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Wong-Zakai approximations and random attractors for nonlocal stochastic Schrödinger lattice systems in weighted spaces

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Abstract

This paper deals with the long-term behavior of nonlocal Schrödinger lattice systems with multiplicative white noise and their Wong-Zakai approximate systems in weighted spaces. We first prove the existence and uniqueness of tempered pullback attractors for the original stochastic systems and the Wong-Zakai approximations. Then, we establish the upper semicontinuity of these attractors for the Wong-Zakai approximate systems as the step length of the Wiener shift approaches zero.

Mathematics Subject Classification: 37L30; 37L55; 37L60

Keywords: Wong-Zakai approximation; Schrödinger lattice system; Random attractor; Weighted spaces; Upper semicontinuity

1 Introduction

In this paper, we will consider the following nonautonomous stochastic Schrödinger lattice system driven by multiplicative white noise:

$$\begin{cases} i\dot{u}_n = -\sum_{m\in\mathbb{Z}} J(n-m)u_m - i\lambda u_n + f_n(u_n,t) + g_n(t) + iu_n \circ \dot{\omega}(t), \\ u_n(\tau) = u_{\tau,n}, \end{cases} \tag{1.1}$$

and its Wong-Zakai approximations:

$$\begin{cases} i\dot{u}^\delta_n = -\sum_{m\in\mathbb{Z}} J(n-m)u^\delta_m - i\lambda u^\delta_n + f_n(u^\delta_n,t) + g_n(t) + iu^\delta_n \mathcal{G}_\delta(\theta_t\omega), \\ u^\delta_n(\tau) = u^\delta_{\tau,n}, \end{cases} \tag{1.2}$$

where $n \in \mathbb{Z}$, $\tau \in \mathbb{R}$, $t > \tau$, $\delta \in \mathbb{R}$ with $\delta \neq 0$, λ is a positive real constant, i is the unit of imaginary numbers, u_n are complex functions, the coupling parameters $J(m)$ are real numbers and satisfy $J(m) = J(-m)$ for all positive integer m, f_n are nonlinear functions with some conditions, $g(t) = (g_n(t))_{n\in\mathbb{Z}}$ is a time-dependent sequence, $\mathcal{G}_\delta(\theta_t\omega)$ is a random variable defined in (4.2), and ω is a standard Wiener processes on a probability space $(\Omega, \mathcal{F}, P)$. The symbol $\circ$ is interpreted in the sense of Stratonovich's integration.

Note that when $J(m)$ are written as

$$J(m) = \sum_{j=0}^{2k} \binom{2k}{j} (-1)^j \delta_{m,j-k},$$

where k is any positive integer, and $\delta_{m,k}$ is the Kronecker delta, system (1.1) can be changed into the following stochastic system:

$$\begin{cases} i\dot{u}_n = -\triangle^k u_n - i\lambda u_n + f_n(u_n, t) + g_n(t) + iu_n \circ \dot{\omega}(t), \\ u_n(\tau) = u_{n,\tau}, \end{cases} \tag{1.3}$$

where $n \in \mathbb{Z}$, $\tau \in \mathbb{R}$, $t > \tau$, $\triangle^k = \triangle \circ \cdots \circ \triangle$, k times, and $\triangle$ is defined by $\triangle u_n = u_{n+1} + u_{n-1} - 2u_n$.

Lattice dynamical systems, whose spatial structure is discrete, arise from various developments, such as neural networks with applications to image processing, brain science, and others. The deterministic models have been discussed in [10, 27, 38], stochastic models driven by additive white noise in [2, 39, 41], multiplicative white noise systems in [19, 32, 34, 42, 44], and nonlinear white noise systems in [7, 8, 21, 33]. Furthermore, a kind of lattice systems in weighted spaces were considered in [3, 4, 11–13, 20, 25, 30].

Nonlocal lattice systems arise naturally in a wide variety of applications. The dynamics of DNA molecule has been described by nonlocal Schrödinger lattice systems in [24]. Later on, the long-term behavior for nonlocal Schrödinger lattice systems and their delay systems were studied in [25] and [26], respectively. Recently, Chen et al. have proved the existence of random attractors for nonlocal stochastic complex Ginzburg-Landau lattice systems in [4]. At the same time, Wong-Zakai approximations of nonlocal stochastic lattice systems have been investigated in [5]. These results have great significance in this field.

Schrödinger lattice systems are widely applied in physics and biology, see, e.g., [14, 16–18]. Recently, Wang et al. [29] obtained the existence of weak pullback random attractors for Schrödinger lattice systems with nonlinear noise. Jia et al. [15] studied the existence and multiplicity of homoclinic solutions for Schrödinger lattice systems. Furthermore, the existence of nontrivial solutions for stochastic Schrödinger lattice systems has been studied in [40], and the existence of random uniform exponential attractors for stochastic Schrödinger lattice systems with quasi-periodic forces has been studied in [42]. Other properties of solutions for Schrödinger lattice systems have been investigated in [25, 26, 29, 43].

The Wong-Zakai approximation used in this paper was first proposed in [22, 28], where the authors studied the chaotic behavior of the random system with $\mathcal{G}_\delta(\theta_t\omega)$. The work was later extended in [3, 6, 9, 23, 35–37]. However, to our knowledge, the literature about the Wong-Zakai approximations and random attractors for nonlocal stochastic Schrödinger lattice systems with multiplicative white noise in weighted spaces is sparse. In this paper, motivated by [4, 26, 37], we will consider the existence and uniqueness of tempered pullback attractors of Schrödinger lattice system (1.1) and the approximate system (1.2) in weighted space l^2_η. Then, we establish the upper semicontinuity of these attractors when the step length of the Wiener shift approaches zero.

This paper is organized as follows. In Sect. 2, we introduce some definitions and conditions. In the next Section, we prove the existence and uniqueness of random attractors of system (1.1). Section 4 is devoted to the study of the limiting behavior of Wong-Zakai approximations associated with system (1.1). In Sect. 5, we study the upper semicontinuity of random attractors for the Wong-Zakai approximations.

2 Preliminaries

In this section, we recall some definitions and introduce some conditions for the stochastic lattice system (1.1). First, we consider the canonical probability space $(\Omega, \mathcal{F}, P)$, where

$$\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\},$$

$\mathcal{F}$ is the Borel σ-algebra induced by the compact-open topology of Ω, and P is the corresponding Wiener measure on $(\Omega, \mathcal{F})$. Define the time shift by

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad \omega \in \Omega \text{ and } t \in \mathbb{R}.$$

Then, $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$ is a metric dynamical system [1]. Additionally, there exists a $\{\theta_t\}_{t\in\mathbb{R}}$-invariant subset $\tilde{\Omega} \subseteq \Omega$ of full measure such that for each $\omega \in \tilde{\Omega}$,

$$\frac{\omega(t)}{t} \to 0 \text{ as } t \to \pm\infty. \tag{2.1}$$

For the sake of convenience, we will abuse the notation slightly and write the space $\tilde{\Omega}$ as Ω in the sequel.

In the sequel, we use $(\mathbb{X}, d)$ and $\|\cdot\|_{\mathbb{X}}$ to denote a complete separable metric space and the norm of $\mathbb{X}$. Initially, we introduce some fundamental concepts related to random dynamical systems.

Definition 2.1 A mapping $\Psi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times \mathbb{X} \to \mathbb{X}$ is said to be a continuous cocycle on $\mathbb{X}$ over $\mathbb{R}$ and $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$ if for all $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $t, s \in \mathbb{R}^+$, the following conditions are satisfied:

(i) $\Psi(\cdot, \tau, \cdot, \cdot) : \mathbb{R}^+ \times \Omega \times \mathbb{X} \to \mathbb{X}$ is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(\mathbb{X}), \mathcal{B}(\mathbb{X}))$-measurable;
(ii) $\Psi(t, \tau, \omega, \cdot) : \mathbb{X} \to \mathbb{X}$ is continuous;
(iii) $\Psi(0, \tau, \omega, \cdot)$ is the identity on $\mathbb{X}$;
(iv) $\Psi(t + s, \tau, \omega, \cdot) = \Psi(t, \tau + s, \theta_s\omega, \cdot) \circ \Psi(s, \tau, \omega, \cdot)$.

Definition 2.2 Let $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ be a family of non-empty subsets of $\mathbb{X}$. Then, D is called tempered if for every $c > 0$,

$$\lim_{t\to-\infty} e^{ct}\|D(\tau + t, \theta_t\omega)\|_{\mathbb{X}} = 0,$$

where $\|D\|_{\mathbb{X}} = \sup\{\|x\|_{\mathbb{X}} : x \in D\}$.

In the sequel, we denote by $\mathcal{D}$ the collection of all families of tempered non-empty subsets of $\mathbb{X}$, i.e.,

$$\mathcal{D} = \{D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} : D \text{ is tempered in } \mathbb{X}\}.$$

Definition 2.3 Let $\mathcal{D}$ be a collection of some families of non-empty subsets of $\mathbb{X}$ and $K = K(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$. Then, K is called a $\mathcal{D}$-pullback absorbing set for Ψ if for all $\tau \in \mathbb{R}$, $\omega \in \Omega$, and for every $D \in \mathcal{D}$, there exists $T = T(D,\tau,\omega) > 0$ such that

$$\Psi(t,\tau - t,\theta_{-t}\omega, D(\tau - t,\theta_{-t}\omega)) \subseteq K(\tau,\omega) \text{ for all } t \geq T.$$

Definition 2.4 Let $\mathcal{D}$ be a collection of some families of non-empty subsets of $\mathbb{X}$ and $\mathcal{A} = \{\mathcal{A}(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$. Then, $\mathcal{A}$ is called a $\mathcal{D}$-pullback attractor for Ψ if the following conditions are fulfilled:

(i) $\mathcal{A}$ is measurable, $\mathcal{A}(\tau,\omega)$ is compact for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$;

(ii) $\mathcal{A}$ is invariant, i.e., for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\Psi(t,\tau,\omega,\mathcal{A}(\tau,\omega)) = \mathcal{A}(\tau + t,\theta_t\omega), \ \forall t \geq 0;$$

(iii) $\mathcal{A}$ attracts every member of $\mathcal{D}$, i.e., for every $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ and for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\lim_{t\to+\infty} d_{\mathbb{X}}(\Psi(t,\tau - t,\theta_{-t}\omega, D(\tau - t,\theta_{-t}\omega)), \mathcal{A}(\tau,\omega)) = 0,$$

where $d_{\mathbb{X}}(X,Y) = \sup_{x\in X}\inf_{y\in Y} \|x - y\|_{\mathbb{X}}$.

Next, we introduce the weighted space l^p_η. For $\eta = (\eta_n)_{n\in\mathbb{Z}}$ with $\eta_n > 0$, $1 \leq p < \infty$, l^p_η is defined by

$$l^p_\eta = \Big\{u = (u_n)_{n\in\mathbb{Z}} | u_n \in \mathbb{C}, \sum_{n\in\mathbb{Z}} \eta_n |u_n|^p < +\infty\Big\}.$$

Particularly, l^2_η is a Hilbert space with the inner product and norm given by

$$(u,v)_\eta = \sum_{n\in\mathbb{Z}} \eta_n u_n \bar{v}_n, \ \|u\|^2_\eta = (u,u)_\eta, \ u,v \in l^2_\eta.$$

We further assume that weight η_n satisfies the following conditions:

$$\sum_{n\in\mathbb{Z}} \eta_n < +\infty, \tag{2.2}$$

and

$$\alpha_m := \sup_{n\in\mathbb{Z}} \frac{|\eta_{n+m} - \eta_n|}{\eta^{1/2}_{n+m}\eta^{1/2}_n} < +\infty, \ \forall m \geq 1. \tag{2.3}$$

To obtain the existence of pullback random attractors for stochastic system (1.1) and the approximate system (1.2) in l^2_η, the interaction $J(m)$ needs to decrease quickly enough to ensure that

$$\sum_{m=0}^{+\infty}(1+\alpha_m)|J(m)| < +\infty, \tag{2.4}$$

and

$$\beta = \lambda - 2\sum_{m=0}^{+\infty}\alpha_m|J(m)| > 0. \tag{2.5}$$

Moreover, for $u=(u_n)_{n\in\mathbb{Z}} \in l^2$, we introduce the following operator on l^2:

$$(Au)_n = \sum_{m\in\mathbb{Z}} J(n-m)u_m.$$

Using Lemma 3.1 of [4], we have

$$\|Au\|^2 \leq 2|J(0)|^2\|u\|^2 + 8\Big(\sum_{m=1}^{+\infty}|J(m)|\Big)^2\|u\|^2,$$

which along with (2.4) implies that A is a bounded operator on l^2.

