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Qualitative analysis of a harvested predator-prey system with Holling type III functional response

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Abstract

A harvested predator-prey system with Holling type III functional response is considered. By applying qualitative theory of differential equations, we show the instability and global stability properties of the equilibria and the existence and uniqueness of limit cycles for the model. The possibility of existence of a bionomic equilibrium is discussed. The optimal harvesting policy is studied from the view point of control theory. Numerical simulations are carried out to illustrate the validity of our results.

Keywords: limit cycles; global stability; bionomic equilibrium; optimal harvest policy; Holling type III

1 Introduction

The dynamical relationship between predators and their preys is one of the dominant subjects in ecology and mathematical ecology due to its universal importance, see [1].

Harvesting generally has a strong impact on the population dynamics of a harvested species. The severity of this impact depends on the harvesting strategy implemented which in turn may range from rapid depletion to complete preservation of the population. Problems related to the exploitation of multispecies systems are interesting and difficult both theoretically and practically. The problem of inter-specific competition between two species which obey the law of logistic growth was considered by Gause [2]. But he did not study the effect of harvesting. Clark [3] considered harvesting of a single species in a two-species ecologically competing population model. Modifying Clark's model, Chaudhuri [4, 5] studied combined harvesting and considered the perspectives of bioeconomics and dynamic optimization of a two-species fishery. Matsuda [6] showed that the total yield and lowest stock levels of harvested resources increase if fishers may focus their effort on a temporally abundant stock. Matsuda [7] showed that the maximum sustainable yield from the multispecies systems does not guarantee persistence of all resources. Dai [8] analyzed the global behavior of a predator-prey system with some functional response in the presence of constant harvesting. Xiao [9] studied Bogdanov-Takens bifurcations in a predator-prey system with constant rate harvesting.

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Kar [10] studied global dynamics and controllability of a harvested prey-predator system with Holling type III functional response:

$$\begin{cases} \frac{dx}{d\tau} = x(\alpha - x) - \frac{\beta x^2 y}{1+x^2} - E_1 x, \\ \frac{dy}{d\tau} = \frac{\beta_1 x^2 y}{1+x^2} - \gamma y - E_2 y. \end{cases} \tag{1.1}$$

The aim of this paper is to consider the instability and global stability properties of the equilibria and the optimal harvesting policy, the existence and uniqueness of limit cycles of a harvested predator-prey system with Holling type III functional response:

$$\begin{cases} \frac{dx}{dt} = x(\alpha - x) - \frac{\beta x^2 y}{1+x^2} - E_1 x^2, \\ \frac{dy}{dt} = \frac{\beta_1 x^2 y}{1+x^2} - ry - E_2 y, \end{cases} \tag{1.2}$$

where x, y denote prey and predator population respectively at any time t; α, β, β_1, r, E_1 and E_2 are positive constants. Here α, β, β_1 and r are the same with system (1.1), E_1 and E_2 denote the harvesting efforts for the prey and predator respectively, the term $\beta x^2/(1 + x^2)$ denotes the functional response of the predator, which is known as Holling type III response function [11].

This work is motivated by the paper by Kar and Matsuda. They studied the above system (1.1) and showed the local stability of equilibria and uniqueness of limit cycle, but they did not discuss the global stability of the positive equilibrium, the possibility of existence of a bionomic equilibrium, and the optimal harvesting policy. This paper gives the complete qualitative analysis for model (1.2) and the results improve and extend the corresponding results of the above system (1.1).

This paper is organized as follows. Basic properties such as the boundedness, existence, stability and instability of the equilibria of the model are given in Section 2. In Section 3, sufficient conditions for the global stability of the unique positive equilibrium are obtained. In Section 4, we derive the existence and uniqueness of limit cycle. In Section 5, we study a bionomic equilibrium. In Section 6, we study optimal harvest policy. In Section 7, we give numerical stimulations of system (1.2).

2 Basic properties of the model

Let $\overline{R_2^+} = \{(x,y) \mid x \geq 0, y \geq 0\}$. For practical biological meaning, we simply study system (1.2) in $\overline{R_2^+}$.

Lemma 2.1 *All the solutions* $(x(t), y(t))$ *of system* (1.2) *with the initial values* $x(0) > 0, y(0) > 0$ *are uniformly bounded.*

Proof Let us consider the function

$$w = x + \frac{\beta}{\beta_1} y.$$

Differentiating w with respect to τ and using (1.2), we get

$$\frac{dw}{d\tau} = \frac{dx}{d\tau} + \frac{\beta}{\beta_1}\frac{dy}{d\tau} = x(\alpha - x) - E_1 x^2 - \frac{\beta r}{\beta_1} y - \frac{\beta r}{\beta_1} E_2 y.$$

Now we have

$$\frac{dw}{dt} + rw = x(\alpha - x) + rx - E_1x^2 - \frac{\beta r}{\beta_1}E_2 y \leq (\alpha + r)x - (1 + E_1)x^2 \leq \frac{(\alpha + r)^2}{4(1 + E_1)^3} = L \quad \text{(say)}.$$

Applying the theory of differential inequality [9], we obtain

$$0 < w(x,y) < \frac{L}{r}\bigl(1 - e^{-rt}\bigr) + w\bigl(x(0), y(0)\bigr)e^{-rt},$$

which, upon letting $t \to \infty$, yields $0 < w < (L/r)$. So, we have that all the solutions of system (1.2) that start in R_2^+ are confined to the region B, where

$$B = \left\{(x,y) \in R_2^+ : w = \frac{L}{r} + \varepsilon \text{ for any } \varepsilon > 0\right\}.$$

This completes the proof. □

Now we find all the equilibria admitted by the system and study their stability properties. The equilibria of (1.2) are the intersection points of the prey isocline on which $\dot{x} = 0$ and the predator isocline on which $\dot{y} = 0$. Obviously, $P_0(0,0)$ is the trivial equilibrium and $P_1(\frac{\alpha}{1+E_1}, 0)$ is the only axial equilibrium of system (1.2). The third and the most interesting equilibrium point is $P_2(x^*, y^*)$, where x^* and y^* are given by

$$x^* = \sqrt{\frac{r + E_2}{\beta_1 - r - E_2}}, \qquad y^* = \frac{[\alpha - (1 + E_1)x^*](1 + x^{*2})}{\beta x^*}. \tag{2.1}$$

Thus the existence condition for the positive interior equilibrium point P_2 depends upon the restrictions

$$\beta_1 > r + E_2 \tag{2.2}$$

and

$$0 < x^* < \frac{\alpha}{1 + E_1}, \tag{2.3}$$

namely

$$\alpha > (1 + E_1)\sqrt{\frac{r + E_2}{\beta_1 - r - E_2}}. \tag{2.4}$$

We assume that the system parameters are such that they satisfy conditions (2.2) and (2.3). From the expressions for x^* and y^*, we observe that x^* increases with E_2 and y^* decreases with E_1. This is natural because an increase in E_2 decreases the predator population and hence enhances the survival rate of the prey; on the other hand, an increase in E_1 results in the loss of food for the predator.

