\lambda-1. \tag{26}$$

Remark 15 The algebraic identity (25) is just a q-analogue of Chu's result (5).

Taking $M = 0$ in Corollary 14, we have

Corollary 16 *If z is a complex variable, n and λ are two positive integers, then the following q-algebraic identity holds*:

$$\frac{(q;q)_n^\lambda}{(q^z;q)_{n+1}^\lambda} = \sum_{k=0}^{n} (-1)^{\lambda k} q^{\lambda\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q^\lambda \sum_{j=0}^{\lambda-1} q^{-kj} \frac{\mathbf{B}_j(y_1,\dots,y_j)}{j!(1-q^{z+k})^{\lambda-j}}, \tag{27}$$

where

$$y_i = \lambda (i-1)! q^{ki} \left(H_{k;q}^{(i)} + (-1)^i \tilde{H}_{n-k;q}^{(i)} \right), \quad i = 1,2,\dots,\lambda-1. \tag{28}$$

Remark 17 The algebraic identity (27) is just a q-analogue of Chu's result (4).

Taking $\lambda = 1$ in (27), we have

$$\prod_{k=1}^{n} \frac{1-q^k}{1-q^{z+k}} = \sum_{k=0}^{n} (-1)^k q^{\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{1-q^z}{1-q^{z+k}}, \tag{29}$$

which is a q-analogue of the well-known binomial identity (e.g., see [14]):

$$\prod_{k=1}^{n} \frac{k}{z+k} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{z}{z+k}.$$

Taking $M = 0$, $n \mapsto n+1, z \mapsto z-1$, and then letting $z \mapsto zq^{-1}$ in Theorem 4, we obtain the following corollary.

Corollary 18 *If z is a complex variable, n and λ are two positive integers such that $N = \lambda(n+1)$, then the following q-algebraic identity holds*:

$$\frac{(q;q)_n^\lambda}{(z;q)_{n+1}^\lambda} = \sum_{k=0}^{n}(-1)^{\lambda k} q^{\lambda\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q^\lambda \sum_{j=0}^{\lambda-1} q^{-(k+1)j} \frac{\mathbf{B}_j(y_1,\ldots,y_j)}{j!(1-zq^k)^{\lambda-j}}, \tag{30}$$

where

$$y_i = \lambda(i-1)! q^{(k+1)i}\bigl(H_{k;q}^{(i)} + (-1)^i \tilde{H}_{n-k;q}^{(i)}\bigr), \quad i = 1,2,\ldots,\lambda-1. \tag{31}$$

Remark 19 The algebraic identity (30) is just another q-analogue of Chu's result (4).

Taking $\lambda = 1$ and letting $M \mapsto m$ in Theorem 4, we have

Corollary 20 *If z is a complex variable, m is a nonnegative integer, and n is a positive integer, then the following q-algebraic identity holds*:

$$\begin{aligned} \frac{z^m}{((z+1)q;q)_n} &= (-1)^n q^{-\binom{n+1}{2}} \sum_{j=0}^{m-n} \frac{\mathbf{B}_{m-n-j}(x_1,\ldots,x_{m-n-j})}{(m-n-j)!} z^j \\ &\quad + \sum_{k=1}^{n}(-1)^{k-1} q^{\binom{k}{2}-km} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(1-q^k)^{m+1}}{(q;q)_n} \frac{1}{1-(z+1)q^k}, \end{aligned} \tag{32}$$

where

$$x_i = \lambda(i-1)!(1-q)^i \sum_{j=1}^{n} \bigl(q^{-j}[j]\bigr)^i, \quad i = 1,2,\ldots,m-n. \tag{33}$$

Taking $m = 0$ in (32), we obtain the following q-algebraic identity:

$$\frac{(q;q)_n}{((z+1)q;q)_n} = \sum_{k=1}^{n}(-1)^{k-1} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{1-q^k}{1-(z+1)q^k}. \tag{34}$$

3 Further q-combinatorial identities

In this section, we will give the q-analogues of some combinatorial identities of Chu.

Taking $z = 0$ in (7), we obtain the following q-combinatorial identities involving q-harmonic numbers.

