Danh et al. *Advances in Difference Equations* (2019) 2019:255
https://doi.org/10.1186/s13662-019-2191-4

Advances in Difference Equations
a SpringerOpen Journal

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Regularization of an initial inverse problem for a biharmonic equation

Hua Quoc Nam Danh[1,2], Donal O'Regan[3], Van Au Vo[4], Binh Thanh Tran[5] and Can Huu Nguyen[6*]

*Correspondence:
nguyenhuucan@tdtu.edu.vn
[6]Applied Analysis Research Group, Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam
Full list of author information is available at the end of the article

Abstract

In this paper, we consider the problem of finding the initial distribution for the linear inhomogeneous biharmonic equation. The problem is severely ill-posed in the sense of Hadamard. In order to obtain a stable numerical solution, we propose two regularization methods to solve the problem. We show rigourously, with error estimates provided, that the corresponding regularized solutions converge to the true solution strongly in $\mathcal{L}^2$ uniformly with respect to the space coordinate under some *a priori* assumptions on the solution. Finally, in order to increase the significance of the study, numerical results are presented and discussed illustrating the theoretical findings in terms of accuracy and stability.

MSC: 35K99; 47J06; 47H10; 35K05

Keywords: Backward problem; Biharmonic equation; Polyharmonic problem; Regularization method; Error estimate

1 Introduction

In this paper, we consider the non-homogeneous biharmonic equation

$$\begin{cases} \Delta^2 u = \frac{\partial^4 u}{\partial y^4} + 2\frac{\partial^4 u}{\partial y^2 \partial x^2} + \frac{\partial^4 u}{\partial x^4} = \rho(y,x), & \text{in } Q_L := (0,L) \times \Omega, \\ u(y,x) = \Delta u(y,x) = 0, & \text{on } \Sigma_L := (0,L) \times \partial\Omega, \\ u(L,x) = g(x), \quad \frac{\partial u}{\partial y} u(L,x) = 0, & \text{in } \Omega, \\ \Delta u(L,x) = h(x), \quad \frac{\partial \Delta u}{\partial y}(L,x) = 0, & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^d$, $d \geq 1$ is an open bounded domain with a smooth boundary $\partial\Omega$, and the linear source function $\rho \in \mathcal{L}^\infty(0,L;\mathcal{L}^2(\Omega))$. In practice, the data $g,h \in \mathcal{L}^2(\Omega)$ are noisy and are represented by the observation data $g^\alpha, h^\alpha \in \mathcal{L}^2(\Omega)$ satisfying

$$\|g^\alpha - g\|_{\mathcal{L}^2(\Omega)} \leq \alpha, \qquad \|h^\alpha - h\|_{\mathcal{L}^2(\Omega)} \leq \alpha; \tag{1.2}$$

here $\alpha > 0$ is a small positive number representing the level of noise.

There are many papers on different methods for approximating solutions to boundary value problems for elliptic partial differential equations and most are centered on second order equations where maximum principles are used to obtain asymptotic estimates for the error [1–5, 7, 8, 11, 13–15, 17–19]. The theory for elliptic equations of order greater than two is less developed [8] (note that such equations arise in physics and in engineering design and they also appear naturally in many areas of mathematics, including conformal geometry, and nonlinear elasticity [1, 4, 5]).

The prototypical example of a higher-order elliptic operator, well known from the theory of elasticity, is the biharmonic $\Delta^2 = \Delta(\Delta) = \nabla^4$, and a more general example is the polyharmonic operator $\Delta^p = \underbrace{\Delta(\Delta...\Delta)}_{p \text{ times}}$, $p > 2$. The biharmonic equation arises in many engineering applications such as the deformation of thin plates, the motion of fluids, free boundary problems, nonlinear elasticity and for historical details we refer the reader to [2, 3, 7, 14] (for a more elaborate history of the biharmonic problem and the relation with elasticity from an engineering point of view we refer the reader to the survey of Meleshko [11]).