Lemma 2.2 (1) $P_0(0,0)$ *is a saddle point*;

(2) *If* $E_2 > \frac{\beta_1\alpha^2}{(1+E_1)^2+\alpha^2} - r$ *holds,* $P_1(\frac{\alpha}{1+E_1},0)$ *is a stable focus or a stable node. If* $E_2 < \frac{\beta_1\alpha^2}{(1+E_1)^2+\alpha^2} - r$ *holds, then* $P_1(\frac{\alpha}{1+E_1},0)$ *is a saddle point;* $P_1(\frac{\alpha}{1+E_1},0)$ *is a stable node point for* $E_2 = \frac{\beta_1\alpha^2}{(1+E_1)^2+\alpha^2} - r$; $P_2(x^*,y^*)$ *is a stable focus or a stable node for* $\alpha < \frac{2(1+E_1)x^{*3}}{x^{*2}-1}$; $P_2(x^*,y^*)$ *is an unstable focus or an unstable node for* $\alpha > \frac{2(1+E_1)x^{*3}}{x^{*2}-1}$; $P_2(x^*,y^*)$ *is a center or a focus for* $\alpha = \frac{2(1+E_1)x^{*3}}{x^{*2}-1}$.

Proof The Jacobian matrix of system (1.2) is given by

$$J(x,y) = \begin{pmatrix} \alpha - 2(1+E_1)x - \frac{2\beta xy}{(1+x^2)^2} & -\frac{\beta x^2}{1+x^2} \\ \frac{2\beta_1 xy}{(1+x^2)^2} & \frac{\beta_1 x^2}{1+x^2} - r - E_2 \end{pmatrix}.$$

(1) Since $\det J(0,0) = -\alpha(r+E_2) < 0$, we derive that $P_0(0,0)$ is a saddle point;

(2) The Jacobian matrix of system (1.2) for the equilibrium point $P_1(\frac{\alpha}{1+E_1},0)$ is given by

$$J\left(\frac{\alpha}{1+E_1},0\right) = \begin{pmatrix} -\alpha & -\frac{\beta\alpha^2}{(1+E_1)^2+\alpha^2} \\ 0 & \frac{\beta_1\alpha^2}{(1+E_1)^2+\alpha^2} - r - E_2 \end{pmatrix}.$$

If $\frac{\beta_1 x^2}{1+x^2} - r - E_2 < 0$ holds, namely $E_2 > \frac{\beta_1\alpha^2}{(1+E_1)^2+\alpha^2} - r$, then $\det J(\frac{\alpha}{1+E_1},0) > 0$, $p = -\operatorname{tr} J(\frac{\alpha}{1+E_1},0) > 0$, we can derive that $P_1(\frac{\alpha}{1+E_1},0)$ is a stable focus or a stable node. If $\frac{\beta_1 x^2}{1+x^2} - r - E_2 > 0$ holds, namely $E_2 < \frac{\beta_1\alpha^2}{(1+E_1)^2+\alpha^2} - r$, then $\det J(\frac{\alpha}{1+E_1},0) < 0$, we can derive that $P_1(\frac{\alpha}{1+E_1},0)$ is a saddle point. If $E_2 = \frac{\beta_1\alpha^2}{(1+E_1)^2+\alpha^2} - r$, then $\det J(\frac{\alpha}{1+E_1},0) = 0$, $P_1(\frac{\alpha}{1+E_1},0)$ is a higher-order singular point. At this time, point $P_1(\frac{\alpha}{1+E_1},0)$ and point $P_2(x^*,y^*)$ are the same point. So, we can derive that $P_1(\frac{\alpha}{1+E_1},0)$ is a stable node point.

The Jacobian matrix of system (1.2) for the equilibrium point $P_2(x^*,y^*)$ is given by

$$J(x^*,y^*) = \begin{pmatrix} \alpha - 2(1+E_1)x^* - \frac{2\beta x^* y^*}{(1+x^{*2})^2} & -\frac{\beta x^{*2}}{1+x^{*2}} \\ \frac{2\beta_1 x^* y^*}{(1+x^{*2})^2} & 0 \end{pmatrix}.$$

As $\det J(x^*,y^*) = \frac{2\beta\beta_1 x^{*3} y^*}{(1+x^{*2})^3}$,

$$P = -\operatorname{tr} J(x^*,y^*) = -\left[\alpha - 2(1+E_1)x^* - \frac{2\beta x^* y^*}{(1+x^{*2})^2}\right] = -\frac{-\alpha + \alpha x^{*2} - 2(1+E_1)x^{*3}}{1+x^{*2}},$$

we know that if $-\alpha + \alpha x^{*2} - 2(1+E_1)x^{*3} > 0$, namely $\alpha < \frac{2(1+E_1)x^{*3}}{x^{*2}-1}$, we derive that $P > 0$, $P_2(x^*,y^*)$ is a stable focus or a stable node. If $-\alpha + \alpha x^{*2} - 2(1+E_1)x^{*3} < 0$, namely $\alpha > \frac{2(1+E_1)x^{*3}}{x^{*2}-1}$, we derive that $P < 0$, $P_2(x^*,y^*)$ is an unstable focus or an unstable node. If $-\alpha + \alpha x^{*2} - 2(1+E_1)x^{*3} = 0$, namely $\alpha = \frac{2(1+E_1)x^{*3}}{x^{*2}-1}$, $P_2(x^*,y^*)$ is a center or a focus. The proof is completed. □

We make the following transformation:

$$dt = \left[1 + x^2\right] d\tau.$$

Substituting this into system (1.2), then replacing τ with t gives

$$\begin{cases} \frac{dx}{dt} = x[\alpha - (1+E_1)x](1+x^2) - \beta x^2 y, \\ \frac{dy}{dt} = y[(\beta_1 - r - E_2)x^2 - r - E_2]. \end{cases} \tag{2.5}$$

From Lemma 2.2 we know that if $P = -\frac{-\alpha+\alpha x^{*2}-2(1+E_1)x^{*3}}{1+x^{*2}} = 0$ holds, namely $\alpha = \frac{2(1+E_1)x^{*3}}{x^{*2}-1}$, $P_2(x^*, y^*)$ is a center focus. We can make further conclusions as follows.

Lemma 2.3

(1) *If* $C_0 > 0$ *holds, namely* $\alpha = \frac{2(1+E_1)x^{*3}}{x^{*2}-1} > \frac{[3A\beta+11(1+E_1)]x^*}{3}$, $P_2(x^*, y^*)$ *is a stable fine focus with order one*;

(2) *If* $C_0 < 0$ *holds, namely* $\alpha = \frac{2(1+E_1)x^{*3}}{x^{*2}-1} < \frac{[3A\beta+11(1+E_1)]x^*}{3}$, $P_2(x^*, y^*)$ *is an unstable fine focus with order one*,

where $C_0 = \frac{\Pi(\beta_1-r-E_2)y^*[-3A\beta x^*-11(1+E_1)x^*+3\alpha]}{2A^3\beta^2x^{*3}}$.

Proof First use the coordinate translation, that is, translate the origin of coordinates into the point $P_2(x^*, y^*)$. Then we make the following substitutions for model (2.5):

$$\overline{x} = x - x^*, \qquad \overline{y} = y - y^*, \qquad d\overline{t} = \beta x^{*2}\,dt.$$

Replacing $\overline{x}$, $\overline{y}$, $\overline{t}$ with x, y, t respectively, we have

$$\begin{cases} \frac{dx}{dt} = -y - \frac{2}{x^*}xy + \frac{1}{\beta x^{*2}}[3\alpha x^* - (1+E_1) - 6(1+E_1)x^{*2} - \beta y^*]x^2 \\ \qquad - \frac{1}{x^{*2}}x^2 y + \frac{1}{\beta x^{*2}}[\alpha - 4(1+E_1)x^*]x^3 - \frac{1+E_1}{\beta x^{*2}}x^4, \\ \frac{dy}{dt} = \frac{2(\beta_1-r-E_2)y^*}{\beta x^*}x + \frac{(\beta_1-r-E_2)y^*}{\beta x^{*2}}x^2 + \frac{2(\beta_1-r-E_2)}{\beta x^*}xy + \frac{\beta_1-r-E_2}{\beta x^{*2}}x^2 y. \end{cases} \tag{2.6}$$