Corollary 21 *If M is a nonnegative integers, n and λ are two positive integers such that $N = n\lambda$, then we have the following q-combinatorial identities*:

$$\begin{aligned} &\sum_{k=1}^{n}(-1)^{\lambda k} q^{\lambda\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q^\lambda \sum_{j=0}^{\lambda-1} \frac{q^{-k(M+j)}(1-q^k)^{M+j}\mathbf{B}_j(y_1,y_2,\ldots,y_j)}{j!} \\ &\quad = \begin{cases} (-1)^\lambda, & M = 0, \\ 0, & 1 \le M < N, \\ (-1)^{N-\lambda+1} q^{-\lambda\binom{n+1}{2}} (q;q)_n^\lambda \frac{\mathbf{B}_{M-N}(x_1,\ldots,x_{M-N})}{(M-N)!}, & M \ge N, \end{cases} \end{aligned} \tag{35}$$

where

$$x_i = \lambda(i-1)!(1-q)^i \sum_{j=1}^{n} \left(q^{-j}[j]\right)^i, \quad i = 1,2,\ldots,M-N, \tag{36}$$

$$y_i = (i-1)!q^{ki}\left[\lambda\left(H_{k;q}^{(i)} + (-1)^i \tilde{H}_{n-k;q}^{(i)}\right) - \frac{\lambda+M}{(1-q^k)^i}\right], \quad i = 1,2,\ldots,\lambda-1. \tag{37}$$

Taking $z = 1$ in (7), we obtain the following q-combinatorial identities involving q-harmonic numbers:

$$\begin{aligned}&\sum_{k=1}^{n}(-1)^{\lambda(k-1)}q^{\lambda\binom{k}{2}-kM}\frac{(1-q^k)^{\lambda+M}}{(q;q)_n^{\lambda}}\begin{bmatrix}n\\k\end{bmatrix}_q^{\lambda}\sum_{j=0}^{\lambda-1}q^{-kj}\frac{\mathbf{B}_j(y_1,\ldots,y_j)}{j!(1-2q^k)^{\lambda-j}}\\&\quad=\frac{1}{(2q;q)_n^{\lambda}}+(-1)^{N-1}q^{-\lambda\binom{n+1}{2}}\sum_{j=0}^{M-N}\frac{\mathbf{B}_{M-N-j}(x_1,\ldots,x_{M-N-j})}{(M-N-j)!}.\end{aligned} \tag{38}$$

Taking $z = -1$ in (7), we obtain the following q-combinatorial identities involving q-harmonic numbers:

$$\begin{aligned}&\sum_{k=1}^{n}(-1)^{\lambda k}q^{\lambda\binom{k}{2}-kM}\frac{(1-q^k)^{\lambda+M}}{(q;q)_n^{\lambda}}\begin{bmatrix}n\\k\end{bmatrix}_q^{\lambda}\sum_{j=0}^{\lambda-1}\frac{q^{-kj}\mathbf{B}_j(y_1,\ldots,y_j)}{j!}\\&\quad=(-1)^{M+\lambda}+(-1)^{N+\lambda-1}q^{-\lambda\binom{n+1}{2}}\sum_{j=0}^{M-N}\frac{\mathbf{B}_{M-N-j}(x_1,\ldots,x_{M-N-j})}{(M-N-j)!}(-1)^j.\end{aligned} \tag{39}$$

Taking $z = 1$ in (22), we obtain the following q-combinatorial identities involving q-harmonic numbers:

$$\begin{aligned}&\sum_{k=1}^{n}(-1)^{\lambda k+M}q^{\lambda\binom{k}{2}-kM}\frac{(1-q^k)^{M}}{(q;q)_n^{\lambda}}\begin{bmatrix}n\\k\end{bmatrix}_q^{\lambda}\sum_{j=0}^{\lambda-1}q^{-kj}\frac{\mathbf{B}_j(y_1,\ldots,y_j)}{j!(1-q^{1+k})^{\lambda-j}}\\&\quad=\frac{(1-q)^M}{(q;q)_{n+1}^{\lambda}}-(-1)^{M-N-\lambda}q^{-\lambda\binom{n+1}{2}}\sum_{j=0}^{M-N-\lambda}\frac{\mathbf{B}_{M-N-\lambda-j}(x_1,\ldots,x_{M-N-\lambda-j})}{(M-N-\lambda-j)!}(q-1)^j.\end{aligned} \tag{40}$$

Multiplying both sides of (10) by z and then letting $z \to \infty$, we immediately establish the following combinatorial identities:

Corollary 22 *If M is a nonnegative integer, n and λ are two positive integers such that $N = n\lambda$, then we have the following q-combinatorial identities*:

$$\begin{aligned}&\sum_{k=1}^{n}(-1)^{\lambda k}q^{\lambda\binom{k}{2}-kM-k}\left(1-q^k\right)^{\lambda+M}\begin{bmatrix}n\\k\end{bmatrix}_q^{\lambda}\frac{\mathbf{B}_{\lambda-1}(y_1,y_2,\ldots,y_{\lambda-1})}{(\lambda-1)!}\\&\quad=\begin{cases}0, & 0\le M<N-1,\\(-1)^{\lambda+M}\frac{(q;q)_n^{\lambda}}{q^{\lambda\binom{n+1}{2}}}, & M=N-1,\end{cases}\end{aligned} \tag{41}$$

where

$$y_i = (i-1)!q^{ki}\left[\lambda\left(H_{k;q}^{(i)} + (-1)^i \tilde{H}_{n-k;q}^{(i)}\right) - \frac{\lambda + M}{(1-q^k)^i}\right], \quad i = 1,2,\ldots,\lambda-1.$$

Noting that $\sum_{k=1}^{n} = \sum_{k=0}^{n}$ in (41), we say that the above q-combinatorial identities are the corresponding q-analogues of Chu's result [3, Corollary 7]:

$$\sum_{k=0}^{n}(-1)^{\lambda k}k^{\lambda+M}\binom{n}{k}^{\lambda}\frac{\mathbf{B}_{\lambda-1}(y_1,y_2,\ldots,y_{\lambda-1})}{(\lambda-1)!} = \begin{cases} 0, & 0 \le M < N-1, \\ (-1)^{\lambda+M}(n!)^{\lambda}, & M = N-1, \end{cases} \tag{42}$$

where

$$y_i = (i-1)!\left[\lambda\left(H_k^{(i)} + (-1)^i H_{n-k}^{(i)}\right) - \frac{\lambda+M}{k^i}\right], \quad i = 1,2,\ldots,\lambda-1.$$

Taking $\lambda = 1,2,3,4$ in (41), respectively, we obtain the following q-combinatorial identities involving q-harmonic numbers:

$$\sum_{k=0}^{n}(-1)^k q^{\frac{(k^2-3k)}{2}-kM}\left(1-q^k\right)^{1+M}\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} 0, & 0 \le M < n-1, \\ (-1)^n \frac{(q;q)_n}{q^{\binom{n+1}{2}}}, & M = n-1, \end{cases} \tag{43}$$

$$\begin{aligned}
&\sum_{k=0}^{n} q^{k(k-2-M)}\left(1-q^k\right)^{M+2}\begin{bmatrix} n \\ k \end{bmatrix}_q^2\left[2(H_{k;q} - \tilde{H}_{n-k;q}) - \frac{M+2}{1-q^k}\right] \\
&\quad = \begin{cases} 0, & 0 \le M < 2n-1, \\ \frac{(q;q)_n^2}{q^{2\binom{n+1}{2}}}, & M = 2n-1, \end{cases}
\end{aligned} \tag{44}$$

$$\begin{aligned}
&\sum_{k=0}^{n}(-1)^k q^{\frac{3(k^2-3k)}{2}-kM}\frac{(1-q^k)^{M+3}}{2!}\begin{bmatrix} n \\ k \end{bmatrix}_q^3\left\{\left[3(H_{k;q} - \tilde{H}_{n-k;q}) - \frac{M+3}{(1-q^k)}\right]^2\right. \\
&\quad \left. + \left[3\left(H_{k;q}^{(2)} + \tilde{H}_{n-k;q}^{(2)}\right) - \frac{M+3}{(1-q^k)^2}\right]\right\} \\
&\quad = \begin{cases} 0, & 0 \le M < N-1, \\ (-1)^n \frac{(q;q)_n^3}{q^{3\binom{n+1}{2}}}, & M = N-1, \end{cases}
\end{aligned} \tag{45}$$

$$\begin{aligned}
&\sum_{k=1}^{n} q^{k(2k-3-M)}\frac{(1-q^k)^{M+4}}{3!}\begin{bmatrix} n \\ k \end{bmatrix}_q^4\left[\left[4(H_{k;q} - \tilde{H}_{n-k;q}) - \frac{M+4}{(1-q^k)}\right]^3\right. \\
&\quad + 3\left[4(H_{k;q} - \tilde{H}_{n-k;q}) - \frac{M+4}{(1-q^k)}\right] \\
&\quad \left. \times \left[4\left(H_{k;q}^{(2)} + \tilde{H}_{n-k;q}^{(2)}\right) - \frac{M+4}{(1-q^k)^2}\right] + 2!\left[4\left(H_{k;q}^{(3)} - \tilde{H}_{n-k;q}^{(3)}\right) - \frac{M+4}{(1-q^k)^3}\right]\right] \\
&\quad = \begin{cases} 0, & 0 \le M < N-1, \\ \frac{(q;q)_n^4}{q^{4\binom{n+1}{2}}}, & M = N-1. \end{cases}
\end{aligned} \tag{46}$$

Taking $z = 0$ in (30), we obtain the following q-combinatorial identity:

$$\sum_{k=0}^{n}(-1)^{\lambda k}q^{\lambda\binom{k+1}{2}}\begin{bmatrix}n\\k\end{bmatrix}_q^{\lambda}\sum_{j=0}^{\lambda-1}q^{-(k+1)j}\frac{\mathbf{B}_j(y_1,\ldots,y_j)}{j!} = (q;q)_n^{\lambda}, \tag{47}$$

where y_i is given by (31).