In 1928, Covrant et al. [6] posed a difference analog for the first boundary value problem for the homogeneous biharmonic equation

$$\mathbb{L}u = \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x^2}\right)^2 = 0 \tag{1.3}$$

and proved that the approximate solutions converge to the exact solution as the mesh is refined (however, no estimates for the error were given). In [10], the authors obtained necessary and sufficient conditions for existence of a solution for the biharmonic equation (1.3) in a rectangular domain $[0,\pi] \times [0,L]$ in the space $\mathcal{L}^2(0,\pi)$. In [9] using a nonlocal boundary value problem method, convergence of regularized approximation with a priori parameter choice was proven, provided data noise level tends to zero (however, the authors did not investigate error estimates). The method of nonlocal boundary value problems for second order elliptic equations was used by several authors (see [13, 15, 17–19]). There are many papers on the linear homogeneous case for the biharmonic equation, but, however, very little is known on regularization theory and numerical simulation for the linear inhomogeneous case. Our main aim in this paper is to discuss regularized solutions for problem (1.1). Using the Fourier truncation method introduced in [16], we propose the regularized solution and give an error estimate.

The paper is organized as follows. In Sect. 2, the formulation of the problem and its ill-posed property are given. In Sect. 3, stability estimates are proved under a priori conditions on the solution. Numerical results are presented and discussed in Sect. 4 and, finally, conclusions are summarized in Sect. 5.

2 Preliminaries

2.1 Notations and assumptions

We begin this section by introducing some notations and assumptions that are needed for our analysis in the next sections.

Definition 2.1 Without loss of generality, we assume that $-\Delta$ has the eigenvalues λ_m ($m \in \mathbb{N}^*$):

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \nearrow \infty, \tag{2.1}$$

and the corresponding eigenelements $\xi_m(x)$, which form an orthonormal basis in $\mathcal{L}^2(\Omega)$.

Definition 2.2 (Hilbert scale space, see [12]) The Hilbert scale space $\mathbb{H}^p$, $(p > 0)$ defined by

$$\mathbb{H}^p := \left\{ f \in \mathcal{L}^2(\Omega) : \sum_{m=1}^{\infty} \lambda_m^{2p} \langle f, \xi_m(x) \rangle_{\mathcal{L}^2(\Omega)}^2 \leq \infty \right\}, \tag{2.2}$$

is equipped with the norm defined by

$$\|f\|_{\mathbb{H}^p}^2 = \sum_{m=1}^{\infty} \lambda_m^{2p} \left| \langle f, \xi_m(x) \rangle_{\mathcal{L}^2(\Omega)} \right|^2 \leq \infty. \tag{2.3}$$

For a Hilbert space X, we denote by $\mathcal{L}^p(0, L; X)$ (respectively, $C([0, L]; X)$) the Banach space of measurable (respectively, continuous) functions $f : [0, L] \to X$, such that

$$\|f\|_{\mathcal{L}^p(0,L;X)} = \left(\int_0^L \|f(y)\|_X^p \, dy \right)^{1/p} < \infty, \quad 1 \leq p < \infty,$$

$$\|f\|_{\mathcal{L}^\infty(0,L;X)} = \operatorname*{ess\,sup}_{0 \leq y \leq L} \|f(y)\|_X < \infty, \quad p = \infty,$$

respectively,

$$\|f\|_{C([0,L];X)} = \sup_{0 \leq y \leq L} \|f(y)\|_X < \infty.$$

Throughout this paper, the function ρ is perturbed so as to contain errors in the form of noisy $\rho^\alpha \in \mathcal{L}^\infty(0, L; \mathcal{L}^2(\Omega))$ satisfying

$$\left\| \rho^\alpha - \rho \right\|_{\mathcal{L}^\infty(0,L;\mathcal{L}^2(\Omega))} \leq \alpha. \tag{2.4}$$

2.2 Mild solution and ill-posed of problem (1.1)

The solution to problem (1.1) can be represented in the form of an expansion in the orthogonal series

$$u(y, x) = \sum_{m=1}^{\infty} u_m(y) \xi_m(x), \quad \text{with } u_m(y) = \langle u(y, x), \xi_m(x) \rangle_{\mathcal{L}^2(\Omega)}. \tag{2.5}$$

By considering that the series (2.5) converges and allows a term by term differentiation (the required number of times), we construct a formal solution to the problem. We obtain the problems

$$\begin{cases} u_m^{(4)}(y) - 2\lambda_m u_m''(y) + \lambda_m^2 u_m(y) = \rho_m(y), & y \in (0, L), \\ u_m(L) = g_m, \qquad u_m'(L) = 0, \\ u_m''(L) - \lambda_m u_m(L) = h_m, \qquad u_m'''(L) - \lambda_m u_m'(L) = 0. \end{cases} \tag{2.6}$$