We denote $A = \sqrt{\frac{2(\beta_1-r-E_2)y^*}{\beta x^*}} > 0$ and make the following transformations: $u = x$, $v = \frac{1}{A}y$, $d\tau = -A\,dt$. Replacing u, v, τ with x, y, t, respectively, gives

$$\begin{cases} \frac{dx}{dt} = y + Dxy - Ex^2 + Nx^2y - Fx^3 + Lx^4 = y + \sum_{j=2}^{4} P_j(x,y) \equiv \hat{P}(x,y), \\ \frac{dy}{dt} = -x - Gx^2 - Hxy - Ix^2y = -x + \sum_{j=2}^{3} Q_j(x,y) \equiv \hat{Q}(x,y), \end{cases} \tag{2.7}$$

where $D = \frac{2}{x^*}$, $E = \frac{1}{A\beta x^{*2}}[3\alpha x^* - (1+E_1) - 6(1+E_1)x^{*2} - \beta y^*]$, $N = \frac{1}{x^{*2}}$, $F = \frac{1}{A\beta x^{*2}}[\alpha - 4(1+E_1)x^*]$, $L = \frac{1+E_1}{A\beta x^{*2}}$, $G = \frac{(\beta_1-r-E_2)y^*}{A^2\beta x^{*2}}$, $H = \frac{2(\beta_1-r-E_2)}{A\beta x^*}$, $I = \frac{\beta_1-r-E_2}{A\beta x^{*2}}$.

It is obvious that $D^2 = 4N$, and $HD = 4I$. Then we make use of the Poincare method to calculate the focus value.

Construct a form progression $F(x,y) = x^2 + y^2 + \sum_{k=3}^{\infty} F_k(x,y)$, where $F_k(x,y)$ is the kth homogeneous multinomial with x and y. Considering $\frac{dF}{dt}|_{(2.7)} = \frac{\partial F}{\partial t} \cdot \hat{P}(x,y) + \frac{\partial F}{\partial t}\hat{Q}(x,y) = 0$, we can obtain that three multinomials and four multinomials of $F(x,y)$ are equal to zero separately. Noting that $2xP_2(x,y) + 2yQ_2(x,y) = -H_3(x,y)$, we can obtain

$$H_3 = 2Ex^3 + (2G - 2D)x^2y + 2Hxy^2.$$

Let

$$F_3(x,y) = a_0x^3 + a_1x^2y + a_2xy^2 + a_3y^3,$$

then we can obtain the following form:

$$y\frac{\partial F_3}{\partial x} - x\frac{\partial F_3}{\partial y} = y\left(3a_0x^2 + 2a_1xy + a_2y^2\right) - x\left(a_1x^2 + 2a_2xy + 3a_3y^2\right)$$
$$= -a_1x^3 + (3a_0 - 2a_2)x^2y + (2a_1 - 3a_3)xy^2 + a_2y^3.$$

From $y\frac{\partial F_3}{\partial x} - x\frac{\partial F_3}{\partial y} = H_3$, we can derive that

$$-a_1x^3 + 3(a_0 - 2a_2)x^2y + (2a_1 - 3a_3)xy^2 + a_2y^3 = 2Ex^3 + (2G - 2D)x^2y + 2Hxy^2.$$

By the comparison method of correlates, we can obtain that

$$\begin{cases} a_0 = \frac{2G-2D}{3}, \\ a_1 = -2E, \\ a_2 = 0, \\ a_3 = -\frac{2H+4E}{3}, \end{cases}$$

then $F_3(x,y) = \frac{2G-2D}{3}x^3 - 2Ex^2y - \frac{2H+4E}{3}y^3$ and

$$-H_4 = 2xP_3 + 2yQ_3 + \frac{\partial F_3}{\partial x}\cdot P_2 + \frac{\partial F_3}{\partial y}\cdot Q_2$$
$$= (2EG + 2D - 2G - 2F)x^4 + (2HG + 4EG - 2I - 4DE)x^2y^2$$
$$+ \left(2N + 2GD + 4E^2 + 2EH - 2D^2\right)x^3y + \left(2H^2 + 4EH\right)xy^3.$$

Let $x = r\cos\theta$, $y = r\sin\theta$, then we can derive that

$$C_0 = \int_0^{2\Pi} H_4(\cos\theta, \sin\theta)\,d\theta = \frac{\Pi(\beta_1 - r - E_2)y^*[-3A\beta x^* - 11(1+E_1)x^* + 3\alpha]}{2A^3\beta^2x^{*3}}.$$

Hence the point $O(0,0)$ of system (2.5) is an unstable fine focus with order one when $C_0 > 0$, namely $\alpha = \frac{2(1+E_1)x^{*3}}{x^{*2}-1} > \frac{[3A\beta+11(1+E_1)]x^*}{3}$. But considering the time change $d\tau = -A\,dt$, $P_2(x^*,y^*)$ is a stable fine focus with order one. And $P_2(x^*,y^*)$ is an unstable fine focus with order one when $C_0 < 0$, namely $\alpha = \frac{2(1+E_1)x^{*3}}{x^{*2}-1} < \frac{[3A\beta+11(1+E_1)]x^*}{3}$. The proof is completed. □

3 Global stability of the unique positive equilibrium

Theorem 3.1 *Suppose that* $\alpha < 3\sqrt{3}(1+E_1)$ *holds, there is no close orbit around system* (2.5) *in the first quadrant. And if* $x^* < 3\sqrt{3}$ *holds, the positive equilibrium* $P_2(x^*,y^*)$ *of system* (2.5) *is globally asymptotically stable for* $x^*(1+E_1) < \alpha < \min\{\frac{2(1+E_1)x^{*3}}{x^{*2}-1}, 3\sqrt{3}(1+E_1)\}$.

Proof By Lemmas 2.1 and 2.2, the solution $(x(t),y(t))$ of system (1.2) with the initial values $x(0) > 0$, $y(0) > 0$ is unanimous bounded for all $t \geq 0$, and the point $P_2(x^*,y^*)$ is globally asymptotically stable under conditions of Theorem 3.1. We should prove that system (2.5) does not have limit cycle if $\alpha < 3\sqrt{3}(1+E_1)$ holds.

Define a Dulac function $B(x,y) = x^{-2}y^{-1}$, then along system (2.5), we have

$$W = \frac{\partial(BP)}{\partial x} + \frac{\partial(BQ)}{\partial y} = \frac{x^2}{y}\left[\alpha x^2 - \alpha - 2(1+E_1)x^3\right].$$

Let $f(x) = \alpha x^2 - \alpha - 2(1+E_1)x^3$, then $f'(x) = 2\alpha x - 6(1+E_1)x^2$, we can derive that the roots of $f'(x) = 0$ are $x_1 = 0$, and $x_2 = \frac{\alpha}{3(1+E_1)}$.

As $f(0) = -\alpha < 0$, we know that $f(x)$ reaches the maximum value $-\alpha + \frac{\alpha^3}{27(1+E_1)^2}$ at x_2.

Hence if $\alpha < 3\sqrt{3}(1+E_1)$ holds, $W < 0$ for all $x \geq 0$, then system (2.5) does not have any close orbit. Then we can obtain that if $x^* < 3\sqrt{3}$ holds, the positive equilibrium $P_2(x^*, y^*)$ exists and is globally asymptotically stable for $x^*(1+E_1) < \alpha < \min\{\frac{2(1+E_1)x^{*3}}{x^{*2}-1}, 3\sqrt{3}(1+E_1)\}$. This completes the proof. □

4 Existence and uniqueness of limit cycle

Theorem 4.1 *Suppose that* $\alpha > \frac{2(1+E_1)x^{*3}}{x^{*2}-1}$, *then system* (2.5) *has at least one limit cycle in the first quadrant.*

Proof We see that $\frac{dx}{dt}|_{x=\frac{\alpha}{1+E_1}} = -\frac{\beta x^2 y}{1+x^2} < 0$ with $y > 0$, so we have $x = \frac{\alpha}{1+E_1}$ is an untangent line of the system. And the positive trajectory of system (1.2) goes through from its right side to its left side when it meets the line $x = \frac{\alpha}{1+E_1}$.