Taking $z = -1$ in (30), we have

$$\sum_{k=0}^{n}(-1)^{\lambda k}q^{\lambda\binom{k+1}{2}}\begin{bmatrix}n\\k\end{bmatrix}_q^{\lambda}\sum_{j=0}^{\lambda-1}q^{-(k+1)j}\frac{\mathbf{B}_j(y_1,\ldots,y_j)}{j!(1+q^k)^{\lambda-j}} = \frac{(q;q)_n^{\lambda}}{[2(1+q)(1+q^2)\cdots(1+q^n)]^{\lambda}}. \tag{48}$$

Taking $z = q$ in (30), we have

$$\sum_{k=0}^{n}(-1)^{\lambda k}q^{\lambda\binom{k+1}{2}}\begin{bmatrix}n\\k\end{bmatrix}_q^{\lambda}\sum_{j=0}^{\lambda-1}q^{-(k+1)j}\frac{\mathbf{B}_j(y_1,\ldots,y_j)}{j!(1-q^{k+1})^{\lambda-j}} = \frac{1}{(1-q^{n+1})^{\lambda}}. \tag{49}$$

Taking $z = -q$ in (30), we have

$$\sum_{k=0}^{n}(-1)^{\lambda k}q^{\lambda\binom{k+1}{2}}\begin{bmatrix}n\\k\end{bmatrix}_q^{\lambda}\sum_{j=0}^{\lambda-1}q^{-(k+1)j}\frac{\mathbf{B}_j(y_1,\ldots,y_j)}{j!(1+q^{k+1})^{\lambda-j}} = \frac{(q;q)_n^{\lambda}}{(-q;q)_{n+1}^{\lambda}}. \tag{50}$$

Multiplying both sides of (30) by z and then letting $z \to \infty$, we establish the following q-combinatorial identities:

Corollary 23 *Suppose that λ and n are positive integers. We have the following q-combinatorial identity*:

$$\sum_{k=0}^{n}(-1)^{\lambda k}q^{\lambda\binom{k+1}{2}}\begin{bmatrix}n\\k\end{bmatrix}_q^{\lambda}\mathbf{B}_{\lambda-1}(y_1,y_2,\ldots,y_{\lambda-1}) = 0, \tag{51}$$

where y_i are given by (31).

Remark 24 The combinatorial identity (51) is a q-analogue of Chu's result [3, Corollary 3]:

$$\sum_{k=0}^{n}(-1)^{\lambda k}\binom{n}{k}^{\lambda}\mathbf{B}_{\lambda-1}(y_1,y_2,\ldots,y_{\lambda-1}) = 0, \tag{52}$$

where

$$y_i = \lambda(i-1)!\big(H_k^{(i)} + (-1)^i H_{n-k}^{(i)}\big), \quad i = 1,2,\ldots,\lambda-1.$$

Taking $\lambda = 1$ in (51), we have

$$\sum_{k=0}^{n}(-1)^k q^{-k}\begin{bmatrix}n\\k\end{bmatrix}_q = 0,$$

which is a q-analogue of the well-known identity

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0.$$

4 Proof of Theorem 4

Lemma 25 *If z is a complex variable, M is a nonnegative integer, n and λ are two positive integers such that $N = n\lambda$, then the following q-algebraic identity holds*:

$$\begin{aligned}\frac{z^M}{((z+1)q;q)_n^\lambda} &= (-1)^N q^{-\lambda\binom{n+1}{2}}\frac{\mathbf{B}_{M-N}(z_1,z_2,\ldots,z_{M-N})}{(M-N)!}\\ &\quad+\sum_{k=1}^{n}(-1)^{\lambda(k-1)-1}q^{\frac{\lambda(k^2-3k)}{2}-kM}\frac{(1-q^k)^{\lambda+M}}{(q;q)_n^\lambda}\begin{bmatrix}n\\k\end{bmatrix}_q^\lambda\frac{\mathbf{B}_{\lambda-1}(y_1,y_2,\ldots,y_{\lambda-1})}{(\lambda-1)!(z-q^{-k}+1)},\end{aligned}\tag{53}$$