Using the above notation, we can rewrite systems (1.1) and (1.2) in l^2 as follows:

$$\begin{cases} i\dot{u} = -Au - i\lambda u + f(u,t) + g(t) + iu \circ \dot{\omega}(t), \\ u(\tau) = u_\tau, \end{cases} \tag{2.6}$$

and

$$\begin{cases} i\dot{u}^\delta = -Au^\delta - i\lambda u^\delta + f(u^\delta,t) + g(t) + iu^\delta \mathcal{G}_\delta(\theta_t\omega), \\ u^\delta(\tau) = u^\delta_\tau, \end{cases} \tag{2.7}$$

where $t>\tau$, $\tau\in\mathbb{R}$, $u=(u_n)_{n\in\mathbb{Z}}$, $f(u,t)=(f_n(u_n,t))_{n\in\mathbb{Z}}$, $g(t)=(g_n(t))_{n\in\mathbb{Z}}$.

For all $n\in\mathbb{Z}$ and $z\in\mathbb{C}$, we assume that $f_n(z,t)$ is Lipschitz continuous with respect to z, i.e, there is a constant $L>0$ such that for all $z_1,z_2\in\mathbb{C}$,

$$|f_n(z_1,t) - f_n(z_2,t)| \leq L|z_1 - z_2|. \tag{2.8}$$

We further assume that for all $n\in\mathbb{Z}$ and $z\in\mathbb{C}$,

$$\mathrm{Im} f_n(z,t)\bar{z} = 0, \quad \text{and} \quad |f_n(z,t)| \leq h_{1,n}(t)|z| + h_{2,n}(t), \tag{2.9}$$

where $h_{1,n}$ and $h_{2,n}$ are nonnegative, $h_1=(h_{1,n}(t))_{n\in\mathbb{Z}} \in L^\infty_{loc}(\mathbb{R},l^\infty)$, $h_2=(h_{2,n}(t))_{n\in\mathbb{Z}} \in L^2_{loc}(\mathbb{R},l^2_\eta)$.

Example 2.1 Consider the real-valued function $\pi : \mathbb{Z} \to \mathbb{R}$ and assume that $\pi = (\pi_n)_{n\in\mathbb{Z}} \in l^p$ for some $1 \le p \le \infty$. Define f by $f_n(u_n, t) = \frac{\pi_n u_n}{1+t^2}$, for all $n \in \mathbb{Z}$, $u = (u_n)_{n\in\mathbb{Z}} \in l^2_\eta$ and $t \in \mathbb{R}$. A simple calculation shows that the function $f(u,t)$ satisfies (2.8) and (2.9).

In this paper, we need the following conditions to derive uniform estimates of solutions, for every $\tau \in \mathbb{R}$,

$$\int_{-\infty}^{\tau} e^{\frac{1}{2}\beta s}\|g(s)\|^2_\eta ds < +\infty, \tag{2.10}$$

and for any $\zeta > 0$,

$$\lim_{r\to-\infty} e^{\zeta r}\int_{-\infty}^{0} e^{\frac{1}{2}\beta s}\|g(s+r)\|^2_\eta ds = 0. \tag{2.11}$$

It is clear that condition (2.11) is stronger than (2.10), and both conditions only impose restrictions on $g(\xi)$ as ξ approaches $-\infty$, not as ξ approaches $+\infty$. As discussed in the following section, condition (2.10) proves highly useful for constructing an absorbing set of solutions in l^2_η, while condition (2.11) plays a crucial role in ensuring the temperedness of the absorbing set. In order to investigate the existence of tempered random attractors, a tempered pullback absorbing set must be constructed. To guarantee the existence of tempered absorbing sets, a temperedness condition needs to be imposed on $g(\xi)$ as given by (2.11). Since the positive number ζ can vary arbitrarily, condition (2.11) roughly implies that the growth rate of $g(\xi)$ should be subexponential in l^2_η as ξ approaches $-\infty$. In other words, $g(\xi)$ could exhibit behavior similar to a polynomial of arbitrary order but not like an exponential function as ξ approaches $-\infty$. Extensive studies have been conducted on tempered attractors for autonomous stochastic equations in [1].

3 Pullback attractors of lattice systems

In this section, we will show the existence and uniqueness of pullback attractors for stochastic Schrödinger lattice system (2.6) in l^2_η. To this end, we need to convert system (2.6) into a pathwise deterministic one by $v(t,\tau,\omega) = e^{-\omega(t)}u(t,\tau,\omega)$, where u is a solution of system (2.6). Then, for $t > \tau$ and $\tau \in \mathbb{R}$, v satisfies

$$\begin{cases} i\dot{v} = -Av - i\lambda v + e^{-\omega(t)}f(e^{\omega(t)}v,t) + e^{-\omega(t)}g(t), \\ v(\tau) = v_\tau, \end{cases} \tag{3.1}$$

where $v_\tau = e^{-\omega(\tau)}u_\tau$. For every $\omega \in \Omega$, $\tau \in \mathbb{R}$ and $v_\tau \in l^2$, system (3.1) is a deterministic equation and the nonlinearity in (3.1) is Lipschitz continuous. Therefore, given $T > 0$, one can show that system (3.1) has a unique solution $v(\cdot,\tau,\omega,v_\tau) \in C([\tau,\tau+T), l^2)$. Furthermore, one may find that $v(\cdot,\tau,\omega,v_\tau)$ is $(\mathcal{F},\mathcal{B}(l^2))$-measurable in $\omega \in \Omega$ and continuous in v_τ with respect to the norm of l^2. As shown below, this local solution is actually defined for all $t > \tau$.

Lemma 3.1 *Suppose that $g \in L^2_{loc}(\mathbb{R}, l^2)$ and (2.8)–(2.9) hold. For every $\tau \in \mathbb{R}$, $\omega \in \Omega$, $v_\tau \in l^2$ and $T > 0$, there exists $M_1 = M_1(\tau,\omega,T) > 0$ such that for all $t \in [\tau,\tau+T]$, the solution v*

of system (3.1) *satisfies*

$$\|v(t,\tau,\omega,v_\tau)\|^2 \le M_1\|v_\tau\|^2 + M_1\int_\tau^t \|g(s)\|^2 ds.$$

Proof By (3.1), we have

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 + \mathrm{Im}\Big(Av,v\Big) + \lambda\|v\|^2 = e^{-\omega(t)}\mathrm{Im}\Big(f(e^{\omega(t)}v,t),v\Big) + e^{-\omega(t)}\mathrm{Im}\Big(g(t),v\Big). \tag{3.2}$$

Note that

$$\begin{aligned}\mathrm{Im}\Big(Av,v\Big) &= \mathrm{Im}\Big\{\sum_{n\in\mathbb{Z}}\Big(\sum_{m\in\mathbb{Z}}J(n-m)v_m\Big)\Big\}\bar{v}_n \\ &= \mathrm{Im}\Big\{\sum_{n\in\mathbb{Z}}J(0)|v_n|^2 + \sum_{n\in\mathbb{Z}}\sum_{m=1}^{+\infty}J(m)(v_{n-m}+v_{n+m})\bar{v}_n\Big\} \\ &= \mathrm{Im}\Big\{\sum_{n\in\mathbb{Z}}J(0)|v_n|^2 + 2\sum_{n\in\mathbb{Z}}\sum_{m=1}^{+\infty}J(m)\mathrm{Re}\big(\bar{v}_{n+m}v_n\big)\Big\} = 0.\end{aligned} \tag{3.3}$$

From (2.9), we get

$$e^{-\omega(t)}\mathrm{Im}\Big(f(e^{\omega(t)}v,t),v\Big) = 0. \tag{3.4}$$

For the last term in (3.2), using Young's inequality, we have

$$e^{-\omega(t)}\Big|\mathrm{Im}\Big(g(t),v\Big)\Big| \le \frac{\lambda}{2}\|v\|^2 + \frac{1}{2\lambda}e^{-2\omega(t)}\|g(t)\|^2. \tag{3.5}$$

It follows from (3.2)–(3.5) that

$$\frac{d}{dt}\|v\|^2 + \lambda\|v\|^2 \le \frac{1}{\lambda}e^{-2\omega(t)}\|g(t)\|^2. \tag{3.6}$$

Multiplying (3.6) by $e^{\lambda t}$ and then integrating over (τ,t) with $t\in[\tau,\tau+T]$, we obtain

$$\|v(t,\tau,\omega,v_\tau)\|^2 \le e^{-\lambda(t-\tau)}\|v_\tau\|^2 + \frac{1}{\lambda}\int_\tau^t e^{-\lambda(t-s)-2\omega(s)}\|g(s)\|^2 ds,$$

which along with the continuity of ω implies the desired estimates. □

Using Lemma 3.1, we find that the solution of system (3.1) is globally defined in l^2, and so is the solution of system (2.6). Then, we can define a mapping $\Psi_0 : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times l^2 \to l^2$ associated with system (2.6). For every $t\in\mathbb{R}^+$, $\tau\in\mathbb{R}$ and $\omega\in\Omega$, let

$$\Psi_0(t,\tau,\omega,u_\tau) = u(t+\tau,\tau,\theta_{-\tau}\omega,u_\tau) = e^{\omega(t)-\omega(-\tau)}v(t+\tau,\tau,\theta_{-\tau}\omega,v_\tau), \tag{3.7}$$

where $u_\tau = e^{-\omega(-\tau)}v_\tau$. We can derive that the mapping Ψ_0 is a continuous cocycle over $\mathbb{R}$ and $(\Omega,\mathcal{F},P,\{\theta_t\}_{t\in\mathbb{R}})$. In order to study the long-term behavior of stochastic system (2.6) in the weighted space l^2_η, we need to extend the continuous cocycle Ψ_0 from l^2 to l^2_η. Next, we obtain the Lipschitz continuity of solutions in l^2_η as stated below.

Lemma 3.2 *Suppose that $g_1, g_2 \in L^2_{loc}(\mathbb{R}, l^2)$, (2.2)–(2.4) and (2.8)–(2.9) hold. Let v_1 and v_2 be the solutions of system (3.1) with g replaced by g_1 and g_2, respectively. For every $\tau \in \mathbb{R}$, $\omega \in \Omega$, $v_{1,\tau}, v_{2,\tau} \in l^2$ and $T > 0$, there exists a positive constant $M_2 = M_2(\tau, \omega, T)$ such that for all $t \in [\tau, \tau + T]$,*

$$
\begin{aligned}
&\|v_1(t,\tau,\omega,v_{1,\tau}) - v_2(t,\tau,\omega,v_{2,\tau})\|_\eta^2 \\
&\le e^{M_2(t-\tau)}\|v_{1,\tau} - v_{2,\tau}\|_\eta^2 + M_2\int_\tau^t e^{M_2(t-s)}e^{-2\omega(s)}\|g_1(s) - g_2(s)\|_\eta^2 ds.
\end{aligned}
$$

Proof Let $V = v_1 - v_2$. Using (3.1), we have

$$
i\frac{dV}{dt} + AV + i\lambda V = e^{-\omega(t)}\Big(f(e^{\omega(t)}v_1, t) - f(e^{\omega(t)}v_2, t)\Big) + e^{-\omega(t)}\Big(g_1(t) - g_2(t)\Big),
$$

which implies

$$
\begin{aligned}
\frac{d}{dt}\|V\|_\eta^2 + 2\mathrm{Im}\Big(AV, V\Big)_\eta + 2\lambda\|V\|_\eta^2 = &2e^{-\omega(t)}\mathrm{Im}\Big(f(e^{\omega(t)}v_1, t) - f(e^{\omega(t)}v_2, t), V\Big)_\eta \\
&+ 2e^{-\omega(t)}\mathrm{Im}\Big(g_1(t) - g_2(t), V\Big)_\eta.
\end{aligned}
\tag{3.8}
$$

Note that

$$
\begin{aligned}
2\mathrm{Im}\Big(AV, V\Big)_\eta &= 2\mathrm{Im}\sum_{n\in\mathbb{Z}}\eta_n\sum_{m\in\mathbb{Z}}J(m)V_{n-m}\bar{V}_n \\
&= 2\mathrm{Im}\left\{\sum_{n\in\mathbb{Z}}J(0)\eta_n|V_n|^2 + \sum_{n\in\mathbb{Z}}\eta_n\sum_{m=1}^{+\infty}J(m)(V_{n-m} + V_{n+m})\bar{V}_n\right\} \\
&= 2\mathrm{Im}\sum_{n\in\mathbb{Z}}\sum_{m=1}^{+\infty}J(m)\eta_n V_{n+m}\bar{V}_n + 2\mathrm{Im}\sum_{n\in\mathbb{Z}}\sum_{m=1}^{+\infty}J(m)\eta_{n+m}V_n\bar{V}_{n+m} \\
&= 2\mathrm{Im}\sum_{n\in\mathbb{Z}}\sum_{m=1}^{+\infty}J(m)(\eta_{n+m} - \eta_n)\bar{V}_{n+m}V_n,
\end{aligned}
$$

which along with (2.3) implies that

$$
2\Big|\mathrm{Im}\Big(AV, V\Big)_\eta\Big| \le 2\sum_{n\in\mathbb{Z}}\sum_{m=1}^{+\infty}\alpha_m|J(m)|\eta_{n+m}^{1/2}\eta_n^{1/2}|V_{n+m}||V_n| \le 2\sum_{m=1}^{+\infty}\alpha_m|J(m)|\|V\|_\eta^2. \tag{3.9}
$$

Using (2.8), we have

$$
\begin{aligned}
&2e^{-\omega(t)}\Big|\mathrm{Im}\Big(f(e^{\omega(t)}v_1, t) - f(e^{\omega(t)}v_2, t), V\Big)_\eta\Big| \\
&\le 2e^{-\omega(t)}\sum_{n\in\mathbb{Z}}\eta_n\Big|f_n(e^{\omega(t)}v_{1,n}, t) - f_n(e^{\omega(t)}v_{2,n}, t)\Big|\Big|\bar{V}_n\Big| \le 2L\|V\|_\eta^2.
\end{aligned}
\tag{3.10}
$$

As to the last term of (3.8), using Young's inequality, we obtain

$$
2e^{-\omega(t)}\Big|\mathrm{Im}\Big(g_1(t) - g_2(t), V\Big)_\eta\Big| \le \|V\|_\eta^2 + e^{-2\omega(t)}\|g_1(t) - g_2(t)\|_\eta^2. \tag{3.11}
$$

It follows from (3.8)–(3.11) that

$$\frac{d}{dt}\|V\|_\eta^2 \le \Big(2\sum_{m=1}^{+\infty}\alpha_m|J(m)| + 2L + 1\Big)\|V\|_\eta^2 + e^{-2\omega(t)}\|g_1(t) - g_2(t)\|_\eta^2,$$

which along with (2.4) and Gronwall's inequality implies the desired estimates. □

Next, we will extend the continuous cocycle Ψ_0 from l^2 to l_η^2.