Construct a Dulac function $w(x,y) = y + \frac{\beta_1}{\beta}x - l$, computing $w = 0$ along the trajectories of system (1.2), we have

$$\begin{aligned}\frac{dw}{dt} &= \frac{dy}{dt} + \frac{\beta_1}{\beta}\frac{dx}{dt} = -(r+E_2)y + \frac{\beta_1 x}{\beta}\bigl[\alpha - (1+E_1)x\bigr] \\ &= \frac{\beta_1 x}{\beta}\bigl[\alpha - (1+E_1)x\bigr] - (r+E_2)\left(l - \frac{\beta_1}{\beta}x\right).\end{aligned}$$

If $l > 0$ is large enough, we have $\frac{dw}{dt} < 0$, where $0 < x < \frac{\alpha}{1+E_1}$. So, the line $y + \frac{\beta_1}{\beta}x = l$ goes through upper to lower part in the region $\{(x,y) \mid 0 < x < \frac{\alpha}{1+E_1}, 0 < y < l - \frac{\beta_1}{\beta}\}$. For system (1.2), construct a Bendixson ring $\widehat{OABC}$ including $P_2(x^*, y^*)$. Define $\overline{OA}$, $\overline{AB}$, $\overline{BC}$ as the lengths of line $L_1 = y = 0$, $L_2 = x - \frac{\alpha}{1+E_1} = 0$, $L_3 = y + \frac{\beta_1}{\beta}x - l$ separately. The outer boundary line of Bendixson ring $\widehat{OABC}$ is J. Due to Lemma 2.1, we know that there exists a unique unstable singular point $P_2(x^*, y^*)$ in the Bendixson ring $\widehat{OABC}$. By the Poincare-Bendixson theorem, system (2.5) has at least one limit cycle in the first quadrant. This completes the proof. □

Note $\alpha = \frac{2(1+E_1)x^{*3}}{x^{*2}-1} = \alpha^*$. From the proof of Lemma 2.2, we know that

$$\frac{d}{dt}(\operatorname{trace} J_{P_2})_{\alpha=\alpha^*} = 1 - \frac{1}{1+x^{*2}} = \frac{r+E_2}{\beta_1} \neq 0. \tag{4.1}$$

Hence by the Hopf bifurcation theorem [12], the system enters into a Hopf-type small amplitude periodic solution at the parametric value $\alpha = \alpha^*$ near the positive interior equilibrium point P_2.

Now we consider the problem of uniqueness of the limit cycle arising from Hopf bifurcation at the parametric value $\alpha = \alpha^*$. There are different techniques for studying the uniqueness of limit cycles. Kuang [13] gave the following result on the uniqueness of limit cycles for the system:

$$\begin{aligned}\frac{dx}{dt} &= x\rho(x) - y\phi(x), \quad x(0) > 0, \\ \frac{dy}{dt} &= y\bigl(-\nu + \varphi(x)\bigr), \quad y(0) > 0,\end{aligned} \tag{4.2}$$

where $\nu > 0$; all the functions are sufficiently smooth on $[0,\infty]$ and satisfy

$$\phi(0) = 0, \qquad \varphi(0) = 0 \quad \text{and} \quad \phi'(x) > 0, \qquad \varphi'(x) > 0 \quad \text{for } x > 0. \tag{4.3}$$

Theorem 4.2 *Assume that* (4.3) *holds. If there exist constants* ξ *and* η *with* $0 < \xi < \eta$ *such that*

$$\phi(\xi) = \nu \quad \textit{and} \quad (x-\eta)\rho(x) < 0 \quad \textit{for } x \neq \eta; \tag{4.4}$$

$$\frac{d}{dx}\left(\frac{x\rho(x)}{\phi(x)}\right)_{x=\xi} > 0; \tag{4.5}$$

$$\frac{d}{dx}\left(\frac{x\rho'(x) + \rho(x) - x\rho(x)\frac{\phi'(x)}{\phi(x)}}{-\nu + \varphi(x)}\right) \leq 0 \quad \textit{for } x \neq \xi. \tag{4.6}$$

Then system (4.2) *has exactly one limit cycle which is globally asymptotically stable* [13].

Theorem 4.3 *If conditions* (2.2), (2.3) *and* $\alpha > \frac{2(1+E_1)x^{*3}}{x^{*2}-1} > \frac{[3A\beta+11(1+E_1)]x^*}{3}$ *hold, then system* (1.2) *has a unique limit cycle.*

Proof We can rewrite system (1.2) as system (4.2) with $\rho(x) = \alpha - (1+E_1)x$,

$$\phi(x) = \frac{\beta x^2}{1+x^2} \quad \text{and} \quad \varphi(x) = \frac{\beta_1 x^2}{1+x^2} = e\phi(x).$$

It is clear that $\phi(x)$ and $\varphi(x)$ satisfy assumption (4.3). Let $\xi = x^* = \sqrt{\frac{r+E_2}{\beta_1-r-E_2}}$, and $\eta = \frac{\alpha}{1+E_1}$. Then by (2.3) we see that $\xi < \frac{\alpha}{1+E_1}$. Assumption (4.4) is satisfied. In fact, we have $\varphi(\xi) = r + E_2$; $\rho(x) > 0$ if $0 < x < \frac{\alpha}{1+E_1}$ and $\rho(x) < 0$ if $x > \frac{\alpha}{1+E_1}$. For the sake of convenience, let

$$S(x) = x\rho'(x) + \rho(x) - x\rho(x)\frac{\phi'}{\phi}$$

and

$$T(x) = -\frac{S(x)}{\varphi(x) - r - E_2} \quad \text{for } x \neq x^*.$$

Since

$$\frac{d}{dx}\left(\frac{x\rho(x)}{\phi(x)}\right) = \frac{S(x)}{\phi(x)},$$

we get $\frac{d}{dx}(\frac{x\rho(x)}{\phi(x)})_{x=x^*} = \frac{-\alpha+\alpha x^{*2}-2(1+E_1)x^{*3}}{\beta x^{*2}} > 0$. Hence, assumption (4.5) is satisfied.

Now

$$T(x) = -\frac{S(x)}{\varphi(x) - r - E_2} = -\frac{1}{(\beta_1 - r - E_2)x^2 - r - E_2}\left[-\alpha + \alpha x^2 - 2(1+E_1)x^3\right].$$

Differentiating this equality and using the fact that $r + E_2 = (\beta_1 - r - E_2)x^{*2}$, we obtain

$$T'(x) = \frac{(\beta_1 - r - E_2)}{[(\beta_1 - r - E_2)x^2 - r - E_2]}\left[2(1+E_1)x^4 - 6(1+E_1)x^{*2}x^2 + 2\alpha x^{*2}x - 2\alpha x\right].$$

Taking $\alpha > \frac{2(1+E_1)x^{*3}}{x^{*2}-1}$ into account, we have

$$2\alpha\left(x^{*2}-1\right) > 2\frac{2(1+E_1)x^{*3}}{x^{*2}-1}\left(x^{*2}-1\right) = 4(1+E_1)x^{*3}.$$

Hence

$$T'(x) > \frac{(\beta_1 - r - E_2)}{[(\beta_1 - r - E_2)x^2 - r - E_2]}\left[2(1+E_1)xU(x)\right],$$

where $U(x) = x^3 - 3(x^*)^2x + 2x^{*3}$.