Here g_m, h_m and $\rho_m(y)$ are Fourier coefficients of the expansion according to the orthonormal basis $\{\xi_m(x)\}_{m \in \mathbb{N}^*}$ of the functions $g(x)$, $h(x)$ and $\rho(y, x)$, respectively:

$$g(x) = \sum_{m=1}^{\infty} g_m \xi_m(x), \qquad h(x) = \sum_{m=1}^{\infty} h_m \xi_m(x),$$

$$\rho(y,x) = \sum_{m=1}^{\infty} \rho_m(y)\xi_m(x).$$

By direct calculation, the solution to problem (2.6) has the form

$$\begin{aligned} u_m(y) &= \cosh\bigl(\sqrt{\lambda_m}(L-y)\bigr)g_m + \frac{(L-y)\sinh(\sqrt{\lambda_m}(L-y))}{2\sqrt{\lambda_m}}h_m \\ &\quad + \int_y^L \frac{(\sigma-y)\cosh(\sqrt{\lambda_m}(\sigma-y))}{2\lambda_m}\rho_m(\sigma)\,d\sigma \\ &\quad + \int_y^L \frac{\sinh(\sqrt{\lambda_m}(\sigma-y))}{2\lambda_m\sqrt{\lambda_m}}\rho_m(\sigma)\,d\sigma. \end{aligned} \tag{2.7}$$

Substituting the result into (2.5), we obtain the formal solution to problem (1.1).

Next, we give an example which shows that the solution of problem (1.1) does not depend continuously on the final data.

Example For any $j \in \mathbb{N}^*$, let $\widetilde{g}_j$, $\widetilde{h}_j$ and $\widetilde{\rho}_j$ be as follows:

$$\begin{cases} \widetilde{g}_j(x) := \frac{\xi_j(x)}{\sqrt{\lambda_j}}, \qquad \widetilde{h}_j(x) = 0, \\ \widetilde{\rho}(y,x) := \frac{e^{-\lambda_j L}}{L}\xi_j(x), \quad \forall y \in [0,L]. \end{cases} \tag{2.8}$$

Let $\widetilde{u}_j$ be the solution of (1.1) with $\widetilde{g}_j$, $\widetilde{h}_j$ and $\widetilde{\rho}_j$. One has

$$\begin{aligned} &\bigl\|\widetilde{u}_j(y,\cdot)\bigr\|_{\mathcal{L}^2(\Omega)} \\ &\quad \ge \Biggl\| \sum_{m=1}^{\infty} \cosh\bigl(\sqrt{\lambda_m}(L-y)\bigr)\bigl\langle \widetilde{g}(x),\xi_m(x)\bigr\rangle_{\mathcal{L}^2(\Omega)}\xi_m(x)\Biggr\|_{\mathcal{L}^2(\Omega)} \\ &\qquad - \Biggl\| \sum_{m=1}^{\infty} \int_y^L \frac{(\sigma-y)\cosh(\sqrt{\lambda_m}(\sigma-y))}{2\lambda_m}\bigl\langle \widetilde{\rho}(\sigma,x),\xi_m(x)\bigr\rangle_{\mathcal{L}^2(\Omega)}\,d\sigma\,\xi_m(x)\Biggr\|_{\mathcal{L}^2(\Omega)} \\ &\qquad - \Biggl\| \sum_{m=1}^{\infty} \int_y^L \frac{\sinh(\sqrt{\lambda_m}(\sigma-y))}{2\lambda_m\sqrt{\lambda_m}}\bigl\langle \widetilde{\rho}(\sigma,x),\xi_m(x)\bigr\rangle_{\mathcal{L}^2(\Omega)}\,d\sigma\,\xi_m(x)\Biggr\|_{\mathcal{L}^2(\Omega)} \\ &\quad = \biggl\| \cosh\bigl(\sqrt{\lambda_j}(L-y)\bigr)\frac{\xi_j(x)}{\sqrt{\lambda_j}}\biggr\|_{\mathcal{L}^2(\Omega)} \\ &\qquad - \biggl\| \int_y^L \frac{(\sigma-y)\cosh(\sqrt{\lambda_j}(\sigma-y))}{2\lambda_j}\frac{e^{-\lambda_j L}}{L}\,d\sigma\,\xi_j(x)\biggr\|_{\mathcal{L}^2(\Omega)} \\ &\qquad - \Biggl\| \sum_{j=1}^{\infty} \int_y^L \frac{\sinh(\sqrt{\lambda_j}(\sigma-y))}{2\lambda_j\sqrt{\lambda_j}}\frac{e^{-\lambda_j L}}{L}\,d\sigma\,\xi_j(x)\Biggr\|_{\mathcal{L}^2(\Omega)}. \end{aligned}$$