Lemma 3.3 *Suppose that* $g \in L_{loc}^2(\mathbb{R}, l_\eta^2)$, (2.2)–(2.4) *and* (2.8)–(2.9) *hold. Then, there exists a continuous cocycle* Ψ_0 *in* l_η^2 *over* $\mathbb{R}$ *and* $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$ *such that for every* $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ *and* $u_\tau \in l^2$, *the value* $\Psi_0(t, \tau, \omega, u_\tau)$ *is the unique solution of system* (2.6).

Proof Given $\tau \in \mathbb{R}$, $\omega \in \Omega$, $T > 0$ and $(v_\tau, g) \in l_\eta^2 \times L^2((\tau, \tau + T), l_\eta^2)$. $l^2 \times L^2((\tau, \tau + T), l^2)$ is dense in $l_\eta^2 \times L^2((\tau, \tau + T), l_\eta^2)$, then there is a sequence $(v_n, g_n) \in l^2 \times L^2((\tau, \tau + T), l^2)$ such that $(v_n, g_n) \to (v_\tau, g)$ in $l_\eta^2 \times L^2((\tau, \tau + T), l_\eta^2)$. Lemma 3.2 implies that $\{v(\cdot, \tau, \theta_{-\tau}\omega, (v_n, g_n))\}_{n=1}^{+\infty}$ is a Cauchy sequence in $C([\tau, \tau + T], l_\eta^2)$, and hence $\lim_{n\to+\infty} v(\cdot, \tau, \theta_{-\tau}\omega, (v_n, g_n))$ exists in $C([\tau, \tau + T], l_\eta^2)$. Note that this limit is independent of the choice of (v_n, g_n) by Lemma 3.2. Define a mapping ϕ: $l_\eta^2 \times L^2((\tau, \tau + T), l_\eta^2) \to C([\tau, \tau + T], l_\eta^2)$ by, for every $(v_\tau, g) \in l_\eta^2 \times L^2((\tau, \tau + T), l_\eta^2)$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\phi(\tau, \omega, (v_\tau, g)) = \lim_{n\to+\infty} v(\cdot, \tau, \theta_{-\tau}\omega, (v_n, g_n)), \tag{3.12}$$

where $(v_n, g_n) \in l^2 \times L^2((\tau, \tau + T), l^2)$ with $(v_n, g_n) \to (v_\tau, g)$ in $l_\eta^2 \times L^2((\tau, \tau + T), l_\eta^2)$. By Lemma 3.2, $\phi(\tau, \omega, (v_\tau, g))$ is Lipschitz continuous in (v_τ, g) in $l_\eta^2 \times L^2((\tau, \tau + T), l_\eta^2)$. For every $t \ge \tau$, we find that $v(t, \tau, \theta_{-\tau}\omega, (v_n, g_n))$ is $(\mathcal{F}, \mathcal{B}(l^2))$-measurable and the embedding $l^2 \hookrightarrow l_\eta^2$ is continuous. Then, $v(t, \tau, \theta_{-\tau}\omega, (v_n, g_n))$ is $(\mathcal{F}, \mathcal{B}(l_\eta^2))$-measurable, which along with (3.12) implies that $\phi(\tau, \omega, (v_\tau, g))(t)$ is $(\mathcal{F}, \mathcal{B}(l_\eta^2))$-measurable for all $t \ge \tau$. We fix $g \in L^2((\tau, \tau + T), l_\eta^2)$ and define a map $\tilde{\Psi}_0 : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times l_\eta^2 \to l_\eta^2$ by, for every $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $v_\tau \in l_\eta^2$,

$$\tilde{\Psi}_0(t, \tau, \omega, v_\tau) = e^{\omega(t)-\omega(-\tau)}\phi(\tau, \omega, (v_\tau, g))(t + \tau).$$

Therefore, $\tilde{\Psi}_0(t, \tau, \omega, v_\tau)$ is continuous in $t \in \mathbb{R}^+$ and in $v_\tau \in l_\eta^2$. Using the measurability of ϕ, we find that $\tilde{\Psi}_0(t, \tau, \omega, v_\tau)$ is measurable in $\omega \in \Omega$. Note that $\tilde{\Psi}_0$ is a continuous cocycle in l_η^2 over $\mathbb{R}$ and $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$. Actually, $\tilde{\Psi}_0$ is an extension of Ψ_0 to l_η^2, and we will not distinguish $\tilde{\Psi}_0$ and Ψ_0 from now on. Moreover, the uniqueness of Ψ_0 is ensured by the uniqueness of the solution of the system (2.6). This completes the proof. □

The next Lemma is concerned with the uniform estimates of the solutions for stochastic system (2.6), which is necessary to prove the existence and uniqueness of pullback attractors.

Lemma 3.4 *Suppose that* (2.2)–(2.5) *and* (2.8)–(2.10) *hold. For every* $\tau \in \mathbb{R}$, $\omega \in \Omega$ *and* $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, *there exists* $T = T(\tau, \omega, D) > 0$ *such that for all* $t \ge T$, *the* *solution u of system* (2.6) *satisfies*

$$\|u(\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|_\eta^2+\frac{\lambda}{2}\int_{-t}^{0}e^{\beta s-2\omega(s)}\|u(s+\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|_\eta^2ds \\ \le\frac{4}{\lambda}\int_{-\infty}^{0}e^{\beta s-2\omega(s)}\|g(s+\tau)\|_\eta^2ds,$$

where $u_{\tau-t}\in D(\tau-t,\theta_{-t}\omega)$.

Proof Using (3.1), we have

$$\frac{1}{2}\frac{d}{dt}\|v\|_\eta^2+\mathrm{Im}\Big(Av,v\Big)_\eta+\lambda\|v\|_\eta^2=e^{-\omega(t)}\mathrm{Im}\Big(f(e^{\omega(t)}v,t),v\Big)_\eta+e^{-\omega(t)}\mathrm{Im}\Big(g(t),v\Big)_\eta. \quad (3.13)$$

Similar to (3.9), we get

$$|\mathrm{Im}(Av,v)_\eta|\le\sum_{n\in\mathbb{Z}}\sum_{m=1}^{+\infty}\alpha_m|J(m)|\eta_{n+m}^{1/2}\eta_n^{1/2}|v_{n+m}||v_n|\le\sum_{m=1}^{+\infty}\alpha_m|J(m)|\|v\|_\eta^2. \quad (3.14)$$

Using (2.9), we get

$$e^{-\omega(t)}\mathrm{Im}\Big(f(e^{\omega(t)}v,t),v\Big)_\eta=0. \quad (3.15)$$

For the last term in (3.13), using Young's inequality, we have

$$e^{-\omega(t)}\Big|\mathrm{Im}\Big(g(t),v\Big)_\eta\Big|\le\frac{\lambda}{4}\|v\|_\eta^2+\frac{1}{\lambda}e^{-2\omega(t)}\|g(t)\|_\eta^2. \quad (3.16)$$

It follows from (3.13)–(3.16) and (2.5) that

$$\frac{d}{dt}\|v\|_\eta^2+\frac{\lambda}{2}\|v\|_\eta^2+\beta\|v\|_\eta^2\le\frac{2}{\lambda}e^{-2\omega(t)}\|g(t)\|_\eta^2,$$

which implies that for every $\omega\in\Omega$ and $t\in\mathbb{R}^+$,

$$\|v(\tau,\tau-t,\omega,v_{\tau-t})\|_\eta^2+\frac{\lambda}{2}\int_{\tau-t}^{\tau}e^{\beta(s-\tau)}\|v(s,\tau-t,\omega,v_{\tau-t})\|_\eta^2ds \\ \le e^{-\beta t}\|v_{\tau-t}\|_\eta^2+\frac{2}{\lambda}\int_{\tau-t}^{\tau}e^{\beta(s-\tau)-2\omega(s)}\|g(s)\|_\eta^2ds. \quad (3.17)$$

Replacing ω by $\theta_{-\tau}\omega$ in (3.17) and by

$$u(s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})=e^{\omega(s-\tau)-\omega(-\tau)}v(s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t}), \quad (3.18)$$

we obtain for every $\omega\in\Omega$,

$$\|u(\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|_\eta^2+\frac{\lambda}{2}\int_{\tau-t}^{\tau}e^{\beta(s-\tau)-2\omega(s-\tau)}\|u(s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|_\eta^2ds \\ \le e^{-\beta t-2\omega(-t)}\|u_{\tau-t}\|_\eta^2+\frac{2}{\lambda}\int_{\tau-t}^{\tau}e^{\beta(s-\tau)-2\omega(s-\tau)}\|g(s)\|_\eta^2ds \\ \le e^{-\beta t-2\omega(-t)}\|u_{\tau-t}\|_\eta^2+\frac{2}{\lambda}\int_{-\infty}^{0}e^{\beta s-2\omega(s)}\|g(s+\tau)\|_\eta^2ds. \quad (3.19)$$

Using (2.1) and (2.10), we get

$$\frac{2}{\lambda}\int_{-\infty}^{0} e^{\beta s-2\omega(s)}\|g(s+\tau)\|_{\eta}^{2}ds < +\infty. \tag{3.20}$$

By $u_{\tau-t} \in D(\tau - t, \theta_{-t}\omega) \in \mathcal{D}$, we find that

$$\lim_{t\to+\infty}\sup e^{-\beta t-2\omega(-t)}\|u_{\tau-t}\|_{\eta}^{2} \leq \lim_{t\to+\infty}\sup e^{-\beta t-2\omega(-t)}\|D(\tau-t,\theta_{-t}\omega)\|_{\eta}^{2} = 0,$$

which implies that there is a $T = T(\tau,\omega,D) > 0$ such that for all $t \geq T$,

$$e^{-\beta t-2\omega(-t)}\|u_{\tau-t}\|_{\eta}^{2} \leq \frac{2}{\lambda}\int_{-\infty}^{0} e^{\beta s-2\omega(s)}\|g(s+\tau)\|_{\eta}^{2}ds,$$

which along with (3.19) and (3.20) shows the desired estimate. □

The next step involves deriving uniform estimates on the tails of solutions as $t \to +\infty$, which will play a crucial role in establishing the asymptotic compactness of solutions.

Lemma 3.5 *Suppose that* (2.2)–(2.5) *and* (2.8)–(2.10) *hold. For every* $\tau \in \mathbb{R}$, $\omega \in \Omega$, $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ *and* $\varepsilon > 0$, *there exist* $T = T(\tau,\omega,D,\varepsilon) > 0$ *and* $N = N(\tau,\omega,\varepsilon) > 0$ *such that for all* $t \geq T$, *the solution* u *of system* (2.6) *satisfies*

$$\sum_{|n|\geq N}\eta_n|u_n(\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})|^2 \leq \varepsilon,$$

where $u_{\tau-t} \in D(\tau - t, \theta_{-t}\omega)$.