Since $U'(x) = 3(x^2 - x^{*2})$, $U(x)$ has a minimum value at $x = x^*$.

Therefore, $T'(x) \geq 0$. Thus assumption (4.6) is satisfied.

Thus we observe that whenever the nontrivial equilibrium of system (1.2) is unstable, then all the solutions of the system, initiating in the interior of the positive quadrant of the (x, y) plane, except at the equilibrium, approach a unique limit cycle eventually. But by Lemma 2.3, we know that when $\alpha = \frac{2(1+E_1)x^{*3}}{x^{*2}-1} > \frac{[3A\beta+11(1+E_1)]x^*}{3}$, $P_2(x^*, y^*)$ is a stable fine focus with order one. So, when and only when $\alpha > \frac{2(1+E_1)x^{*3}}{x^{*2}-1} > \frac{[3A\beta+11(1+E_1)]x^*}{3}$, system (1.2) has a unique limit cycle. The proof is completed. □

Remark 1 System (1.2) has no limit cycle whenever the harvesting efforts E_1 and E_2 satisfy the condition $\alpha < \frac{2(1+E_1)x^{*3}}{x^{*2}-1}$. So, local stability of the equilibrium point (x^*, y^*) implies global asymptotic stability. From the point of view of ecological managers, it may be desirable to have a unique positive equilibrium which is globally asymptotically stable in order to plan harvesting and maintain sustainable development of an ecosystem.

Remark 2 If we wish, we can prevent the cycles in the system considered. Let $(\hat{E}_1, \hat{E}_2)$ be the harvesting efforts for which system (1.2) admits a limit cycle. Then these efforts must satisfy the condition $\alpha > \frac{2(1+E_1)x^{*3}}{x^{*2}-1} > \frac{[3A\beta+11(1+E_1)]x^*}{3}$. Now let $(\tilde{x}, \tilde{y})$ be the required limiting value for the solutions of the system. Let (E_1^*, E_2^*) be such that $(x^*(E_1^*), y^*(E_2^*)) = (\tilde{x}, \tilde{y})$. Hence $(\tilde{x}, \tilde{y})$ will be asymptotically stable only if the harvesting efforts (E_1^*, E_2^*) satisfy the condition $\alpha < \frac{2(1+E_1)x^{*3}}{x^{*2}-1}$. Therefore, by choosing the effort functions $(E_1(t), E_2(t))$ as $(E_1(0), E_2(0)) = (\hat{E}_1, \hat{E}_2)$ and $(E_1(\infty), E_2(\infty)) = (E_1^*, E_2^*)$, it is possible to prevent cycles and drive to the steady state $(\tilde{x}, \tilde{y})$.

5 Bionomic equilibrium

Motivated by the paper by Kar [14], we consider the bionomic equilibrium and optimal harvest policy of system (1.2). Let q_1 be the constant prey species cost per unit effort. Let q_2 be the constant predator species cost per unit effort, let p_1 be the constant price per unit biomass of the prey species, and let p_2 be the constant price per unit biomass of the predator species.

The economic rent (net revenue) at any time is given by

$$N = \left(p_1x^2 - q_1\right)E_1 + (p_2y - q_2)E_2 = N_1 + N_2, \tag{5.1}$$

where $N_1 = (p_1x^2 - q_1)E_1$ and $N_2 = (p_2y - q_2)E_2$, they denote the economic rent of the prey species and the predator species separately.

The bionomic equilibrium of the predator-prey system implies both a biological equilibrium and an economic equilibrium. The biological equilibrium occurs when $\frac{dx}{dt} = \frac{dy}{dt} = 0$, and the economic equilibrium is defined to be achieved when the economic rent is completely dissipated (*i.e.*, when $N = 0$).

Hence we can derive the bionomic equilibrium from the following system:

$$\begin{aligned} &x(\alpha - x) - \frac{\beta x^2 y}{1+x^2} - E_1 x^2 = 0, \\ &\frac{\beta_1 x^2 y}{1+x^2} - ry - E_2 y = 0, \\ &N = \left(p_1 x^2 - q_1\right)E_1 + (p_2 y - q_2)E_2 = 0. \end{aligned} \tag{5.2}$$

Now the following cases may arise.

Case 1: If $x \le \sqrt{\frac{q_1}{p_1}}$ and $y > \frac{q_2}{p_2}$, then $N_1 \le 0$ and $N_2 > 0$. In this case, we should stop harvesting the prey species, but we can harvest the predator species. Hence we can derive that $E_{1\infty} = 0$, and from $N = (p_1x^2 - q_1)E_1 + (p_2y - q_2)E_2 = 0$, we have $y_\infty = \frac{q_2}{p_2}$. We substitute $E_{1\infty} = 0$ and $y_\infty = \frac{q_2}{p_2}$ into $x(\alpha - x) - \frac{\beta x^2 y}{1+x^2} - E_1x^2 = 0$, then we have

$$x^3 - \alpha x^2 + \left(1 + \frac{\beta q_2}{p_2}\right) - \alpha = 0. \tag{5.3}$$

Let $f(x) = x^3 - \alpha x^2 + (1 + \frac{\beta q_2}{p_2}) - \alpha = (x^2 - 1)(x - \alpha) + \frac{\beta q_2}{p_2}$, we know that $f(0) = -\alpha < 0$, $f(\alpha) = \frac{\beta q_2}{p_2}\alpha > 0$. So, $f(x) = 0$ has at least one positive root in $(0, \alpha)$.

Now we discuss the roots of $f(x) = 0$ as follows.

For $f'(x) = 3x^2 - 2\alpha x + (1 + \frac{\beta q_2}{p_2})$, we can derive that $f'(x) > 0$ when $x \ge \alpha$, and $f(\alpha) = \frac{\beta q_2}{p_2}\alpha > 0$, then $f(x) = 0$ does not have any roots in $[\alpha, +\infty)$. Hence we discuss the roots of $f(x) = 0$ only in $(0, \alpha)$.

(1) If $\alpha^2 \le 3(1 + \frac{\beta q_2}{p_2})$, we have $f'(x) \ge 0$, then $f(x) = 0$ has only one positive root in $(0, \alpha)$.

(2) If $\alpha^2 > 3(1 + \frac{\beta q_2}{p_2})$, we can derive that $x_1 = \frac{\alpha - \sqrt{\alpha^2 - 3(1+\frac{\beta q_2}{p_2})}}{3}$ and $x_2 = \frac{\alpha + \sqrt{\alpha^2 - 3(1+\frac{\beta q_2}{p_2})}}{3}$ are the roots of $f'(x) = 0$. It is obvious that $0 < x_1 < x_2 < \frac{2\alpha}{3} < \alpha$, $f(x_i) = -(2x_i{}^3 - \alpha x_i{}^2 + \alpha)$. We can further obtain that:

   (i) If $f(x_1) > 0$ and $f(x_2) < 0$, then $f(x) = 0$ has three positive roots in $(0, \alpha)$.

   (ii) If $f(x_1) = 0$ or $f(x_2) = 0$, then $f(x) = 0$ has two positive roots in $(0, \alpha)$.

   (iii) If $f(x_1) < 0$ or $f(x_2) > 0$, then $f(x) = 0$ has one positive root in $(0, \alpha)$.