Since $z > 0$, we have

$$\frac{e^z}{2} \le \cosh(z) \le e^z, \qquad \frac{e^z-1}{2} \le \sinh(z) \le e^z, \tag{2.9}$$

and it follows that

$$\begin{aligned}\big\|\widetilde{u}_j(y,\cdot)\big\|_{\mathcal{L}^2(\Omega)} &\geq \frac{e^{(L-y)\sqrt{\lambda_j}}}{2\sqrt{\lambda_j}} - \int_y^L \left(\frac{1}{2\lambda_j} + \frac{1}{2L\lambda_j\sqrt{\lambda_j}}\right) d\sigma \\ &= \frac{e^{(L-y)\sqrt{\lambda_j}}}{2\sqrt{\lambda_j}} - \left(\frac{L}{2\lambda_j} + \frac{1}{2\lambda_j\sqrt{\lambda_j}}\right) \\ &\geq \frac{e^{(L-y)\sqrt{\lambda_j}}}{2\sqrt{\lambda_j}} - \left(\frac{L}{2\lambda_j} + \frac{1}{2\lambda_j\sqrt{\lambda_j}}\right), \quad 0 \leq y < L. \end{aligned} \tag{2.10}$$

Hence, we deduce that

$$\begin{aligned}\|\widetilde{u}_j\|_{\mathcal{L}^\infty(0,L;\mathcal{L}^2(\Omega))} &= \operatorname*{ess\,sup}_{0\leq y\leq L}\big\|\widetilde{u}(y,\cdot)\big\|_{\mathcal{L}^2(\Omega)} \\ &\geq \frac{e^{L\sqrt{\lambda_j}}}{2\sqrt{\lambda_j}} - \left(\frac{L}{2\lambda_j} + \frac{1}{2\lambda_j\sqrt{\lambda_j}}\right), \end{aligned} \tag{2.11}$$

as $j \to \infty$, we see that (for $0 \leq y < L$)

$$\lim_{j\to\infty}\left(\|\widetilde{g}\|_{\mathcal{L}^2(\Omega)} + \|\widetilde{h}\|_{\mathcal{L}^2(\Omega)}\right) = 0, \tag{2.12}$$

$$\begin{aligned}\lim_{j\to\infty}\|\widetilde{u}_j\|_{\mathcal{L}^\infty(0,L;\mathcal{L}^2(\Omega))} &\geq \lim_{j\to\infty}\left[\frac{e^{L\sqrt{\lambda_j}}}{2\sqrt{\lambda_j}} - \left(\frac{L}{2\lambda_j} + \frac{1}{2\lambda_j\sqrt{\lambda_j}}\right)\right] \\ &= +\infty. \end{aligned} \tag{2.13}$$

Thus our problem is ill-posed in the Hadamard sense in the $\mathcal{L}^2(\Omega)$-norm.

3 Regularization and error estimate

In order to obtain stable numerical solutions, we propose two regularization methods to solve the problem. As was shown in the previous section, for the linear biharmonic problem (1.1), its solution (true solution) can be represented as an integral equation which contains some instability terms. Indeed, we find that the four functions

$$\cosh(\sqrt{\lambda_m}z), \sinh(\sqrt{\lambda_m}z), \quad z > 0,$$

in (2.7) are unbounded, as functions of the variable m, for $y \in (0,L)$. Consequently, small errors in high frequency components can blow up and completely destroy the solution for $y \in (0,L)$. A natural idea to stabilize the problem is to eliminate all high frequencies (truncation method) or to replace them by a bounded approximation (quasi-boundary value method). We introduce two bounded operators as follows:

- For $f \in C([0,L];\mathcal{L}^2(\Omega))$, we define

$$\widehat{\mathbf{Q}}^{\gamma(\alpha)}f(y,x) = \sum_{m=1}^{\infty}\mathcal{I}_{L,m}^{\gamma(\alpha)}\big\langle f(y,x),\xi_m(x)\big\rangle_{\mathcal{L}^2(\Omega)}\xi_m(x), \tag{3.1}$$

where

$$\mathcal{I}_{L,m}^{\gamma(\alpha)} = \left(1 + \gamma(\alpha)\sqrt{\lambda_m} e^{\sqrt{\lambda_m} L}\right)^{-1}, \quad \forall m \in \mathbb{N}^*,$$

and $\gamma(\alpha) > 0$ is the parameter regularization which satisfies

$$\lim_{\alpha\to 0^+} \gamma(\alpha) = 0. \tag{3.2}$$

- For $f \in C([0,L];\mathcal{L}^2(\Omega))$, we define

$$\widehat{\mathbf{B}}^{M_\alpha} f(y,x) = \sum_{m\in\mathbb{T}_\alpha^\dagger} \langle f(y,x), \xi_m(x)\rangle_{\mathcal{L}^2(\Omega)} \xi_m(x), \tag{3.3}$$

where

$$\mathbb{T}_\alpha^\dagger := \{m \in \mathbb{N}^* | \lambda_m \le M_\alpha\},$$

and $M_\alpha > 0$ is the parameter regularization which satisfies

$$\lim_{\alpha\to 0^+} M_\alpha = +\infty. \tag{3.4}$$

3.1 The main results

3.1.1 Result for quasi-boundary value method

Let us consider the following well-posed problem:

$$\begin{cases} \Delta^2 u^{\gamma(\alpha)} = \widehat{\mathbf{Q}}^{\gamma(\alpha)} \rho^\alpha(y,x), & & \text{in } Q_L, \\ u^{\gamma(\alpha)} = \Delta u^{\gamma(\alpha)} = 0, & & \text{on } \Sigma_L, \\ u^{\gamma(\alpha)}(L,x) = \widehat{\mathbf{Q}}^{\gamma(\alpha)} g^\alpha(x), & \frac{\partial u^\alpha}{\partial y}(L,x) = 0, & \text{in } \Omega, \\ \Delta u^{\gamma(\alpha)}(L,x) = \widehat{\mathbf{Q}}^{\gamma(\alpha)} h^\alpha(x), & \frac{\partial \Delta u^{\gamma(\alpha)}}{\partial y}(L,x) = 0, & \text{in } \Omega. \end{cases} \tag{3.5}$$

Theorem 3.1 ((QBV) method) *Assume that the exact solution u of (1.1) satisfies*

$$\|u\|_{\mathcal{L}^\infty(0,L;\mathbb{H}^{p+1}(\Omega))} \le E_1, \tag{3.6}$$

where p, E_1 are positive constants. Choose $\gamma(\alpha) \in (0,1)$ such that

$$\begin{cases} \lim_{\alpha\to 0^+} \gamma(\alpha) = 0, \\ \lim_{\alpha\to 0^+} \frac{\alpha}{\gamma(\alpha)} = \textit{finite}. \end{cases} \tag{3.7}$$

Then the estimate

$$\left\| u^{\gamma(\alpha)}(y,\cdot) - u(y,\cdot)\right\|_{\mathcal{L}^2(\Omega)} \le \frac{C(\lambda_1, L)}{\log(\frac{L}{\gamma(\alpha)})}\left(\frac{\alpha}{\gamma(\alpha)} + E_1\right) \tag{3.8}$$

holds.

Remark 3.1 From condition (3.7), if we choose $\gamma(\alpha) = \alpha^k$ for some $k \in (0,1)$, then the error estimate in (3.8) is of order $\frac{\alpha^{1-k}}{\log(\frac{L}{\alpha^k})}$, which tends to zero as $\alpha \to 0^+$.