Proof Let ϑ be a smooth function satisfying $0 \leq \vartheta(s) \leq 1$ for $s \geq 0$ and

$$\vartheta(s) = \begin{cases} 0, & 0 \leq s \leq 1, \\ 1, & s \geq 2. \end{cases}$$

Let k be a fixed positive integer, which will be specified later, and set $y = (y_n)_{n\in\mathbb{Z}}$ with $y_n = \vartheta(\frac{|n|}{k})v_n$. Using (3.1), we have

$$\begin{aligned}\frac{d}{dt}\sum_{n\in\mathbb{Z}}\vartheta\Big(\frac{|n|}{k}\Big)\eta_n|v_n|^2 &= -2\lambda\sum_{n\in\mathbb{Z}}\vartheta\Big(\frac{|n|}{k}\Big)\eta_n|v_n|^2 - 2\mathrm{Im}\Big(Av,y\Big)_{\eta} \\ &\quad + 2e^{-\omega(t)}\mathrm{Im}\Big(f(e^{\omega(t)}v,t),y\Big)_{\eta} + 2e^{-\omega(t)}\mathrm{Im}\Big(g(t),y\Big)_{\eta} \\ &= -2\lambda\sum_{n\in\mathbb{Z}}\vartheta\Big(\frac{|n|}{k}\Big)\eta_n|v_n|^2 + \sum_{i=1}^{3}\mathfrak{K}_i.\end{aligned} \tag{3.21}$$

The subsequent procedure entails the individual estimation of $\mathfrak{K}_i$ $(i = 1, 2, 3)$. First, we estimate $\mathfrak{K}_1$ as follows:

$$\begin{aligned}\left(Av, y\right)_\eta &= \sum_{n\in\mathbb{Z}} \vartheta\left(\frac{|n|}{k}\right)\eta_n \sum_{m\in\mathbb{Z}} J(m)v_{n-m}\bar{v}_n \\ &= J(0)\sum_{n\in\mathbb{Z}} \vartheta\left(\frac{|n|}{k}\right)\eta_n|v_n|^2 + \sum_{n\in\mathbb{Z}}\sum_{m=1}^{+\infty} J(m)\vartheta\left(\frac{|n|}{k}\right)\eta_n v_{n+m}\bar{v}_n \\ &\quad + \sum_{n\in\mathbb{Z}}\sum_{m=1}^{+\infty} J(m)\vartheta\left(\frac{|n+m|}{k}\right)\eta_{n+m}v_n\bar{v}_{n+m}.\end{aligned}$$

Then,

$$\begin{aligned}\operatorname{Im}\left(Av, y\right)_\eta &= \operatorname{Im}\sum_{n\in\mathbb{Z}}\sum_{m=1}^{+\infty} J(m)\left(\vartheta\left(\frac{|n+m|}{k}\right)\eta_{n+m} - \vartheta\left(\frac{|n|}{k}\right)\eta_n\right)\bar{v}_{n+m}v_n \\ &= \mathfrak{I}_1 + \mathfrak{I}_2,\end{aligned} \tag{3.22}$$

where

$$\mathfrak{I}_1 = \operatorname{Im}\sum_{n\in\mathbb{Z}}\sum_{m=1}^{+\infty} J(m)\left(\vartheta\left(\frac{|n+m|}{k}\right) - \vartheta\left(\frac{|n|}{k}\right)\right)\eta_{n+m}\bar{v}_{n+m}v_n,$$

and

$$\mathfrak{I}_2 = \operatorname{Im}\sum_{n\in\mathbb{Z}}\sum_{m=1}^{+\infty} J(m)\vartheta\left(\frac{|n|}{k}\right)(\eta_{n+m} - \eta_n)\bar{v}_{n+m}v_n.$$

By the definition of $\vartheta(s)$, for any $n \in \mathbb{Z}$ and $m \in \mathbb{N}^+$, there exists a constant $c_0 > 0$ such that

$$\left|\vartheta\left(\frac{|n+m|}{k}\right) - \vartheta\left(\frac{|n|}{k}\right)\right| \le \frac{m}{k}c_0 \qquad \text{and} \qquad \left|\vartheta\left(\frac{|n+m|}{k}\right) - \vartheta\left(\frac{|n|}{k}\right)\right| \le 2. \tag{3.23}$$

It follows from (2.3) that for all $n \in \mathbb{Z}$ and $m \ge 1$,

$$\eta_{n+m}^{1/2} \le (2 + \alpha_m)\eta_n^{1/2},$$

which along with (3.23) shows that for any $l > 1$,

$$\begin{aligned}|\mathfrak{I}_1| &\le \sum_{n\in\mathbb{Z}}\sum_{m=1}^{+\infty} |J(m)|\left|\vartheta\left(\frac{|n+m|}{k}\right) - \vartheta\left(\frac{|n|}{k}\right)\right|\eta_{n+m}|v_{n+m}||v_n| \\ &\le \frac{c_0}{k}\sum_{m=1}^{l} m(2+\alpha_m)|J(m)|\sum_{n\in\mathbb{Z}} \eta_{n+m}^{1/2}\eta_n^{1/2}|v_{n+m}||v_n| \\ &\quad + 2\sum_{m=l+1}^{+\infty}(2+\alpha_m)|J(m)|\sum_{n\in\mathbb{Z}} \eta_{n+m}^{1/2}\eta_n^{1/2}|v_{n+m}||v_n| \\ &\le \frac{c_0}{k}\sum_{m=1}^{l} m(2+\alpha_m)|J(m)|\|v\|_\eta^2 + 2\sum_{m=l+1}^{+\infty}(2+\alpha_m)|J(m)|\|v\|_\eta^2.\end{aligned} \tag{3.24}$$

Using (2.3), we get

$$|\mathfrak{I}_2| \le \sum_{m=1}^{+\infty} \alpha_m |J(m)| \sum_{n\in\mathbb{Z}} \vartheta\left(\frac{|n|}{k}\right) \eta_{n+m}^{1/2} \eta_n^{1/2} |v_{n+m}||v_n|$$
$$\le \frac{1}{2} \sum_{m=1}^{+\infty} \alpha_m |J(m)| \Big(\sum_{n\in\mathbb{Z}} \vartheta\left(\frac{|n|}{k}\right) \eta_{n+m} |v_{n+m}|^2 + \sum_{n\in\mathbb{Z}} \vartheta\left(\frac{|n|}{k}\right) \eta_n |v_n|^2 \Big),$$

which along with (3.23) implies that for any $l > 1$,

$$\begin{aligned} |\mathfrak{I}_2| \le{}& \sum_{m=1}^{+\infty} \alpha_m |J(m)| \sum_{n\in\mathbb{Z}} \vartheta\left(\frac{|n|}{k}\right) \eta_n |v_n|^2 \\ &+ \frac{1}{2} \sum_{m=1}^{l} \alpha_m |J(m)| \sum_{n\in\mathbb{Z}} \eta_{n+m} \left| \vartheta\left(\frac{|n+m|}{k}\right) - \vartheta\left(\frac{|n|}{k}\right) \right| |v_{n+m}|^2 \\ &+ \frac{1}{2} \sum_{m=l+1}^{+\infty} \alpha_m |J(m)| \sum_{n\in\mathbb{Z}} \eta_{n+m} \left| \vartheta\left(\frac{|n+m|}{k}\right) - \vartheta\left(\frac{|n|}{k}\right) \right| |v_{n+m}|^2 \\ \le{}& \sum_{m=1}^{+\infty} \alpha_m |J(m)| \sum_{n\in\mathbb{Z}} \vartheta\left(\frac{|n|}{k}\right) \eta_n |v_n|^2 + \frac{c_0}{2k} \sum_{m=1}^{l} m \alpha_m |J(m)| \|v\|_\eta^2 \\ &+ \sum_{m=l+1}^{+\infty} \alpha_m |J(m)| \|v\|_\eta^2. \end{aligned} \tag{3.25}$$

For any $l > 1$, it follows from (3.24)–(3.25) and (3.22) that

$$\begin{aligned} |\mathfrak{K}_1| \le{}& 2 \sum_{m=1}^{+\infty} \alpha_m |J(m)| \sum_{n\in\mathbb{Z}} \vartheta\left(\frac{|n|}{k}\right) \eta_n |v_n|^2 + \frac{3c_0}{k} \sum_{m=1}^{l} m(2+\alpha_m) |J(m)| \|v\|_\eta^2 \\ &+ 6 \sum_{m=l+1}^{+\infty} (2+\alpha_m) |J(m)| \|v\|_\eta^2. \end{aligned} \tag{3.26}$$

Second, we estimate $\mathfrak{K}_2$. From (2.9), we obtain

$$\mathfrak{K}_2 = 2e^{-\omega(t)} \mathrm{Im} \sum_{n\in\mathbb{Z}} \vartheta\left(\frac{|n|}{k}\right) \eta_n f_n\left(e^{\omega(t)} v_n, t\right) \bar{v}_n = 0. \tag{3.27}$$

Lastly, we estimate $\mathfrak{K}_3$. Using Young's inequality, we have

$$\begin{aligned} |\mathfrak{K}_3| \le{}& 2e^{-\omega(t)} \sum_{n\in\mathbb{Z}} \vartheta\left(\frac{|n|}{k}\right) \eta_n |g_n(t)||v_n| \\ \le{}& \lambda \sum_{n\in\mathbb{Z}} \vartheta\left(\frac{|n|}{k}\right) \eta_n |v_n|^2 + \frac{1}{\lambda} e^{-2\omega(t)} \sum_{n\in\mathbb{Z}} \vartheta\left(\frac{|n|}{k}\right) \eta_n |g_n(t)|^2. \end{aligned} \tag{3.28}$$

Then, it follows from (3.21), (3.26)–(3.28) and (2.5) that for $l > 1$,

$$\begin{aligned}&\frac{d}{dt}\sum_{n\in\mathbb{Z}}\vartheta\Big(\frac{|n|}{k}\Big)\eta_n|v_n|^2+\beta\sum_{n\in\mathbb{Z}}\vartheta\Big(\frac{|n|}{k}\Big)\eta_n|v_n|^2\\&\leq\frac{3c_0}{k}\sum_{m=1}^{l}m(2+\alpha_m)|J(m)|\|v\|_\eta^2+6\sum_{m=l+1}^{+\infty}(2+\alpha_m)|J(m)|\|v\|_\eta^2\\&\quad+\frac{1}{\lambda}e^{-2\omega(t)}\sum_{|n|\geq k}\eta_n|g_n(t)|^2,\end{aligned}$$

which implies that for $t\in\mathbb{R}^+$, $\tau\in\mathbb{R}$, $\omega\in\Omega$ and $l > 1$,

$$\begin{aligned}&\sum_{n\in\mathbb{Z}}\vartheta\Big(\frac{|n|}{k}\Big)\eta_n|v_n(\tau,\tau-t,\omega,v_{\tau-t})|^2-e^{-\beta t}\sum_{n\in\mathbb{Z}}\vartheta\Big(\frac{|n|}{k}\Big)\eta_n|v_{\tau-t,n}|^2\\&\leq\frac{3c_0}{k}\sum_{m=1}^{l}m(2+\alpha_m)|J(m)|\int_{\tau-t}^{\tau}e^{\beta(s-\tau)}\|v(s,\tau-t,\omega,v_{\tau-t})\|_\eta^2ds\\&\quad+6\sum_{m=l+1}^{+\infty}(2+\alpha_m)|J(m)|\int_{\tau-t}^{\tau}e^{\beta(s-\tau)}\|v(s,\tau-t,\omega,v_{\tau-t})\|_\eta^2ds\\&\quad+\frac{1}{\lambda}\int_{\tau-t}^{\tau}e^{\beta(s-\tau)}e^{-2\omega(s)}\sum_{|n|\geq k}\eta_n|g_n(s)|^2ds.\end{aligned}\tag{3.29}$$

Using (3.18) and (3.29), we get

$$\begin{aligned}&\sum_{n\in\mathbb{Z}}\vartheta\Big(\frac{|n|}{k}\Big)\eta_n|u_n(\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})|^2\\&\leq e^{-\beta t-2\omega(-t)}\sum_{n\in\mathbb{Z}}\vartheta\Big(\frac{|n|}{k}\Big)\eta_n|u_{\tau-t,n}|^2+\frac{1}{\lambda}\int_{-\infty}^{0}e^{\beta s-2\omega(s)}\sum_{|n|\geq k}\eta_n|g_n(s+\tau)|^2ds\\&\quad+\frac{3c_0}{k}\sum_{m=1}^{l}m(2+\alpha_m)|J(m)|\int_{-t}^{0}e^{\beta s-2\omega(s)}\|u(s+\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|_\eta^2ds\\&\quad+6\sum_{m=l+1}^{+\infty}(2+\alpha_m)|J(m)|\int_{-t}^{0}e^{\beta s-2\omega(s))}\|u(s+\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|_\eta^2ds\\&=\sum_{i=1}^{4}\mathfrak{L}_i.\end{aligned}\tag{3.30}$$

According to $u_{\tau-t}\in D(\tau-t,\theta_{-t}\omega)\in\mathcal{D}$ and (2.1), there exists $T_1=T_1(\tau,\omega,D,\varepsilon)>0$ such that for all $t\geq T_1$,

$$\mathfrak{L}_1\leq\frac{\varepsilon}{4}.\tag{3.31}$$

Using (3.20), we find that there exists $N_1=N_1(\tau,\omega,\varepsilon)>0$ such that for all $k\geq N_1$,

$$\mathfrak{L}_2\leq\frac{\varepsilon}{4}.\tag{3.32}$$

Furthermore, it follows from (2.4) and Lemma 3.4 that there are $T_2 = T_2(\tau,\omega,D,\varepsilon) > 0$ and $N_2 = N_2(\tau,\omega,\varepsilon) > 0$ such that for all $t \geq T_2$ and $k \geq N_2$,

$$\mathfrak{L}_3 \leq \frac{\varepsilon}{4}. \tag{3.33}$$

Using (2.4) and Lemma 3.4, we can choose $l = l(\varepsilon)$ large enough and $t \geq T_2$,

$$\mathfrak{L}_4 \leq \frac{\varepsilon}{4}. \tag{3.34}$$

Let $N(\tau,\omega,\varepsilon) = \max\{N_1,N_2\}$ and $T(\tau,\omega,D,\varepsilon) = \max\{T_1,T_2\}$. It follows from (3.30)–(3.34) that for all $t \geq T$ and $k \geq N$,

$$\sum_{|n|\geq 2k} \eta_n |u_n(\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})|^2 \leq \sum_{n\in\mathbb{Z}} \vartheta\Big(\frac{|n|}{k}\Big)\eta_n |u_n(\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})|^2 \leq \varepsilon.$$

This concludes the proof. □

Next, we show that stochastic system (2.6) has a tempered $\mathcal{D}$-pullback absorbing set as stated below.