Let the root of $f(x) = 0$ be x_∞, then we substitute x_∞ and y_∞ into $\frac{\beta_1 x^2 y}{1+x^2} - ry - E_2y = 0$. We can obtain that $E_{2\infty} = \frac{\beta_1 x_\infty{}^2}{1+x_\infty{}^2} - r$. For $E_{2\infty} > 0$, then we must have $r < \frac{\beta_1 x_\infty{}^2}{1+x_\infty{}^2}$. At last, we have the bionomic equilibrium $(x_\infty, y_\infty, 0, E_{2\infty})$.

Case 2: If $x > \sqrt{\frac{q_1}{p_1}}$ and $y \le \frac{q_2}{p_2}$, then $N_1 > 0$ and $N_2 \le 0$. In this case, we should stop harvesting the predator species, but we can harvest the prey species. Hence we can derive that $E_{2\infty} = 0$, and from $N = (p_1x^2 - q_1)E_1 + (p_2y - q_2)E_2 = 0$, we have $x_\infty = \sqrt{\frac{q_1}{p_1}}$. Substituting $E_{2\infty} = 0$ and $x_\infty = \sqrt{\frac{q_1}{p_1}}$ into $\frac{\beta_1 x^2 y}{1+x^2} - ry - E_2y = 0$, we have $r = \frac{\beta_1 p_1}{p_1+q_1}$ and y_∞ is any positive number. Then we substitute x_∞ and y_∞ into $x(\alpha - x) - \frac{\beta x^2 y}{1+x^2} - E_1x^2 = 0$, we can obtain that $E_{1\infty} = \frac{(\alpha - x_\infty)(1+x_\infty{}^2) - \beta x_\infty y_\infty}{x_\infty(1+x_\infty{}^2)}$. For $E_{1\infty} > 0$, then we must have $y_\infty < \frac{(\alpha - \sqrt{\frac{q_1}{p_1}})(1+\frac{q_1}{p_1})}{\beta\sqrt{\frac{q_1}{p_1}}}$. Hence y_∞ is any positive number but it must satisfy $y_\infty < \frac{(\alpha-\sqrt{\frac{q_1}{p_1}})(1+\frac{q_1}{p_1})}{\beta\sqrt{\frac{q_1}{p_1}}}$. At last, we have the bionomic equilibrium $(x_\infty, y_\infty, E_{1\infty}, 0)$.

Case 3: If $x \le \sqrt{\frac{q_1}{p_1}}$ and $y \le \frac{q_2}{p_2}$, then $N_1 \le 0$ and $N_2 \le 0$. In this case, we should stop harvesting the predator species and the prey species. Hence we can derive that $E_{1\infty} = 0$ and $E_{2\infty} = 0$. Then the bionomic equilibrium does not exist.

Case 4: If $x > \sqrt{\frac{q_1}{p_1}}$ and $y > \frac{q_2}{p_2}$, then $N_1 > 0$ and $N_2 > 0$. In this case, we can harvest both the predator species and the prey species. Hence, from $N = (p_1x^2 - q_1)E_1 + (p_2y - q_2)E_2 = 0$, we have $x_\infty = \sqrt{\frac{q_1}{p_1}}$ and $y_\infty = \frac{q_2}{p_2}$. Then we substitute x_∞ and y_∞ into $x(\alpha - x) - \frac{\beta x^2 y}{1+x^2} - E_1x^2 = 0$, $\frac{\beta_1 x^2 y}{1+x^2} - ry - E_2y = 0$. We can obtain that $E_{1\infty} = \frac{(\alpha - x_\infty)(1+x_\infty{}^2) - \beta x_\infty y_\infty}{x_\infty(1+x_\infty{}^2)}$ and $E_{2\infty} = \frac{\beta_1 x_\infty{}^2}{1+x_\infty{}^2} - r$. For $E_{1\infty} > 0$ and $E_{2\infty} > 0$, then we must have $\beta < \frac{(p_1+q_1)(\alpha\sqrt{p_1} - \sqrt{q_1})p_2}{p_1q_2\sqrt{q_1}}$ and $\beta_1 > \frac{r(p_1+q_1)}{q_1}$. At last, we have the bionomic equilibrium $(x_\infty, y_\infty, E_{1\infty}, E_{2\infty})$.

6 Optimal harvest policy

The fundamental problem in determination of an optimal harvest policy in a commercial predator-prey system is to determine the optimal trade-off between the current and future harvests. So, for determining an optimal harvesting policy, we consider the present value J of a continuous time stream of revenues given by

$$J = \int_0^\infty N(x, y, E_1, E_2, t)e^{-\delta t}\,dt,$$

where δ denotes the instantaneous annual rate of discount. N is the economic rent (net revenue) at any time t given by $N(x, y, E_1, E_2, t) = (p_1x^2 - q_1)E_1 + (p_2y - q_2)E_2$, where q_1 and q_2 are the cost per unit biomass of the x and y species, separately; p_1 and p_2 are the prices per unit biomass of the x and y species, respectively.

We shall optimize the objective function

$$J = \int_0^\infty e^{-\delta t}\big[(p_1x^2 - q_1)E_1 + (p_2y - q_2)E_2\big]\,dt, \tag{6.1}$$

subject to the state Eq. (1.2) by using Pontryagin's maximal principle.

Let us now construct the Hamiltonian function

$$\begin{aligned} H^* &= e^{-\delta t}\big[(p_1x^2 - q_1)E_1 + (p_2y - q_2)E_2\big] \\ &\quad + \lambda_1\left[x(\alpha - x) - \frac{\beta x^2 y}{1 + x^2} - E_1x^2\right] + \lambda_2\left[\frac{\beta_1 x^2 y}{1 + x^2} - ry - E_2y\right], \end{aligned} \tag{6.2}$$

where $\lambda_1(t)$ and $\lambda_2(t)$ are the adjoint variables. E_1 and E_2 are the control variables. Now we wish to find the maximum equilibrium $(x_\delta, y_\delta, E_{1\delta}, E_{2\delta})$ of the Hamiltonian function H^*. For the control variables E_1 and E_2 are the linear function of H^*, we have that the optimal equilibrium must occur at the extreme point, namely

$$\frac{\partial H^*}{\partial E_i} = 0, \quad i = 1, 2. \tag{6.3}$$

By the maximal principle, there exist adjoint variables $\lambda_1(t)$ and $\lambda_2(t)$ for all $t \geq 0$ such that

$$\frac{d\lambda_1}{dt} = -\frac{\partial H^*}{\partial x} = -\left\{2e^{-\delta t}p_1E_1x + \lambda_1\left[\alpha - 2x - \frac{2\beta xy}{(1+x^2)^2} - 2E_1x\right] + \frac{2\beta_1\lambda_2 xy}{(1+x^2)^2}\right\} \tag{6.4}$$

and

$$\frac{d\lambda_2}{dt} = -\frac{\partial H^*}{\partial y} = -\left\{e^{-\delta t}p_2E_2 - \frac{\lambda_1\beta x^2}{1+x^2} + \lambda_2\left[\frac{\beta_1 x^2}{1+x^2} - r - E_2\right]\right\}. \tag{6.5}$$

Using Eq. (6.3) we can derive that

$$\lambda_1(t) = e^{-\delta t}\left(p_1 - \frac{q_1}{x^2}\right) \tag{6.6}$$

and

$$\lambda_2(t) = e^{-\delta t}\left(q_1 - \frac{q_2}{y}\right). \tag{6.7}$$

By Eqs. (6.6) and (6.7) we know that $\lambda_i(t)e^{\delta t}$ $(i = 1, 2)$ are invariable by the change of time t.