3.1.2 Result for truncation method

Next, we propose a second regularized solution u^α solving the following problem:

$$\begin{cases} \Delta^2 u^\alpha = \widehat{\mathbf{B}}^{M_\alpha} \rho^\alpha(y,x), & \text{in } Q_L, \\ u^\alpha = \Delta u^\alpha = 0, & \text{on } \Sigma_L, \\ u^\alpha(L,x) = \widehat{\mathbf{B}}^{M_\alpha} g^\alpha(x), \quad \frac{\partial u^\alpha}{\partial y}(L,x) = 0, & \text{in } \Omega, \\ \Delta u^\alpha(L,x) = \widehat{\mathbf{B}}^{M_\alpha} h^\alpha(x), \quad \frac{\partial \Delta u^\alpha}{\partial y}(L,x) = 0, & \text{in } \Omega. \end{cases} \tag{3.9}$$

Theorem 3.2 ((*TR*) *method*). *Suppose that the problem* (1.1) *has a solution u satisfying*

$$\|u\|_{\mathcal{L}^\infty(0,L;\mathbb{H}^p(\Omega))} \le E_2, \tag{3.10}$$

for some known constant $E_2 > 0$. Assume that we can choose $M_\alpha > 0$ such that

$$\begin{cases} \lim_{\alpha\to 0^+} M_\alpha = +\infty, \\ \lim_{\alpha\to 0^+} \alpha e^{L\sqrt{M_\alpha}} = 0. \end{cases} \tag{3.11}$$

Then

$$\big\|u^\alpha(y,\cdot) - u(y,\cdot)\big\|_{\mathcal{L}^2(\Omega)} \le Ce^{\sqrt{M_\alpha}L}\alpha + \frac{E_2}{M_\alpha^p}. \tag{3.12}$$

Remark 3.2 Let us choose $M_\alpha = \frac{1}{L^2}\log^2(\alpha^{-\ell})$, for some $\ell \in (0,1)$. Then the hypothesis

$$\lim_{\alpha\to 0^+} \alpha e^{L\sqrt{M_\alpha}} = 0,$$

is fulfilled and (3.12) is of order

$$\max\left\{\alpha^{1-\ell}; \frac{1}{\log^{2p}(\alpha^{-\ell})}\right\}, \quad p \in \mathbb{N}^*. \tag{3.13}$$

Theorem 3.3 (Estimate $\mathbb{H}^p$) *Let us choose $M_\alpha > 0$ such that $\lim_{\alpha\to 0^+} M_\alpha = \infty$ and*

$$\lim_{\alpha\to 0^+} \alpha M_\alpha^p e^{\sqrt{M_\alpha}L} < \infty. \tag{3.14}$$

Assume further that the problem (1.1) *has a unique exact solution u satisfying $u \in \mathcal{L}^\infty(0,L;\mathbb{H}^{p+q})$, for $p,q > 0$. Then, for all $y \in [0,L]$, we have*

$$\begin{aligned} &\big\|u^\alpha(y,\cdot) - u(y,\cdot)\big\|_{\mathbb{H}^p(\Omega)} \\ &\quad \le C(\lambda_1,L) M_\alpha^p e^{\sqrt{M_\alpha}L}\alpha + M_\alpha^{-q}\|u\|_{\mathcal{L}^\infty(0,L;\mathbb{H}^{p+q}(\Omega))}. \end{aligned} \tag{3.15}$$

Remark 3.3 Let any $\chi \in (0,1)$. We choose

$$M_\alpha = \frac{\log^2(\alpha^{-\chi})}{L^2} \longrightarrow \infty, \quad \text{as } \alpha \text{ goes to } 0^+.$$

Then condition (3.14) is satisfied as $\alpha \to 0^+$ and the right-hand side of (3.15) is of order

$$\max\left\{\log^{2p}\left(\alpha^{-\chi}\right)\alpha^{1-\chi}; \frac{1}{\log^{2q}(\alpha^{-\chi})}\right\}, \quad p,q>0. \tag{3.16}$$