where

$$y_s = (s-1)!\left(\sum_{j=1,j\neq k}^{n}\frac{\lambda}{(q^{-k}-q^{-j})^s}-\frac{M}{(q^{-k}-1)^s}+\frac{(-1)^s}{(z-q^{-k}+1)^s}\right),\quad s=1,2,\ldots,\lambda-1,\tag{54}$$

$$z_s = (s-1)!\left(\sum_{j=1}^{n}\lambda\left(q^{-j}-1\right)^s+z^s\right),\quad s=1,2,\ldots,M-N.\tag{55}$$

Proof We first construct two polynomials $P(z)$ and $Q(z)$ of degree M and $N+1$, respectively, which are given by

$$P(z) = z^M \quad\text{and}\quad Q(z) = (z-\alpha)\prod_{j=1}^{n}\left(z-q^{-j}+1\right)^\lambda,$$

such that $\alpha \neq q^{-1}-1, q^{-2}-1,\ldots,q^{-n}-1$.

We next construct three contour integrals for the rational functions $P(z)/Q(z)$:

- $\oint_\Gamma \frac{P(z)}{Q(z)}\,\mathrm{d}z$, where Γ is a simple closed contour which only surrounds the single pole α of $P(z)/Q(z)$;
- $\oint_{\Gamma'} \frac{P(z)}{Q(z)}\,\mathrm{d}z$, where Γ' is a simple closed contour which only surrounds the poles $q^{-1}-1, q^{-2}-1,\ldots,q^{-n}-1$ of $P(z)/Q(z)$;
- $\oint_{\Gamma''} \frac{P(z)}{Q(z)}\,\mathrm{d}z$, where Γ'' is a simple closed contour which only surrounds the pole ∞ of $P(z)/Q(z)$.

In the extended complex plane, since the total sum of residues of a rational function at all finite poles and that at infinity is equal to zero [13, Theorem 2], we have

$$\oint_{\Gamma+\Gamma'+\Gamma''}\frac{P(z)}{Q(z)}\,\mathrm{d}z = 0,$$

or equivalently,

$$\oint_{\Gamma}\frac{P(z)}{Q(z)}\,\mathrm{d}z = -\oint_{\Gamma''}\frac{P(z)}{Q(z)}\,\mathrm{d}z-\oint_{\Gamma'}\frac{P(z)}{Q(z)}\,\mathrm{d}z.\tag{56}$$

Below we compute the contour integrals $\oint_{\Gamma} \frac{P(z)}{Q(z)}\,\mathrm{d}z, \oint_{\Gamma'} \frac{P(z)}{Q(z)}\,\mathrm{d}z$, and $\oint_{\Gamma''} \frac{P(z)}{Q(z)}\,\mathrm{d}z$, respectively.

We compute the contour integral $\oint_{\Gamma} \frac{P(z)}{Q(z)}\,\mathrm{d}z$ as follows:

$$\begin{aligned}\oint_{\Gamma} \frac{P(z)}{Q(z)}\,\mathrm{d}z &= 2\pi i \operatorname*{Res}_{z=\alpha} \frac{P(z)}{Q(z)} \\ &= 2\pi i \operatorname*{Res}_{z=\alpha} \frac{z^M}{(z-\alpha)\prod_{j=1}^{n}(z-q^{-j}+1)^{\lambda}} \\ &= 2\pi i \frac{\alpha^M}{\prod_{j=1}^{n}(\alpha-q^{-j}+1)^{\lambda}} \\ &= 2\pi i (-1)^N q^{\lambda\binom{n+1}{2}} \frac{\alpha^M}{((\alpha+1)q;q)_n^{\lambda}}. \end{aligned} \tag{57}$$

We calculate the contour integral $\oint_{\Gamma''} \frac{P(z)}{Q(z)}\,\mathrm{d}z$ as follows:

$$\begin{aligned}\oint_{\Gamma''} \frac{P(z)}{Q(z)}\,\mathrm{d}z &= 2\pi i \operatorname*{Res}_{z=\infty} \frac{P(z)}{Q(z)} = -2\pi i \operatorname*{Res}_{t=0} \frac{1}{t^2} \frac{P(\frac{1}{t})}{Q(\frac{1}{t})} \\ &= -2\pi i \operatorname*{Res}_{t=0} \frac{t^{N-M-1}}{(1-t\alpha)\prod_{j=1}^{n}(1-t(q^{-j}-1))^{\lambda}}. \end{aligned}$$