Using (5.3), Eqs. (6.6) and (6.7), we have that Eqs. (6.4) and (6.5) become

$$\begin{aligned} &2p_1\left[\alpha - x - \frac{\beta xy}{1+x^2}\right] + \left(p_1 - \frac{q_1}{x^2}\right)\left[-\alpha - \frac{2\beta xy}{(1+x^2)^2} + \frac{2\beta xy}{1+x^2}\right] \\ &\quad + \frac{2\beta_1 xy}{(1+x^2)^2}\left(p_2 - \frac{q_2}{y}\right) - \delta\left(p_1 - \frac{q_1}{x^2}\right) = 0, \end{aligned} \tag{6.8}$$

$$p_2\left[\frac{\beta_1 x^2}{1+x^2} - r\right] - \left(p_1 - \frac{q_1}{x^2}\right)\frac{\beta x^2}{1+x^2} - \delta\left(p_2 - \frac{q_2}{y}\right) = 0. \tag{6.9}$$

By Eqs. (6.8) and (6.9), we can obtain the positive root (x_δ, y_δ). Then substituting the values of the positive root into (5.2), we get

$$E_{1\delta} = \frac{(\alpha - x_\delta)(1 + {x_\delta}^2) - \beta x_\delta y_\delta}{x_\delta(1 + {x_\delta}^2)}, \tag{6.10}$$

$$E_{2\delta} = \frac{\beta_1 {x_\delta}^2}{1 + {x_\delta}^2} - r. \tag{6.11}$$

7 Numerical simulation

Example 1 For simulation let us take $\alpha = 13$, $\beta = 2$, $\beta_1 = 3$, $r = 0.5$.

We verify the limit cycle, global stability and controllability of system (1.2). (See Figures 1-6.)

To show the controllability of the system, we take examples as follows:

(1) $(E_1, E_2) = (3.5, 2)$ (see Figure 3);

(2) $(E_1, E_2) = (4.5, 2)$ (see Figure 4);

(3) $(E_1, E_2) = (5, 2)$ (see Figure 5);

(4) $(E_1, E_2) = (1{,}200, 2)$ (see Figure 6).

Figures 3-6 show the dependence of the dynamic behavior of system (1.2) on the harvesting efforts for the prey E_1. Figures 3-6 show that when E_1 is small, both the prey and

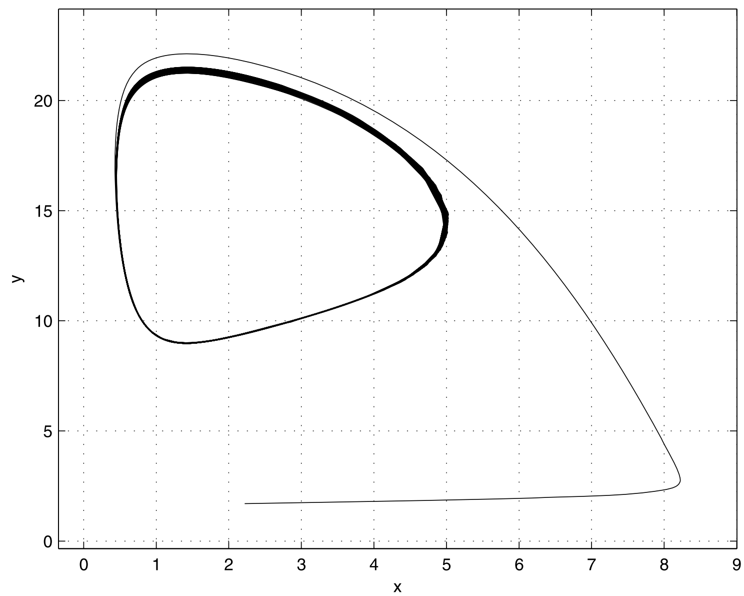

Figure 1 Phase diagram of the limit cycle with different initial conditions (E_1 = 0.5, E_2 = 1.5). Here harvesting efforts lie in the region $\alpha > \frac{2(1+E_1)x^{*3}}{x^{*2}-1} > \frac{[3A\beta+11(1+E_1)]x^*}{3}$.

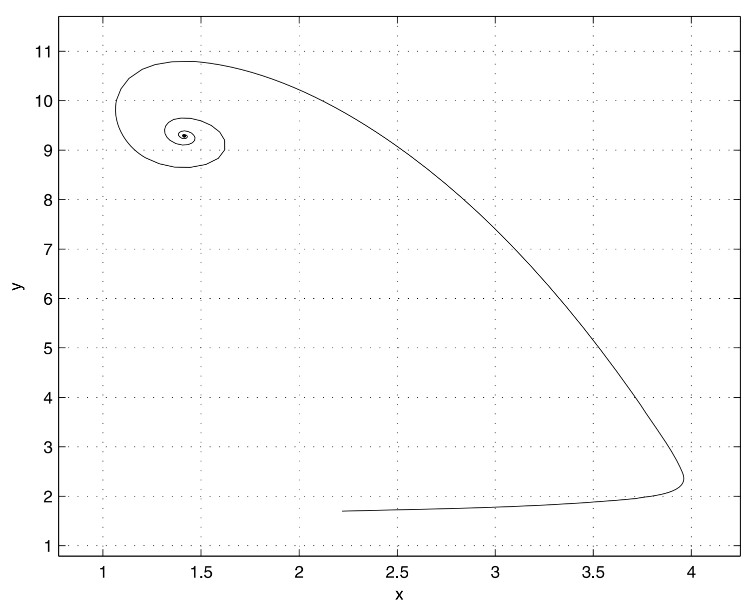

Figure 2 Phase plane trajectories corresponding to different initial levels (E_1 = 2, E_2 = 1.5). The figure clearly indicates that the interior equilibrium point P_2 is globally asymptotically stable.

predator population converge to their equilibrium values respectively, but as E_1 is large, the predator does not have enough food and approaches extermination at last. As E_1 is large enough, both the prey and predator approach extermination at last. Which means that if we change the value of E_1, it is possible to prevent the cyclic behavior of the predator-prey system and to drive it to a required stable state.

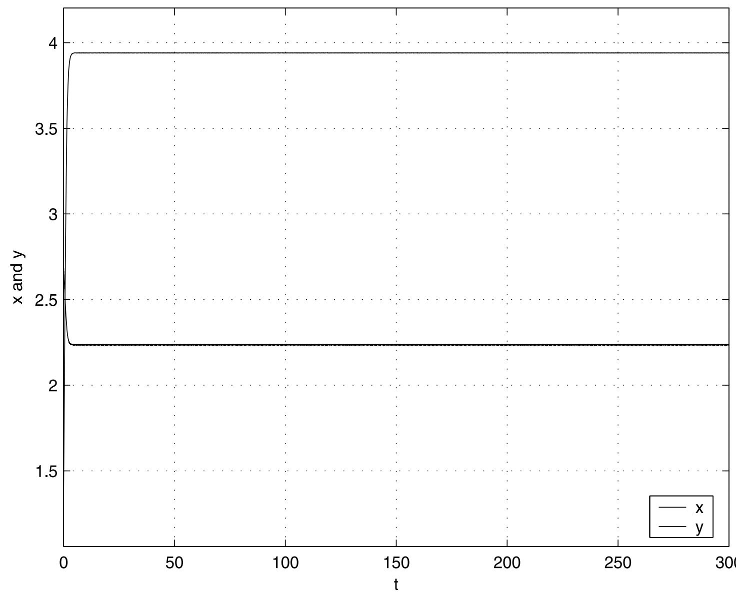

Figure 3 The quantity of predator is larger than the prey when they approach globally asymptotically stable.