Lemma 3.6 *Suppose that* (2.2)–(2.5) *and* (2.8)–(2.11) *hold. Then, the continuous cocycle* Ψ_0 *associated with system* (2.6) *has a closed measurable* $\mathcal{D}$*-pullback absorbing set* $K_0 \in \mathcal{D}$*, which is given by, for each* $\tau \in \mathbb{R}$ *and* $\omega \in \Omega$*,*

$$K_0(\tau,\omega) = \{u \in l^2_\eta : \|u\|^2_\eta \leq R_0(\tau,\omega)\}, \tag{3.35}$$

where $R_0(\tau,\omega)$ *is given by*

$$R_0(\tau,\omega) = \frac{4}{\lambda}\int_{-\infty}^{0} e^{\beta s - 2\omega(s)} \|g(s+\tau)\|^2_\eta ds. \tag{3.36}$$

Proof Note that K_0 given by (3.35) is a closed random set in l^2_η. Using Lemma 3.4 and (3.7), we obtain that for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $D \in \mathcal{D}$, there exists $T_3 = T_3(\tau,\omega,D) > 0$ such that for all $t \geq T_3$,

$$\Psi_0(t,\tau-t,\theta_{-t}\omega,D(\tau-t,\theta_{-t}\omega)) \subseteq K_0(\tau,\omega).$$

This implies that K_0 pullback attracts all elements in $\mathcal{D}$. Next, we prove that $K_0(\tau,\omega)$ is tempered. Given $\zeta > 0$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$, we have

$$\begin{aligned}
&e^{2\zeta r}\|K_0(\tau+r,\theta_r\omega)\|^2_\eta \leq e^{2\zeta r} R_0(\tau+r,\theta_r\omega)\\
&= \frac{4e^{2\zeta r}}{\lambda}\int_{-\infty}^{0} e^{\beta s - 2\theta_r\omega(s)}\|g(s+\tau+r)\|^2_\eta ds\\
&= \frac{4e^{2\zeta r}}{\lambda}\int_{-\infty}^{0} e^{\beta s + 2(\omega(r)-\omega(r+s))}\|g(s+\tau+r)\|^2_\eta ds.
\end{aligned}$$

Let $0 < \alpha < \min\{\frac{\beta}{4}, \frac{\zeta}{2}\}$. By (2.1), for each $\omega \in \Omega$, there exists $T_4 = T_4(\omega) < 0$ such that for all $r \leq T_4$ and $s < 0$,

$$|\omega(r)| \leq -\alpha r, \quad |\omega(r+s)| \leq -\alpha(r+s).$$

Then, we have for each $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\begin{aligned}\limsup_{r\to-\infty} e^{2\zeta r}\|K_0(\tau+r,\theta_r\omega)\|_\eta^2 &\leq \frac{4e^{(2\zeta-4\alpha)r}}{\lambda}\int_{-\infty}^0 e^{(\beta-2\alpha)s}\|g(s+\tau+r)\|_\eta^2 ds\\ &\leq \frac{4e^{(2\zeta-4\alpha)r}}{\lambda}\int_{-\infty}^0 e^{\frac{\beta}{2}s}\|g(s+\tau+r)\|_\eta^2 ds\\ &\leq \frac{4e^{(2\zeta-4\alpha)r}}{\lambda}e^{-\frac{\beta\tau}{2}}\int_{-\infty}^\tau e^{\frac{\beta}{2}s}\|g(s+r)\|_\eta^2 ds,\end{aligned}$$

which along with (2.10) and (2.11) implies that

$$\limsup_{r\to-\infty} e^{\zeta r}\|K_0(\tau+r,\theta_r\omega)\|_\eta = 0.$$

On the other hand, it is evident that, for each $\tau \in \mathbb{R}$, $R_0(\tau,\cdot):\Omega\to\mathbb{R}$ is $(\mathcal{F},\mathcal{B}(\mathbb{R}))$-measurable. Consequently, $K_0(\tau,\omega)$ is a closed measurable $\mathcal{D}$-pullback absorbing set for Ψ_0 in $\mathcal{D}$. This completes the proof. □

We are now ready to present the existence and uniqueness of $\mathcal{D}$-pullback attractors for Ψ_0.

Theorem 3.1 *Suppose that* (2.2)–(2.5) *and* (2.8)–(2.11) *hold. Then, the continuous cocycle* Ψ_0 *associated with system* (2.6) *has a unique* $\mathcal{D}$*-pullback attractor* $\mathcal{A}_0 = \{\mathcal{A}_0(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$ *in* l_η^2.

Proof The key observation is that Lemma 3.6 demonstrates the existence of a closed measurable $\mathcal{D}$-pullback absorbing set K_0 for Ψ_0, while Lemma 3.5 implies that Ψ_0 is asymptotically null in l_η^2 with respect to $\mathcal{D}$. As a result, the existence and uniqueness of $\mathcal{D}$-pullback attractors $\mathcal{A}_0$ can be immediately deduced from Theorem 3.6 in [3]. This completes the proof. □

4 Wong-Zakai approximation of lattice system

In this section, we propose a Wong-Zakai approximation of solutions for nonautonomous stochastic Schrödinger system (2.6) by system (2.7). Given $\delta \neq 0$, define a random variable $\mathcal{G}_\delta$ by

$$\mathcal{G}_\delta(\omega) = \frac{\omega(\delta)}{\delta}, \text{ for all } \omega\in\Omega. \tag{4.1}$$

Using (4.1), we get

$$\mathcal{G}_\delta(\theta_t\omega) = \frac{\omega(t+\delta)-\omega(t)}{\delta} \text{ and } \int_0^t \mathcal{G}_\delta(\theta_s\omega)ds = \int_t^{t+\delta}\frac{\omega(s)}{\delta}ds + \int_\delta^0 \frac{\omega(s)}{\delta}ds, \tag{4.2}$$

which together with the continuity of ω implies that for all $t \in \mathbb{R}$,

$$\lim_{\delta \to 0} \int_0^t \mathcal{G}_\delta(\theta_s \omega) ds = \omega(t). \tag{4.3}$$

From [23], we find that this convergence is uniform on a finite interval as stated below.

Lemma 4.1 *Let $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $T > 0$. Then, for every $\varepsilon > 0$, there exists $\tilde{\delta} = \tilde{\delta}(\varepsilon, \tau, \omega, T) > 0$ such that for all $0 < |\delta| < \tilde{\delta}$ and $t \in [\tau, \tau + T]$,*

$$\left| \int_0^t \mathcal{G}_\delta(\theta_s \omega) ds - \omega(t) \right| < \varepsilon.$$

By Lemma 4.1, for all $0 < |\delta| < \tilde{\delta}$ and $t \in [\tau, \tau + T]$, there exist $c_1 = c_1(\tau, \omega, T) > 0$ and $\tilde{\delta} = \tilde{\delta}(\tau, \omega, T) > 0$ such that

$$\left| \int_0^t \mathcal{G}_\delta(\theta_s \omega) ds \right| \le c_1. \tag{4.4}$$

From (4.3), we see that $\mathcal{G}_\delta(\theta_t \omega)$ is an approximation of the white noise in a sense. Denote by

$$v^\delta(t, \tau, \omega, v_\tau^\delta) = e^{-\int_0^t \mathcal{G}_\delta(\theta_s \omega) ds} u^\delta(t, \tau, \omega, u_\tau^\delta) \ \text{ with } \ v_\tau^\delta = e^{-\int_0^\tau \mathcal{G}_\delta(\theta_s \omega) ds} u_\tau^\delta. \tag{4.5}$$

Then, we get from (2.7) and (4.5) that

$$\begin{cases} i \frac{dv^\delta}{dt} + Av^\delta + i\lambda v^\delta = e^{-\int_0^t \mathcal{G}_\delta(\theta_s \omega) ds} f(u^\delta, t) + e^{-\int_0^t \mathcal{G}_\delta(\theta_s \omega) ds} g(t), \quad t > \tau, \\ v^\delta(\tau) = v_\tau^\delta. \end{cases} \tag{4.6}$$

System (4.6) is a deterministic functional equation and the nonlinearity in (4.6) is Lipschitz continuous from l^2 to l^2. Therefore, for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $v_\tau^\delta \in l^2$, system (4.6) has a unique solution $v^\delta(\cdot, \tau, \omega, v_\tau^\delta) \in C([\tau, \tau + T), l^2)$. As shown below, this local solution is actually defined for all $t > \tau$. In addition, we find that $v^\delta(\cdot, \tau, \omega, v_\tau^\delta)$ is $(\mathcal{F}, \mathcal{B}(l^2))$-measurable in $\omega \in \Omega$ and continuous in v_τ^δ with respect to the norm of l^2. Similar to Lemmas 3.1–3.4, we know for every $\delta \neq 0$, equation (1.2) defines a continuous cocycle Ψ_δ in l_η^2. Given $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $u_\tau^\delta \in l_\eta^2$, let

$$\Psi_\delta(t, \tau, \omega, u_\tau^\delta) = u^\delta(t + \tau, \tau, \theta_{-\tau}\omega, u_\tau^\delta) = e^{\int_0^{t+\tau} \mathcal{G}_\delta(\theta_{s-\tau}\omega) ds} v^\delta(t + \tau, \tau, \theta_{-\tau}\omega, v_\tau^\delta).$$

Lemma 4.2 *Suppose that (2.2)–(2.5) and (2.8)–(2.10) hold. For every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $T > 0$, there exist $\tilde{\delta} = \tilde{\delta}(\tau, \omega, T) > 0$, $M_3 = M_3(\tau, \omega, T) > 0$ such that for all $0 < |\delta| < \tilde{\delta}$ and $t \in [\tau, \tau + T]$, the solution v^δ of system (4.6) satisfies*

$$\|v^\delta(t, \tau, \omega, v_\tau^\delta)\|_\eta^2 + \int_\tau^t \|v^\delta(s, \tau, \omega, v_\tau^\delta)\|_\eta^2 ds \le M_3 \|v_\tau^\delta\|_\eta^2 + M_3 \int_\tau^t \|g(s)\|_\eta^2 ds.$$

Proof Using (4.6), we obtain for every $\omega \in \Omega$,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\sum_{n\in\mathbb{Z}}\eta_n|v_n^\delta|^2 + \mathrm{Im}\sum_{n\in\mathbb{Z}}\eta_n(Av^\delta)_n\bar{v}_n^\delta + \lambda\sum_{n\in\mathbb{Z}}\eta_n|v_n^\delta|^2 \\ &= e^{-\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds}\mathrm{Im}\sum_{n\in\mathbb{Z}}\eta_n f_n(u_n^\delta,t)\bar{v}_n^\delta + e^{-\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds}\mathrm{Im}\sum_{n\in\mathbb{Z}}\eta_n g_n(t)\bar{v}_n^\delta. \end{aligned} \tag{4.7}$$

From (2.9), we have

$$e^{-\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds}\mathrm{Im}\sum_{n\in\mathbb{Z}}\eta_n f_n(u_n^\delta,t)\bar{v}_n^\delta = 0. \tag{4.8}$$

Similar to (3.9), we obtain

$$\left|\mathrm{Im}\sum_{n\in\mathbb{Z}}\eta_n(Av^\delta)_n\bar{v}_n^\delta\right| \le \sum_{m=1}^{+\infty}\alpha_m|J(m)|\|v^\delta\|_\eta^2. \tag{4.9}$$

As to the last term of (4.7), we get

$$e^{-\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds}\left|\mathrm{Im}\sum_{n\in\mathbb{Z}}\eta_n g_n(t)\bar{v}_n^\delta\right| \le \frac{\lambda}{4}\sum_{n\in\mathbb{Z}}\eta_n|v_n^\delta|^2 + \frac{1}{\lambda}e^{-2\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds}\sum_{n\in\mathbb{Z}}\eta_n|g_n(t)|^2. \tag{4.10}$$

It follows from (4.7)–(4.10) and (2.5) that

$$\frac{d}{dt}\sum_{n\in\mathbb{Z}}\eta_n|v_n^\delta|^2 + \frac{\lambda}{2}\sum_{n\in\mathbb{Z}}\eta_n|v_n^\delta|^2 + \beta\sum_{n\in\mathbb{Z}}\eta_n|v_n^\delta|^2 \le \frac{2}{\lambda}e^{-2\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds}\sum_{n\in\mathbb{Z}}\eta_n|g_n(t)|^2. \tag{4.11}$$

Multiplying (4.11) by $e^{\beta t}$ and then integrating over (τ,t) with $t \ge \tau$, we obtain

$$\begin{aligned} &\|v^\delta(t,\tau,\omega,v_\tau^\delta)\|_\eta^2 + \frac{\lambda}{2}\int_\tau^t e^{\beta(s-t)}\|v^\delta(s,\tau,\omega,v_\tau^\delta)\|_\eta^2 ds \\ &\le e^{-\beta(t-\tau)}\|v_\tau^\delta\|_\eta^2 + \frac{2}{\lambda}\int_\tau^t e^{\beta(s-t)}e^{-2\int_0^s \mathcal{G}_\delta(\theta_l\omega)dl}\|g(s)\|_\eta^2 ds, \end{aligned} \tag{4.12}$$

which together with (4.4) completes the proof. □

Next, we establish uniform estimates of the solutions for stochastic system (2.7) in the following lemma.