If $M-N<0$, then $t=0$ is not a pole, and so we have

$$\oint_{\Gamma''} \frac{P(z)}{Q(z)}\,\mathrm{d}z = -2\pi i \operatorname*{Res}_{t=0} \frac{t^{N-M-1}}{(1-t\alpha)\prod_{j=1}^{n}(1-t(q^{-j}-1))^{\lambda}} = 0.$$

If $M-N=0$, then $t=0$ is a single pole of order 1, so we have

$$\begin{aligned}\oint_{\Gamma''} \frac{P(z)}{Q(z)}\,\mathrm{d}z &= = -2\pi i \operatorname*{Res}_{t=0} \frac{1}{t} \frac{1}{(1-t\alpha)\prod_{j=1}^{n}(1-t(q^{-j}-1))^{\lambda}} \\ &= -2\pi i \lim_{t\to 0} \frac{1}{(1-t\alpha)\prod_{j=1}^{n}(1-t(q^{-j}-1))^{\lambda}} \\ &= -2\pi i. \end{aligned}$$

If $M-N>0$, then $t=0$ is a single pole of order $M-N$. By utilizing Cauchy's residue theorem, noting that the power series expansion of the logarithmic function is

$$\log(1+z) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n} \bigl(|z|<1\bigr)$$

and using the definition of complete Bell polynomials, we obtain

$$\begin{aligned}\oint_{\Gamma''} \frac{P(z)}{Q(z)}\,\mathrm{d}z &= -2\pi i \operatorname*{Res}_{t=0} \frac{t^{N-M-1}}{(1-t\alpha)\prod_{j=1}^{n}(1-t(q^{-j}-1))^{\lambda}} \\ &= -2\pi i \bigl[t^{M-N}\bigr] \frac{1}{(1-t\alpha)\prod_{j=1}^{n}(1-t(q^{-j}-1))^{\lambda}} \end{aligned}$$

$$
\begin{aligned}
&= -2\pi i\left[t^{M-N}\right]\exp\left\{\sum_{k=1}^{\infty}\left[(k-1)!\left(\alpha^k+\sum_{j=1}^{n}\lambda\left(q^{-j}-1\right)^k\right)\frac{t^k}{k!}\right]\right\}\\
&= -2\pi i\left[t^{M-N}\right]\sum_{k=0}^{\infty}\mathbf{B}_k(z_1,z_2,\dots,z_k)\frac{t^k}{k!}\\
&= -\frac{2\pi i}{(M-N)!}\mathbf{B}_{M-N}(z_1,z_2,\dots,z_{M-N}).
\end{aligned}
$$

We get

$$
\oint_{\Gamma''}\frac{P(z)}{Q(z)}\,\mathrm{d}z = -\frac{2\pi i}{(M-N)!}\mathbf{B}_{M-N}(z_1,z_2,\dots,z_{M-N}). \tag{58}
$$

We now calculate the contour integral $\oint_{\Gamma'}\frac{P(z)}{Q(z)}\,\mathrm{d}z$. By utilizing Cauchy's residue theorem, noting that the power series expansion of the logarithmic function is

$$
\log(1+z)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{z^n}{n}\quad\left(|z|<1\right)
$$

and using the definition of complete Bell polynomials, we obtain

$$
\begin{aligned}
\oint_{\Gamma'}\frac{P(z)}{Q(z)}\,\mathrm{d}z &= 2\pi i\sum_{k=1}^{n}\operatorname*{Res}_{z=q^{-k}-1}\frac{P(z)}{Q(z)}\\
&= 2\pi i\sum_{k=1}^{n}\operatorname*{Res}_{z=q^{-k}-1}\frac{z^M}{(z-\alpha)\prod_{j=1}^{n}(z-q^{-j}+1)^{\lambda}}\\
&= 2\pi i\sum_{k=1}^{n}\left[\left(z-q^{-k}-1\right)^{\lambda-1}\right]\frac{z^M}{(z-\alpha)\prod_{\substack{j=1\\ j\neq k}}^{n}(z-q^{-j}+1)^{\lambda}}\\
&= 2\pi i\sum_{k=1}^{n}\left[z^{\lambda-1}\right]\frac{(z+q^{-k}-1)^M}{(z+q^{-k}-1-\alpha)\prod_{\substack{j=1\\ j\neq k}}^{n}(z+q^{-k}-q^{-j})^{\lambda}}\\
&= 2\pi i\sum_{k=1}^{n}\left[\frac{(q^{-k}-1)^M}{(q^{-k}-1-\alpha)\prod_{\substack{j=1\\ j\neq k}}^{n}(q^{-k}-q^{-j})^{\lambda}}\right.\\
&\quad\times\left[z^{\lambda-1}\right]\exp\left(M\log\left(1+\frac{z}{q^{-k}-1}\right)-\log\left(1+\frac{z}{q^{-k}-q^{-j}}\right)\right.\\
&\quad\left.\left.-\sum_{j=1,j\neq k}^{n}\lambda\log\left(1+\frac{z}{q^{-k}-q^{-j}}\right)\right)\right]\\
&= 2\pi i\sum_{k=1}^{n}\left\{\frac{(q^{-k}-1)^M}{(q^{-k}-1-\alpha)\prod_{\substack{j=1\\ j\neq k}}^{n}(q^{-k}-q^{-j})^{\lambda}}\right.\\
&\quad\times\left[z^{\lambda-1}\right]\exp\left[\sum_{s=1}^{\infty}(-1)^s(s-1)!\right.\\
&\quad\left.\left.\times\left(\sum_{j=1,j\neq k}^{n}\frac{\lambda}{(q^{-k}-q^{-j})^s}-\frac{M}{(q^{-k}-1)^s}+\frac{(-1)^s}{(\alpha-q^{-k}+1)^s}\right)\right]\frac{z^s}{s!}\right\}.
\end{aligned}
$$