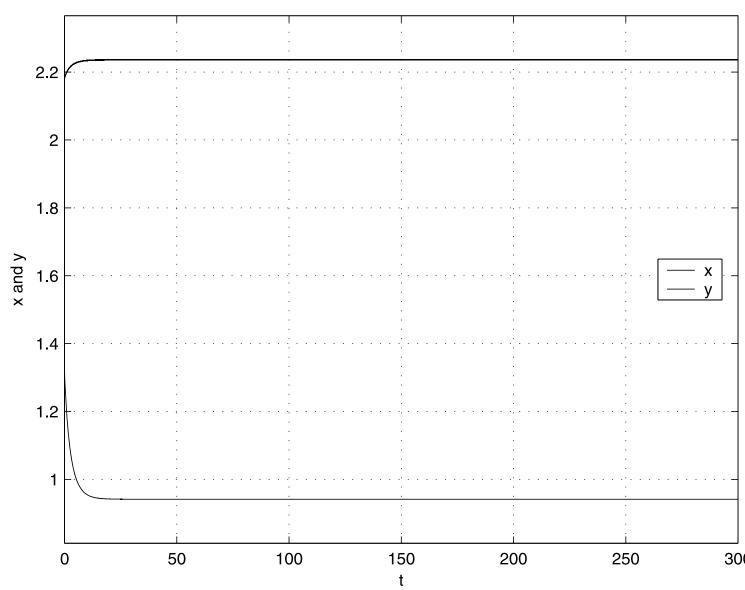

Figure 4 The quantity of predator is less than the prey when they approach globally asymptotically stable.

Example 2 For $p_1 = 0.4$, $q_1 = 0.2$, $p_2 = 0.5$, $q_2 = 0.2$, $\delta = 1.25$, and the remaining parameter values being the same as above, then Eqs. (6.8), (6.9) can be simplified as follows:

$$\begin{aligned} &9.9 - 0.8x - \frac{1.6xy}{1+x^2} + \left(0.4 - \frac{0.2}{x^2}\right)\left(-13 - \frac{4xy}{(1+x^2)^2} + \frac{4xy}{1+x^2}\right) \\ &\quad + \frac{6xy(0.5 - \frac{0.2}{y})}{(1+x^2)^2} + \frac{0.25}{x^2} = 0, \end{aligned} \tag{7.1}$$

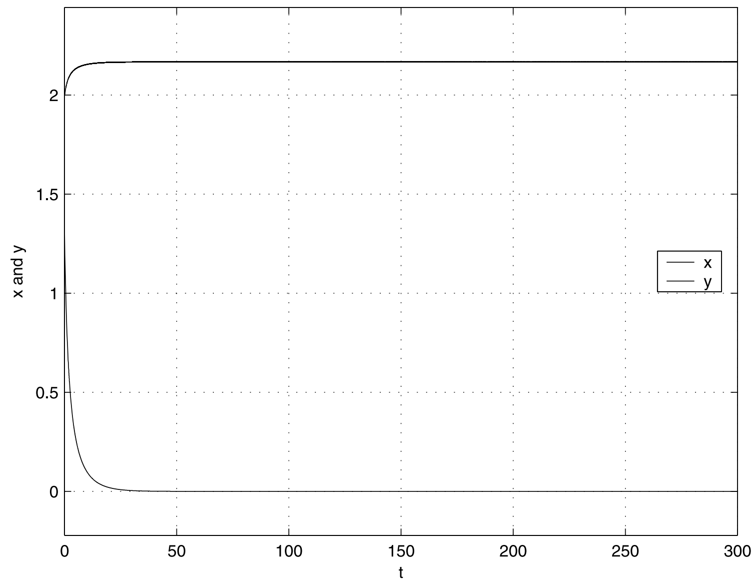

Figure 5 The prey approaches stable state, but the prey approaches extermination at last.

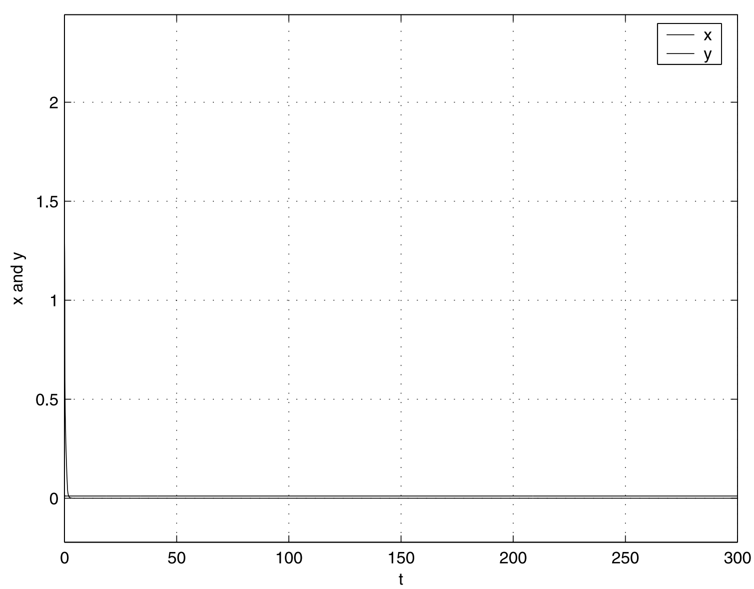

Figure 6 Both the predator and the prey approach extermination at last.

$$\frac{1.5x^2}{1+x^2} - 0.875 - \frac{2x^2(0.4 - \frac{0.2}{x^2})}{1+x^2} + \frac{0.25}{y} = 0. \tag{7.2}$$

We can solve roots of Eqs. (7.1) and (7.2) by the tool of Maple. The roots of Eqs. (7.1) and (7.2) are solved as follows:

$\{x = 5.972747749, y = 1.364775983\}$;

$\{x = 0.2724898191 + 0.8891706524I, y = 0.2709042167 + 0.2807842746I\}$;

$$\{x = 0.4735566438 + 1.677512880I, y = 8.300695365 + 11.19592013I\};$$

$$\{x = -0.1074917926 + 0.6629159229I, y = 0.3616016214 - 0.6650693911I\};$$

$$\{x = -0.2612275652 + 1.185532456I, y = -0.5060530721 - 0.6653290376I\};$$

$$\{x = -0.2612275652 - 1.185532456I, y = -0.5060530721 + 0.6653290376I\};$$

$$\{x = -0.1074917926 - 0.6629159229I, y = 0.3616016214 + 0.6650693911I\};$$

$$\{x = 0.4735566438 - 1.677512880I, y = 8.300695365 - 11.19592013I\};$$

$$\{x = 0.2724898191 - 0.8891706524I, y = 0.2709042167 - 0.2807842746I\}.$$

But only $\{x = 5.972747749, y = 1.364775983\}$ satisfies

$$x_\delta > \sqrt{\frac{q_1}{p_1}} \quad \left(= \frac{\sqrt{2}}{2}\right), \qquad y_\delta > \frac{q_2}{p_2} \quad (= 0.5).$$

So, we can derive that $x_\delta = 5.972747749$, $y_\delta = 1.364775983$. Then, substituting the values of x_δ and y_δ into (5.2), we get $E_{1\delta} = 1.102124642$, $E_{2\delta} = 2.418197544$. So, we can derive that an optimal equilibrium solution is (5.972747749, 1.364775983, 1.102124642, 2.418197544).

8 Conclusion

In this paper, qualitative analysis of a harvested predator-prey system with Holling type III functional response is considered. This work presents analysis of the effect of harvesting efforts on the prey-predator system. We have proved that exactly one stable limit cycle occurs in the system under certain conditions and have proved the global stability of the positive equilibrium. It was also found that it is possible to control the system in such a way that the system approaches a required state, using the efforts E_1 and E_2 as the control. The results we have obtained may be helpful for the fishery managers wishing to maintain a globally sustainable yield.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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