Lemma 4.3 *Suppose that* (2.2)–(2.5)*and* (2.8)–(2.10) *hold. For every* $\delta \neq 0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$, *and* $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, *there exists* $T = T(\tau,\omega,D,\delta) > 0$ *such that for all* $t \ge T$, *the solution* u^δ *of system* (2.7) *satisfies*

$$\begin{aligned} &\|u^\delta(\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t}^\delta)\|_\eta^2 + \frac{\lambda}{2}\int_{-t}^0 e^{\beta s+2\int_s^0 \mathcal{G}_\delta(\theta_l\omega)dl}\|u^\delta(s+\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t}^\delta)\|_\eta^2 ds \\ &\quad\le R_\delta(\tau,\omega), \end{aligned}$$

where $u_{\tau-t}^\delta \in D(\tau-t,\theta_{-t}\omega)$, *and* $R_\delta(\tau,\omega)$ *is determined by*

$$R_\delta(\tau,\omega) = \frac{4}{\lambda}\int_{-\infty}^0 e^{\beta s+2\int_s^0 \mathcal{G}_\delta(\theta_l\omega)dl}\|g(s+\tau)\|_\eta^2 ds. \tag{4.13}$$

Proof For every $\tau \in \mathbb{R}$, $t \in \mathbb{R}^+$ and $\omega \in \Omega$, it follows from (4.12) that

$$\begin{aligned} &\|v^\delta(\tau,\tau-t,\theta_{-\tau}\omega,v^\delta_{\tau-t})\|^2_\eta + \frac{\lambda}{2}\int_{\tau-t}^{\tau} e^{\beta(s-\tau)}\|v^\delta(s,\tau-t,\theta_{-\tau}\omega,v^\delta_{\tau-t})\|^2_\eta ds \\ &\le e^{-\beta t}\|v^\delta_{\tau-t}\|^2_\eta + \frac{2}{\lambda}\int_{\tau-t}^{\tau} e^{\beta(s-\tau)+2\int_s^0 \mathcal{G}_\delta(\theta_{l-\tau}\omega)dl}\|g(s)\|^2_\eta ds, \end{aligned} \tag{4.14}$$

which along with (4.5) shows that

$$\begin{aligned} &\|u^\delta(\tau,\tau-t,\theta_{-\tau}\omega,u^\delta_{\tau-t})\|^2_\eta + \frac{\lambda}{2}\int_{-t}^{0} e^{\beta s+2\int_s^0 \mathcal{G}_\delta(\theta_l\omega)dl}\|u^\delta(s+\tau,\tau-t,\theta_{-\tau}\omega,u^\delta_{\tau-t})\|^2_\eta ds \\ &\quad\le e^{-\beta t+2\int_{-t}^0 \mathcal{G}_\delta(\theta_l\omega)dl}\|u^\delta_{\tau-t}\|^2_\eta + \frac{2}{\lambda}\int_{-t}^{0} e^{\beta s+2\int_s^0 \mathcal{G}_\delta(\theta_l\omega)dl}\|g(s+\tau)\|^2_\eta ds. \end{aligned} \tag{4.15}$$

Using (4.1) and the ergodic theory, we get

$$\lim_{s\to\pm\infty}\frac{1}{s}\int_0^s \mathcal{G}_\delta(\theta_l\omega)dl = E(\mathcal{G}_\delta(\omega)) = 0. \tag{4.16}$$

From (2.10) and (4.16), we obtain

$$\int_{-\infty}^{0} e^{\beta s+2\int_s^0 \mathcal{G}_\delta(\theta_l\omega)dl}\|g(s+\tau)\|^2_\eta ds < +\infty. \tag{4.17}$$

Since $u^\delta_{\tau-t} \in D(\tau-t,\theta_{-t}\omega)$ and $D \in \mathcal{D}$, we find that there exists $T_5 = T_5(\tau,\omega,D,\delta) > 0$ such that for all $t \ge T_5$,

$$e^{-\beta t+2\int_{-t}^0 \mathcal{G}_\delta(\theta_l\omega)dl}\|u^\delta_{\tau-t}\|^2_\eta \le \frac{2}{\lambda}\int_{-\infty}^{0} e^{\beta s+2\int_s^0 \mathcal{G}_\delta(\theta_l\omega)dl}\|g(s+\tau)\|^2_\eta ds,$$

which together with (4.15) and (4.17) concludes the proof. □

Lemma 4.4 *Suppose that* (2.2)–(2.5) *and* (2.8)–(2.11) *hold. Then, the continuous cocycle* Ψ_δ *associated with system* (2.7) *has a closed measurable* $\mathcal{D}$*-pullback absorbing set* $K_\delta = \{K_\delta(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, *where for every* $\tau \in \mathbb{R}$ *and* $\omega \in \Omega$,

$$K_\delta(\tau,\omega) = \{u^\delta \in l^2_\eta : \|u^\delta\|^2_\eta \le R_\delta(\tau,\omega)\}, \tag{4.18}$$

where $R_\delta(\tau,\omega)$ *is given by* (4.13). *Additionally, we have for every* $\tau \in \mathbb{R}$ *and* $\omega \in \Omega$,

$$\lim_{\delta\to 0} R_\delta(\tau,\omega) = R_0(\tau,\omega), \tag{4.19}$$

where $R_0(\tau,\omega)$ *is given by* (3.36).

Proof Note that K_δ given by (4.18) is a closed measurable random set in l^2_η. Given $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $D \in \mathcal{D}$, it follows from Lemma 4.3 that there exists $T_6 = T_6(\tau,\omega,D,\delta)$ such that for all $t \ge T_6$,

$$\Psi_\delta(t,\tau-t,\theta_{-t}\omega,D(\tau-t,\theta_{-t}\omega)) \subseteq K_\delta(\tau,\omega),$$

which implies that K_δ pullback attracts all elements in $\mathcal{D}$. Next, we prove $K_\delta(\tau,\omega)$ is tempered. Given $\zeta > 0$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$, we have

$$\begin{aligned} e^{2\zeta r}\|K_\delta(\tau + r,\theta_r\omega)\|_\eta^2 \le e^{2\zeta r} R_\delta(\tau + r,\theta_r\omega) \\ &= \frac{4e^{2\zeta r}}{\lambda}\int_{-\infty}^0 e^{\beta s+2\int_s^0 \mathcal{G}_\delta(\theta_{l+r}\omega)dl}\|g(s+\tau+r)\|_\eta^2 ds \\ &= \frac{4e^{2\zeta r}}{\lambda}\int_{-\infty}^0 e^{\beta s+2\int_{s+r}^r \mathcal{G}_\delta(\theta_l\omega)dl}\|g(s+\tau+r)\|_\eta^2 ds. \end{aligned}$$

Using (4.2), we have

$$2\int_s^0 \mathcal{G}_\delta(\theta_l\omega)dl = -2\int_s^{s+\delta}\frac{\omega(l)}{\delta}dl + 2\int_0^\delta \frac{\omega(l)}{\delta}dl. \tag{4.20}$$

Since $\lim\limits_{\delta\to 0}\int_0^\delta \frac{\omega(r)}{\delta}dr = 0$, there exists $\delta_1 = \delta_1(\omega) > 0$ such that for all $0 < |\delta| < \delta_1$

$$2\left|\int_0^\delta \frac{\omega(l)}{\delta}dl\right| \le 1. \tag{4.21}$$

Let $0 < \alpha < \min\{\frac{\beta}{2},\zeta\}$. Similarly, there exists l_1 between s and $s+\delta$ such that $\int_s^{s+\delta}\frac{\omega(l)}{\delta}dl = \omega(l_1)$, which along with (2.1) implies that there exists $T_7 = T_7(\omega) < 0$ such that for all $s \le T_7$ and $|\delta| \le 1$,

$$2\left|\int_s^{s+\delta}\frac{\omega(l)}{\delta}dl\right| \le \alpha - \alpha s. \tag{4.22}$$

Let $\delta_2 = \min\{\delta_1, 1\}$. From (4.20)–(4.22), we get for all $0 < |\delta| < \delta_2$ and $s \le T_7$,

$$2\left|\int_s^0 \mathcal{G}_\delta(\theta_l\omega)dl\right| < \alpha - \alpha s + 1. \tag{4.23}$$

According to (4.4), there exist $\tilde\delta = \tilde\delta(\omega) \in (0,\delta_2)$ and $c_2(\omega) > 0$ such that for all $0 < |\delta| < \tilde\delta$ and $T_7 \le s \le 0$,

$$2\left|\int_s^0 \mathcal{G}_\delta(\theta_l\omega)dl\right| \le c_2(\omega),$$

which along with (4.23) implies that for all $0 < |\delta| < \tilde\delta$ and $s \le 0$,

$$2\left|\int_s^0 \mathcal{G}_\delta(\theta_l\omega)dl\right| \le \alpha - \alpha s + c_3(\omega), \tag{4.24}$$

where $c_3(\omega) = 1 + c_2(\omega)$. Using (4.24), we find that for all $0 < |\delta| < \tilde\delta$, $s \le 0$ and $r \le 0$,

$$2\left|\int_{s+r}^r \mathcal{G}_\delta(\theta_l\omega)dl\right| \le 2\left|\int_{s+r}^0 \mathcal{G}_\delta(\theta_l\omega)dl\right| + 2\left|\int_r^0 \mathcal{G}_\delta(\theta_l\omega)dl\right| \le 2\alpha + 2c_3 - \alpha s - 2\alpha r.$$

Consequently, we have for each $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\begin{aligned} e^{2\zeta r}\|K_\delta(\tau+r,\theta_r\omega)\|_\eta^2 &\le \frac{4e^{2\alpha+2c_3}}{\lambda}e^{(2\zeta-2\alpha)r}\int_{-\infty}^0 e^{(\beta-\alpha)s}\|g(s+\tau+r)\|_\eta^2 ds \\ &\le \frac{4e^{2\alpha+2c_3}}{\lambda}e^{(2\zeta-2\alpha)r}\int_{-\infty}^0 e^{\frac{\beta}{2}s}\|g(s+\tau+r)\|_\eta^2 ds \\ &\le \frac{4e^{2\alpha+2c_3-\frac{\beta\tau}{2}}}{\lambda}e^{(2\zeta-2\alpha)r}\int_{-\infty}^\tau e^{\frac{\beta}{2}s}\|g(s+r)\|_\eta^2 ds, \end{aligned}$$

which along with (2.10) and (2.11) implies that

$$\limsup_{r\to-\infty} e^{\zeta r}\|K_\delta(\tau+r,\theta_r\omega)\|_\eta = 0.$$

The convergence of (4.19) can be obtained by Lebesgue's dominated convergence as in [23]. This concludes the proof. □

Lemma 4.5 *Suppose that* (2.2)–(2.5) *and* (2.8)–(2.10) *hold. For every* $\tau \in \mathbb{R}$, $\omega \in \Omega$ *and* $\varepsilon > 0$, *there exist* $\tilde{\delta} = \tilde{\delta}(\omega) > 0$, $T = T(\tau,\omega,\varepsilon) > 0$ *and* $N = N(\tau,\omega,\varepsilon) > 0$ *such that for all* $t \ge T$ *and* $0 < |\delta| < \tilde{\delta}$, *the solution* u^δ *of system* (2.7) *satisfies*

$$\sum_{|n|\ge N}\eta_n|u_n^\delta(\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t}^\delta)|^2 \le \varepsilon,$$

where $u_{\tau-t}^\delta \in K_\delta(\tau-t,\theta_{-t}\omega)$ *with* K_δ *defined by* (4.18).