Noting that

$$\mathbf{B}_n\big(-x_1,(-1)^2x_2,\ldots,(-1)^nx_n\big)=(-1)^n\mathbf{B}_n(x_1,x_2,\ldots,x_n),$$

it follows that

$$\oint_{\Gamma'}\frac{P(z)}{Q(z)}\,\mathrm{d}z=2\pi i\sum_{k=1}^{n}\frac{(-1)^{\lambda}(q^{-k}-1)^{M}\mathbf{B}_{\lambda-1}(y_1,y_2,\ldots,y_{\lambda-1})}{(\lambda-1)!(\alpha-q^{-k}+1)\prod_{\substack{j=1\\ j\neq k}}^{n}(q^{-k}-q^{-j})^{\lambda}}. \tag{59}$$

Therefore, by substituting (57)–(59) into (56), and then changing $\alpha\mapsto z$, we obtain Lemma 25. This proof is complete. □

Lemma 26 *The following recursion formula of complete Bell polynomial holds true*:

$$\frac{\mathbf{B}_{\lambda-1}(y_1,\ldots,y_{\lambda-1})}{(\lambda-1)!}=\sum_{j=0}^{\lambda-1}(-1)^{\lambda-j-1}\frac{\mathbf{B}_j(w_1,\ldots,w_j)}{j!(z-q^{-k}+1)^{\lambda-j-1}}, \tag{60}$$

where

$$w_s=(s-1)!\left[\sum_{j=1,j\neq k}^{n}\frac{\lambda}{(q^{-k}-q^{-j})^s}-\frac{M}{(q^{-k}-1)^s}\right].$$

Proof Write $y_s=w_s+(-1)^s\frac{(s-1)!}{(z-q^{-k}+1)^s}$ in (54). From the definition of a complete Bell polynomial, we obtain

$$\begin{aligned}
\frac{\mathbf{B}_{\lambda-1}(y_1,\ldots,y_{\lambda-1})}{(\lambda-1)!}&=\left[t^{\lambda-1}\right]\sum_{n=0}^{\infty}\mathbf{B}_n(y_1,y_2,\ldots,y_n)\frac{t^n}{n!}=\left[t^{\lambda-1}\right]\exp\left(\sum_{n=1}^{\infty}y_n\frac{t^n}{n!}\right)\\
&=\left[t^{\lambda-1}\right]\exp\left\{\sum_{n=1}^{\infty}\left(w_n+(-1)^n\frac{(n-1)!}{(z-q^{-k}+1)^n}\right)\frac{t^n}{n!}\right\}\\
&=\left[t^{\lambda-1}\right]\exp\left\{\sum_{n=1}^{\infty}w_n\frac{t^n}{n!}\right\}\exp\left\{\sum_{n=1}^{\infty}\frac{(-1)^n(n-1)!}{(z-q^{-k}+1)^n}\frac{t^n}{n!}\right\}\\
&=\sum_{j=0}^{\lambda-1}\left[t^j\right]\exp\left\{\sum_{n=1}^{\infty}w_n\frac{t^n}{n!}\right\}\left[t^{\lambda-1-j}\right]\exp\left\{\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left(\frac{t}{z-q^{-k}+1}\right)^n\right\}\\
&=\sum_{j=0}^{\lambda-1}\frac{\mathbf{B}_j(w_1,\ldots,w_j)}{j!}\left[t^{\lambda-1-j}\right]\exp\left\{-\log\left(1+\frac{t}{z-q^{-k}+1}\right)\right\}\\
&=\sum_{j=0}^{\lambda-1}\frac{\mathbf{B}_j(w_1,\ldots,w_j)}{j!}\left[t^{\lambda-1-j}\right]\sum_{n=0}^{\infty}(-1)^n\left(\frac{t}{z-q^{-k}+1}\right)^n\\
&=\sum_{j=0}^{\lambda-1}(-1)^{\lambda-j-1}\frac{\mathbf{B}_j(w_1,\ldots,w_j)}{j!(z-q^{-k}+1)^{\lambda-j-1}}.
\end{aligned}$$