Proof Let ϑ be the function defined in Lemma 3.5 and $y=(y_n)_{n\in\mathbb{Z}}$ with $y_n = \vartheta(\frac{|n|}{k})v_n^\delta$. From (4.6), we have

$$\begin{aligned} &\frac{d}{dt}\sum_{n\in\mathbb{Z}}\vartheta\Big(\frac{|n|}{k}\Big)\eta_n|v_n^\delta|^2 \\ &= -2\lambda\sum_{n\in\mathbb{Z}}\vartheta\Big(\frac{|n|}{k}\Big)\eta_n|v_n^\delta|^2 - 2\mathrm{Im}\Big(Av^\delta,y\Big)_\eta \\ &\quad + 2e^{-\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds}\mathrm{Im}\Big(f(u^\delta,t),y\Big)_\eta + 2e^{-\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds}\mathrm{Im}\Big(g(t),y\Big)_\eta \\ &= -2\lambda\sum_{n\in\mathbb{Z}}\vartheta\Big(\frac{|n|}{k}\Big)\eta_n|v_n^\delta|^2 + \sum_{i=1}^3\mathfrak{M}_i. \end{aligned} \tag{4.25}$$

The next step involves separate estimation of $\mathfrak{M}_i$ ($i=1,2,3$). Using the same calculations as in (3.24)–(3.26), we have

$$\begin{aligned} |\mathfrak{M}_1| \le & 2\sum_{m=1}^{+\infty}\alpha_m|J(m)|\sum_{n\in\mathbb{Z}}\vartheta\Big(\frac{|n|}{k}\Big)\eta_n|v_n^\delta|^2 + \frac{3c_0}{k}\sum_{m=1}^{l}m(2+\alpha_m)|J(m)|\|v^\delta\|_\eta^2 \\ &+ 6\sum_{m=l+1}^{+\infty}(2+\alpha_m)|J(m)|\|v^\delta\|_\eta^2. \end{aligned} \tag{4.26}$$

From (2.9), we have

$$\mathfrak{M}_2 = 2e^{-\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds}\mathrm{Im}\Big(f(u^\delta,t),y\Big)_\eta = 0. \tag{4.27}$$

For the last term of (4.25), we have

$$\begin{aligned}|\mathfrak{M}_3| &\le 2e^{-\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds}\sum_{n\in\mathbb{Z}}\vartheta\left(\frac{|n|}{k}\right)\eta_n|g_n(t)||v_n^\delta| \\ &\le \frac{\lambda}{2}\sum_{n\in\mathbb{Z}}\vartheta\left(\frac{|n|}{k}\right)\eta_n|v_n^\delta|^2 + \frac{2}{\lambda}e^{-2\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds}\sum_{n\in\mathbb{Z}}\vartheta\left(\frac{|n|}{k}\right)\eta_n|g_n(t)|^2.\end{aligned} \tag{4.28}$$

Then, it follows from (4.25)–(4.28) and (2.5) that

$$\begin{aligned}&\frac{d}{dt}\sum_{n\in\mathbb{Z}}\vartheta\left(\frac{|n|}{k}\right)\eta_n|v_n^\delta|^2 + \beta\sum_{n\in\mathbb{Z}}\vartheta\left(\frac{|n|}{k}\right)\eta_n|v_n^\delta|^2 \\ &\le \frac{3c_0}{k}\sum_{m=1}^{l} m(2+\alpha_m)|J(m)|\|v^\delta\|_\eta^2 + 6\sum_{m=l+1}^{+\infty}(2+\alpha_m)|J(m)|\|v^\delta\|_\eta^2 \\ &\quad + \frac{2}{\lambda}e^{-2\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds}\sum_{|n|\ge k}\eta_n|g_n(t)|^2.\end{aligned} \tag{4.29}$$

Given $t\in\mathbb{R}^+$, $\tau\in\mathbb{R}$ and $\omega\in\Omega$, integrating (4.29) over $(\tau-t,\tau)$, we have

$$\begin{aligned}&\sum_{n\in\mathbb{Z}}\vartheta\left(\frac{|n|}{k}\right)\eta_n|v_n^\delta(\tau,\tau-t,\omega,v_{\tau-t}^\delta)|^2 - e^{-\beta t}\sum_{n\in\mathbb{Z}}\vartheta\left(\frac{|n|}{k}\right)\eta_n|v_{\tau-t,n}^\delta|^2 \\ &\le \frac{3c_0}{k}\sum_{m=1}^{l} m(2+\alpha_m)|J(m)|\int_{\tau-t}^{\tau} e^{\beta(s-\tau)}\|v^\delta(s,\tau-t,\omega,v_{\tau-t}^\delta)\|_\eta^2 ds \\ &\quad + 6\sum_{m=l+1}^{+\infty}(2+\alpha_m)|J(m)|\int_{\tau-t}^{\tau} e^{\beta(s-\tau)}\|v^\delta(s,\tau-t,\omega,v_{\tau-t}^\delta)\|_\eta^2 ds \\ &\quad + \frac{2}{\lambda}\int_{\tau-t}^{\tau} e^{\beta(s-\tau)+2\int_s^0 \mathcal{G}_\delta(\theta_l\omega)dl}\sum_{|n|\ge k}\eta_n|g_n(s)|^2 ds.\end{aligned} \tag{4.30}$$

Replace ω in the above inequality by $\theta_{-\tau}\omega$, then (4.5) and (4.30) yield that

$$\begin{aligned}&\sum_{n\in\mathbb{Z}}\vartheta\left(\frac{|n|}{k}\right)\eta_n|u_n^\delta(\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t}^\delta)|^2 \\ &\le e^{-\beta t+2\int_{-t}^0 \mathcal{G}_\delta(\theta_l\omega)dl}\sum_{n\in\mathbb{Z}}\vartheta\left(\frac{|n|}{k}\right)\eta_n|u_{\tau-t,n}^\delta|^2 \\ &\quad + \frac{3c_0}{k}\sum_{m=1}^{l} m(2+\alpha_m)|J(m)|\int_{-t}^{0} e^{\beta s+2\int_s^0 \mathcal{G}_\delta(\theta_l\omega)dl}\|u^\delta(s+\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t}^\delta)\|_\eta^2 ds \\ &\quad + 6\sum_{m=l+1}^{+\infty}(2+\alpha_m)|J(m)|\int_{-t}^{0} e^{\beta s+2\int_s^0 \mathcal{G}_\delta(\theta_l\omega)dl}\|u^\delta(s+\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t}^\delta)\|_\eta^2 ds \\ &\quad + \frac{2}{\lambda}\int_{-\infty}^{0} e^{\beta s+2\int_s^0 \mathcal{G}_\delta(\theta_l\omega)dl}\sum_{|n|\ge k}\eta_n|g_n(s+\tau)|^2 ds\end{aligned}$$

$$= \sum_{i=1}^{4} \mathfrak{N}_i. \tag{4.31}$$

Assuming $u_{\tau-t} \in K_\delta(\tau - t, \theta_{-t}\omega)$, we get

$$\begin{aligned} \mathfrak{N}_1 &\leq e^{-\beta t+2\int_{-t}^{0}\mathcal{G}_\delta(\theta_l\omega)dl} \frac{4}{\lambda} \int_{-\infty}^{0} e^{\beta s+2\int_s^0 \mathcal{G}_\delta(\theta_{l-t}\omega)dl} \|g(s+\tau-t)\|_\eta^2 ds \\ &\leq e^{-\beta t+2\int_{-t}^{0}\mathcal{G}_\delta(\theta_l\omega)dl} \frac{4}{\lambda} \int_{-\infty}^{-t} e^{\beta(s+t)+2\int_s^{-t} \mathcal{G}_\delta(\theta_l\omega)dl} \|g(s+\tau)\|_\eta^2 ds. \end{aligned} \tag{4.32}$$

According to (4.24), there exists $\tilde{\delta} > 0$ such that for all $0 < |\delta| < \tilde{\delta}$, $s \leq 0$ and $t \geq 0$,

$$2\left|\int_s^{-t} \mathcal{G}_\delta(\theta_l\omega)dl\right| \leq 2\left|\int_s^{0} \mathcal{G}_\delta(\theta_l\omega)dl\right| + 2\left|\int_{-t}^{0} \mathcal{G}_\delta(\theta_l\omega)dl\right| \leq \frac{\beta}{4} + 2c_3 + \frac{\beta}{8}t - \frac{\beta}{8}s, \tag{4.33}$$

which along with (4.32) shows that for all $0 < |\delta| < \tilde{\delta}$,

$$\begin{aligned} \mathfrak{N}_1 &\leq \frac{4e^{-\beta t+2\int_{-t}^{0}\mathcal{G}_\delta(\theta_l\omega)dl}}{\lambda} \int_{-\infty}^{-t} e^{\beta(s+t)+2\int_s^{-t} \mathcal{G}_\delta(\theta_l\omega)dl} \|g(s+\tau)\|_\eta^2 nds \\ &\leq \frac{4e^{\frac{-\beta t}{8}+\frac{3\beta}{8}+3c_3}}{\lambda} \int_{-\infty}^{-t} e^{\frac{\beta}{2}s} \|g(s+\tau)\|_\eta^2 ds \to 0 \ \text{ as } t \to +\infty. \end{aligned} \tag{4.34}$$

Thus, there exists $T_8 = T_8(\tau, \omega, \varepsilon) > 0$ such that for all $t \geq T_8$ and $0 < |\delta| < \tilde{\delta}$,

$$\mathfrak{N}_1 \leq \frac{\varepsilon}{4}. \tag{4.35}$$

According to Lemma 4.3 and (4.24), there exists $T_9 = T_9(\tau, \omega) > 0$ such that for all $t \geq T_9$ and $0 < |\delta| < \tilde{\delta}$,

$$\begin{aligned} &\int_{-t}^{0} e^{\beta s+2\int_s^0 \mathcal{G}_\delta(\theta_l\omega)dl} \|u^\delta(s+\tau, \tau-t, \theta_{-\tau}\omega, u_{\tau-t}^\delta)\|_\eta^2 ds \\ &\leq \frac{8}{\lambda^2} \int_{-\infty}^{0} e^{\beta s+2\int_s^0 \mathcal{G}_\delta(\theta_l\omega)dl} \|g(s+\tau)\|_\eta^2 ds \\ &\leq \frac{8e^{\frac{\beta}{8}+c_3}}{\lambda^2} \int_{-\infty}^{0} e^{\frac{\beta}{2}s} \|g(s+\tau)\|_\eta^2 ds, \end{aligned} \tag{4.36}$$

which along with (2.10) implies that there exists $N_3 = N_3(\tau, \omega, \varepsilon) > 0$ such that for all $k \geq N_3$, $t \geq T_9$ and $0 < |\delta| < \tilde{\delta}$,

$$\mathfrak{N}_2 \leq \frac{\varepsilon}{4}. \tag{4.37}$$

Using (2.4) and (4.36), we can choose $l = l(\varepsilon)$ large enough such that for all $t \geq T_9$ and $0 < |\delta| < \tilde{\delta}$,

$$\mathfrak{N}_3 \leq \frac{\varepsilon}{4}. \tag{4.38}$$

From (2.10) and (4.24), we find that there exists $N_4 = N_4(\tau,\omega,\varepsilon) > 0$ such that for all $k \geq N_4$ and $0 < |\delta| < \tilde{\delta}$,

$$\mathfrak{N}_4 \leq \frac{2}{\lambda} e^{\frac{\beta}{8}+c_3} \int_{-\infty}^{0} e^{\frac{\beta}{2}s} \sum_{|n|\geq k} \eta_n |g_n(s+\tau)|^2 ds \leq \frac{\varepsilon}{4}. \tag{4.39}$$

Let $N = \max\{N_3, N_4\}$ and $T = \max\{T_8, T_9\}$. It follows from (4.31) and (4.35)–(4.39) that for all $t \geq T$, $k \geq N$ and $0 < |\delta| < \tilde{\delta}$,

$$\sum_{|n|\geq 2k} \eta_n |u_n^\delta(\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t}^\delta)|^2 \leq \sum_{n\in\mathbb{Z}} \vartheta\Big(\frac{|n|}{k}\Big)\eta_n |u_n^\delta(\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t}^\delta)|^2 \leq \varepsilon.$$

This concludes the proof. □

As a consequence of Lemmas 4.3–4.5, we obtain the existence and uniqueness of pullback attractors for system (2.7) as stated below.

Theorem 4.1 *Suppose that* (2.2)–(2.5) *and* (2.8)–(2.11) *hold. Then, there exists* $\tilde{\delta} = \tilde{\delta}(\omega) > 0$ *such that for any* $0 < |\delta| < \tilde{\delta}$, *the continuous cocycle* Ψ_δ *associated with system* (2.7) *has a unique* $\mathcal{D}$*-pullback attractor* $\mathcal{A}_\delta = \{\mathcal{A}_\delta(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ *in* l_η^2.

Proof The proof is quite similar to Theorem 3.1, and the details are left to the reader. □

For the attractor $\mathcal{A}_\delta$ of Ψ_δ, we show the uniform compactness in the following theorem.

Theorem 4.2 *Suppose that* (2.2)–(2.5) *and* (2.8)–(2.11) *hold. For every* $\tau \in \mathbb{R}$ *and* $\omega \in \Omega$, *there exists* $\tilde{\delta} = \tilde{\delta}(\omega) > 0$ *such that* $\bigcup_{0<|\delta|<\tilde{\delta}} \mathcal{A}_\delta(\tau,\omega)$ *is precompact in* l_η^2.