In the above process, we apply the binomial theorem:

$$(1-z)^{-r} = \sum_{n=0}^{\infty} \binom{r+n-1}{r-1} z^n \quad (|z| < 1). \qquad \square$$

Lemma 27 *The following recursion formula of complete Bell polynomial holds true*:

$$\frac{\mathbf{B}_{M-N}(z_1, z_2, \dots, z_{M-N})}{(M-N)!} = \sum_{j=0}^{M-N} \frac{\mathbf{B}_{M-N-j}(v_1, \dots, v_{M-N-j})}{(M-N-j)!} z^j, \tag{61}$$

where

$$v_s = (s-1)! \sum_{j=1}^{n} \lambda \left(q^{-j} - 1\right)^s.$$

Proof Write $z_s = v_s + (s-1)!z^s$ in (55). Using the definition of a complete Bell polynomial, we obtain

$$\begin{aligned}
\frac{\mathbf{B}_{M-N}(z_1, z_2, \dots, z_{M-N})}{(M-N)!} &= \left[t^{M-N}\right] \sum_{n=0}^{\infty} \mathbf{B}_n(z_1, z_2, \dots, z_n) \frac{t^n}{n!} = \left[t^{M-N}\right] \exp\left(\sum_{n=1}^{\infty} z_n \frac{t^n}{n!}\right) \\
&= \left[t^{M-N}\right] \exp\left\{\sum_{n=1}^{\infty} \left(v_n + (n-1)!z^n\right) \frac{t^n}{n!}\right\} \\
&= \sum_{j=0}^{M-N} \left[t^{M-N-j}\right] \exp\left\{\sum_{n=1}^{\infty} v_n \frac{t^n}{n!}\right\} \left[t^j\right] \exp\left\{\sum_{n=1}^{\infty} z^n \frac{t^n}{n}\right\} \\
&= \sum_{j=0}^{M-N} \frac{\mathbf{B}_{M-N-j}(v_1, \dots, v_{M-N-j})}{(M-N-j)!} \left[t^j\right] \exp\left\{-\log(1-zt)\right\} \\
&= \sum_{j=0}^{M-N} \frac{\mathbf{B}_{M-N-j}(v_1, \dots, v_{M-N-j})}{(M-N-j)!} \left[t^j\right] \sum_{n=0}^{\infty} (zt)^n \\
&= \sum_{j=0}^{M-N} \frac{\mathbf{B}_{M-N-j}(v_1, \dots, v_{M-N-j})}{(M-N-j)!} z^j.
\end{aligned}$$

In the above process, we apply the binomial theorem:

$$(1-z)^{-r} = \sum_{n=0}^{\infty} \binom{r+n-1}{r-1} z^n \quad (|z| < 1). \qquad \square$$

Proof Proof of Theorem 4 Substituting (60) and (61) into (53), and then changing $w_j \mapsto y_j$ and $v_j \mapsto x_j$, we obtain Theorem 4. This proof is complete. □

5 Conclusions

As it is well known, the basic (or q-) series and basic (or q-) polynomials, especially the basic (or q-) hypergeometric functions and basic (or q-)hypergeometric polynomials, have widespread applications, particularly in several areas of number theory and combinatorial analysis such as the theory of partitions. In [19, p. 340], professor Srivastava points out an important demonstrated observation that any (p,q)-variations of the proposed q-results would be trivially inconsequential because the additional parameter p is obviously redundant.

In the last section (see [24, Conclusions]), Zhu and Luo suggested an open problem which would yield the corresponding basic (or q-) extensions of Theorem 3 (see [24, Theorem 1]). In the present paper, we here have answered this question applying the contour integral and Cauchy's residue theorem and given a q-explicit analogue by decomposing the *general* rational function $\frac{x^M}{(x+1)_n^\lambda}$.

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