Proof Let $\tilde{\delta} = \tilde{\delta}(\omega)$ be the same number in Theorem 4.1. For every $\varepsilon > 0$, one can get that $\bigcup_{0<|\delta|<\tilde{\delta}} \mathcal{A}_\delta(\tau,\omega)$ has a finite covering of balls of radius less than ε. Denote by

$$B(\tau,\omega) = \{u^\delta \in l_\eta^2 : \|u^\delta\|_\eta^2 \leq R(\tau,\omega)\},$$

where $R(\tau,\omega)$ is defined by

$$R(\tau,\omega) = \frac{4e^{\frac{\beta}{8}+c_3}}{\lambda} \int_{-\infty}^{0} e^{\frac{\beta}{2}s} \|g(s+\tau)\|_\eta^2 ds. \tag{4.40}$$

Using (4.13) and (4.24), we get for all $0 < |\delta| < \tilde{\delta}$,

$$R_\delta(\tau,\omega) \leq R(\tau,\omega). \tag{4.41}$$

Using (4.40)–(4.41), for all $0 < |\delta| < \tilde{\delta}$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$, we obtain $K_\delta(\tau,\omega) \subseteq B(\tau,\omega)$. Therefore, for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\bigcup_{0<|\delta|<\tilde{\delta}} \mathcal{A}_\delta(\tau,\omega) \subseteq \bigcup_{0<|\delta|<\tilde{\delta}} K_\delta(\tau,\omega) \subseteq B(\tau,\omega). \tag{4.42}$$

By Lemma 4.5, there exist $T_{10} = T_{10}(\tau,\omega,\varepsilon) > 0$ and $N_5 = N_5(\tau,\omega,\varepsilon) > 0$ such that for all $t \geq T_{10}$ and $0 < |\delta| < \tilde{\delta}$,

$$\sum_{|n|\geq N_5} \eta_n |u_n^\delta(\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t}^\delta)|^2 \leq \frac{\varepsilon}{4} \tag{4.43}$$

for any $u_{\tau-t}^\delta \in K_\delta(\tau-t,\theta_{-t}\omega)$. Using (4.43) and the invariance of $\mathcal{A}_\delta$, we obtain

$$\sum_{|n|\geq N_5} \eta_n |u_n|^2 \leq \frac{\varepsilon}{4}, \text{ for all } u = (u_n)_{n\in\mathbb{Z}} \in \bigcup_{0<|\delta|<\tilde{\delta}} \mathcal{A}_\delta(\tau,\omega). \tag{4.44}$$

We find that (4.42) implies the set $\{(u_n)_{|n|<N_5} : u \in \bigcup_{0<|\delta|<\tilde{\delta}} \mathcal{A}_\delta(\tau,\omega)\}$ is bounded in a finite dimensional space and hence is precompact. This along with (4.44) shows that $\bigcup_{0<|\delta|<\tilde{\delta}} \mathcal{A}_\delta(\tau,\omega)$ has a finite covering of balls of radius less than ε in l_η^2. This completes the proof. □

5 Upper semicontinuity of pullback attractors

In this section, we will study the limiting of solutions for nonlocal stochastic Schrödinger lattice system (2.7) as $\delta \to 0$.

Lemma 5.1 *Suppose that* (2.2)–(2.5) *and* (2.8)–(2.10) *hold. Let u and u^δ be the solutions of* (2.6) *and* (2.7)*, respectively. For every $\tau \in \mathbb{R}$, $\omega \in \Omega$, $T > 0$ and $\varepsilon \in (0,1)$, there exist $\tilde{\delta} = \tilde{\delta}(\tau,\omega,T,\varepsilon) > 0$ and $M_4 = M_4(\tau,\omega,T) > 0$ such that for all $t \in [\tau,\tau+T]$ and $0 < |\delta| < \tilde{\delta}$,*

$$\begin{aligned}
&\|u^\delta(t,\tau,\omega,u_\tau^\delta) - u(t,\tau,\omega,u_\tau)\|_\eta^2 \\
&\quad \leq M_4\|u_\tau^\delta - u_\tau\|_\eta^2 + M_4\varepsilon\Big(\|u_\tau\|^2 + \|u_\tau^\delta\|^2 + \int_\tau^t \|g(s)\|_\eta^2 ds\Big).
\end{aligned}$$

Proof Let $\tilde{v} = v^\delta - v$, where v and v^δ are the solutions of (3.1) and (4.6), respectively. Using (3.1) and (4.6), we get

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\tilde{v}\|_\eta^2 + \mathrm{Im}\Big(A\tilde{v},\tilde{v}\Big)_\eta + \lambda\|\tilde{v}\|_\eta^2 &= \big(e^{-\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds} - e^{-\omega(t)}\big)\mathrm{Im}\Big(g(t),\tilde{v}\Big)_\eta \\
&\quad + \mathrm{Im}\Big(e^{-\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds} f(e^{\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds}v^\delta,t) - e^{-\omega(t)}f(e^{\omega(t)}v,t),\tilde{v}\Big)_\eta.
\end{aligned} \tag{5.1}$$

Note that

$$\begin{aligned}
&\mathrm{Im}\Big(e^{-\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds} f(e^{\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds}v^\delta,t) - e^{-\omega(t)}f(e^{\omega(t)}v,t),\tilde{v}\Big)_\eta \\
&= \mathrm{Im}\sum_{n\in\mathbb{Z}} \eta_n\Big(e^{-\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds} f_n(e^{\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds}v_n^\delta,t) - e^{-\omega(t)}f_n(e^{\omega(t)}v_n,t)\Big)\bar{\tilde{v}}_n \\
&= \mathrm{Im}\sum_{n\in\mathbb{Z}} \eta_n e^{-\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds}\Big(f_n(e^{\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds}v_n^\delta,t) - f_n(e^{\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds}v_n,t)\Big)\bar{\tilde{v}}_n \\
&\quad + \mathrm{Im}\sum_{n\in\mathbb{Z}} \eta_n\Big(e^{-\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds} - e^{-\omega(t)}\Big) f_n(e^{\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds}v_n,t)\bar{\tilde{v}}_n \\
&\quad + \mathrm{Im}\sum_{n\in\mathbb{Z}} \eta_n e^{-\omega(t)}\Big(f_n(e^{\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds}v_n,t) - f_n(e^{\omega(t)}v_n,t)\Big)\bar{\tilde{v}}_n.
\end{aligned} \tag{5.2}$$

According to (4.3)–(4.4) and Lemma 4.1, for every $\varepsilon \in (0,1)$, there exists $\delta_3 = \delta_3(\tau,\omega,T,\varepsilon) > 0$ such that for all $0 < |\delta| < \delta_3$ and $t \in [\tau, \tau+T]$,

$$\left|e^{-\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds} - e^{-\omega(t)}\right| < \varepsilon \text{ and } \left|e^{\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds-\omega(t)} - 1\right| < \varepsilon. \tag{5.3}$$

From (2.8)–(2.9) and (5.2)–(5.3), for all $0 < |\delta| < \delta_3$ and $t \in [\tau, \tau+T]$, we get

$$\begin{aligned}
&\left|\operatorname{Im}\Big(e^{-\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds} f(e^{\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds} v^\delta, t) - e^{-\omega(t)} f(e^{\omega(t)}v,t), \tilde{v}\Big)_\eta\right| \\
&\le L\sum_{n\in\mathbb{Z}} \eta_n|\tilde{v}_n|^2 + \varepsilon\sum_{n\in\mathbb{Z}}\eta_n\Big(e^{\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds} h_{1,n}(t)|v_n| + h_{2,n}(t)\Big)|\tilde{v}_n| + \varepsilon L e^{-\omega(t)}\sum_{n\in\mathbb{Z}}\eta_n|v_n||\tilde{v}_n| \\
&\le L\|\tilde{v}\|_\eta^2 + \frac{\varepsilon}{2} e^{\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds}\|h_1(t)\|_{L^\infty}\|v\|_\eta^2 + \frac{\varepsilon}{2}e^{\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds}\|h_1(t)\|_{L^\infty}\|\tilde{v}\|_\eta^2 + \frac{\varepsilon}{2}\|h_2(t)\|_\eta^2 \\
&\quad + \frac{\varepsilon}{2}\|\tilde{v}\|_\eta^2 + \frac{\varepsilon}{2}Le^{-\omega(t)}\|v\|_\eta^2 + \frac{\varepsilon}{2}Le^{-\omega(t)}\|\tilde{v}\|_\eta^2.
\end{aligned} \tag{5.4}$$

Using (5.3), we get for all $0 < |\delta| < \delta_3$ and $t \in [\tau, \tau+T]$,

$$\left|\Big(e^{-\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds} - e^{-\omega(t)}\Big)\operatorname{Im}\Big(g(t),\tilde{v}\Big)_\eta\right| \le \frac{1}{2}\varepsilon\|\tilde{v}\|_\eta^2 + \frac{1}{2}\varepsilon\|g(t)\|_\eta^2. \tag{5.5}$$

Similar to (3.9), we have

$$\left|\operatorname{Im}\Big(A\tilde{v},\tilde{v}\Big)_\eta\right| \le 2\sum_{m=1}^{+\infty}\alpha_m|J(m)|\|\tilde{v}\|_\eta^2. \tag{5.6}$$

Then, according to (5.1) and (5.4)–(5.6), there exists $c_4 > 0$ such that for all $0 < |\delta| < \delta_3$ and $t \in [\tau, \tau+T]$

$$\frac{d}{dt}\|\tilde{v}\|_\eta^2 \le c_4\|\tilde{v}\|_\eta^2 + c_4\varepsilon(\|v\|_\eta^2 + \|v^\delta\|_\eta^2 + \|g(t)\|_\eta^2 + \|h_2(t)\|_\eta^2),$$

which implies that

$$\|\tilde{v}(t)\|_\eta^2 \le e^{c_4(t-\tau)}\|\tilde{v}(\tau)\|_\eta^2 + c_4\varepsilon e^{c_4(t-\tau)}\int_\tau^t(\|v\|_\eta^2 + \|v^\delta\|_\eta^2 + \|g(s)\|_\eta^2 + \|h_2(s)\|_\eta^2)ds. \tag{5.7}$$

Then, using (5.7), Lemma 3.1 and Lemma 4.2, we find that there exist $\delta_4 \in (0,\delta_3)$ and $c_5 = c_5(\tau,\omega,T) > 0$ such that for all $0 < |\delta| < \delta_4$ and $t \in [\tau, \tau+T]$

$$\begin{aligned}
&\|v^\delta(t,\tau,\omega,v_\tau^\delta) - v(t,\tau,\omega,v_\tau)\|_\eta^2 \\
&\le e^{c_4(t-\tau)}\|v_\tau^\delta - v_\tau\|_\eta^2 + c_5\varepsilon e^{c_4(t-\tau)}\Big(\|v_\tau\|_\eta^2 + \|v_\tau^\delta\|_\eta^2 + \int_\tau^t\|g(s)\|_\eta^2ds\Big).
\end{aligned} \tag{5.8}$$

Notice that, for all $t \in [\tau, \tau+T]$,

$$\begin{aligned}
u^\delta(t,\tau,\omega,u_\tau^\delta) - u(t,\tau,\omega,u_\tau) =& e^{\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds}\Big(v^\delta(t,\tau,\omega,v_\tau^\delta) - v(t,\tau,\omega,v_\tau)\Big) \\
&+ \Big(e^{\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds} - e^{\omega(t)}\Big)v(t,\tau,\omega,v_\tau),
\end{aligned} \tag{5.9}$$

where $v_\tau^\delta = e^{-\int_0^\tau \mathcal{G}_\delta(\theta_s\omega)ds} u_\tau^\delta$ and $v_\tau = e^{-\omega(\tau)} u_\tau$. Then (5.3) and (5.8)–(5.9) imply the desired estimates. □

Finally, we establish the upper semicontinuity of random attractors as $\delta \to 0$.

Theorem 5.1 *Suppose that* (2.2)–(2.5) *and* (2.8)–(2.11) *hold. Then, for every* $\tau \in \mathbb{R}$ *and* $\omega \in \Omega$,

$$\lim_{\delta\to 0} d_{l_\eta^2}(\mathcal{A}_\delta(\tau,\omega), \mathcal{A}_0(\tau,\omega)) = 0.$$

Proof Let $\delta_n \to 0$ and $u_\tau^{\delta_n} \to u_\tau$ in l_η^2. Then, using Lemma 5.1, for all $\tau \in \mathbb{R}$, $t \geq 0$ and $\omega \in \Omega$, we obtain

$$\Psi_{\delta_n}(t,\tau,\omega,u_\tau^{\delta_n}) \to \Psi_0(t,\tau,\omega,u_\tau) \text{ in } l_\eta^2. \tag{5.10}$$

Using (4.18)–(4.19), for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$, we find that

$$\lim_{\delta\to 0} \|K_\delta(\tau,\omega)\|_\eta^2 \leq R_0(\tau,\omega). \tag{5.11}$$

Then, (5.10)–(5.11), Theorem 4.2 and Theorem 3.1 in [31] imply the desired result. □

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Data availability
Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Competing interests
The authors declare that they have no competing interests.

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