3.2 Proof of Theorem 3.1

Problem (3.5) can be rewritten as the following integral equation:

$$\begin{aligned} u^{\gamma(\alpha)}(y,x) &= \sum_{m=1}^{\infty}\left[\cosh^{\gamma(\alpha)}\left(\sqrt{\lambda_m}(L-y)\right)g_m^\alpha\right]\xi_m(x) \\ &+ \sum_{m=1}^{\infty}\left[\frac{(L-y)\sinh^{\gamma(\alpha)}(\sqrt{\lambda_m}(L-y))}{2\sqrt{\lambda_m}}h_m^\alpha\right]\xi_m(x) \\ &+ \sum_{m=1}^{\infty}\left[\int_y^L \frac{(\sigma-y)\cosh^{\gamma(\alpha)}(\sqrt{\lambda_m}(\sigma-y))}{2\lambda_m}\rho_m^\alpha(\sigma)\,d\sigma\right]\xi_m(x) \\ &+ \sum_{m=1}^{\infty}\left[\int_y^L \frac{\sinh^{\gamma(\alpha)}(\sqrt{\lambda_m}(\sigma-y))}{2\lambda_m\sqrt{\lambda_m}}\rho_m^\alpha(\sigma)\,d\sigma\right]\xi_m(x), \end{aligned} \tag{3.17}$$

where we define the operators for $z>0$

$$\cosh^{\gamma(\alpha)}(\sqrt{\lambda_m}z) := \mathcal{I}_{L,m}^{\gamma(\alpha)}\cosh(\sqrt{\lambda_m}z), \tag{3.18}$$

$$\sinh^{\gamma(\alpha)}(\sqrt{\lambda_m}z) := \mathcal{I}_{L,m}^{\gamma(\alpha)}\sinh(\sqrt{\lambda_m}z), \tag{3.19}$$

and

$$g_m^\alpha = \left\langle g^\alpha(x), \xi_m(x)\right\rangle_{\mathcal{L}^2(\Omega)}, \qquad h_m^\alpha = \left\langle h^\alpha(x), \xi_m(x)\right\rangle_{\mathcal{L}^2(\Omega)},$$
$$\rho_m^\alpha(\sigma) = \left\langle \rho^\alpha(x,\sigma), \xi_m(x)\right\rangle_{\mathcal{L}^2(\Omega)}.$$

First, we shall prove some inequalities which will be used in the main part of our proof. The following lemma is proved directly (we omit the proof).

Lemma 3.1 *For $z \ge 0$, we have*

$$\text{(a)}\quad \left|\cosh(\sqrt{\lambda_m}z)\right| \le e^{\sqrt{\lambda_m}z}, \tag{3.20a}$$

$$\text{(b)}\quad \left|\sinh(\sqrt{\lambda_m}z)\right| \le e^{\sqrt{\lambda_m}z}. \tag{3.20b}$$

We need the following lemma.

Lemma 3.2 *For $z \in [0,L]$. The following estimates hold*

$$\text{(a)}\quad \left|\cosh^{\gamma(\alpha)}(\sqrt{\lambda_m}z)\right| \le \left[\frac{L}{\gamma(\alpha)\log(\frac{L}{\gamma(\alpha)})}\right]^{\frac{z}{L}}, \tag{3.21a}$$

$$\text{(b)}\quad \left|\sinh^{\gamma(\alpha)}(\sqrt{\lambda_m}z)\right| \le \left[\frac{L}{\gamma(\alpha)\log(\frac{L}{\gamma(\alpha)})}\right]^{\frac{z}{L}}. \tag{3.21b}$$

Proof (a) We have

$$
\begin{aligned}
\left|\cosh^{\gamma(\alpha)}(\sqrt{\lambda_m}z)\right| &= \left|\mathcal{I}_{L,m}^{\gamma(\alpha)}\right|\left|\cosh(\sqrt{\lambda_m}z)\right|, \\
&\le \frac{e^{\sqrt{\lambda_m}z}}{1+\gamma(\alpha)\sqrt{\lambda_m}e^{\sqrt{\lambda_m}L}} = \frac{e^{-\sqrt{\lambda_m}(L-z)}}{\gamma(\alpha)\sqrt{\lambda_m}+e^{-\sqrt{\lambda_m}L}} \\
&= \frac{e^{-\sqrt{\lambda_m}y}}{(\gamma(\alpha)\sqrt{\lambda_m}+e^{-\sqrt{\lambda_m}L})^{\frac{L-z}{L}}(\gamma(\alpha)\sqrt{\lambda_m}+e^{-\sqrt{\lambda_m}L})^{\frac{z}{L}}} \\
&\le \frac{1}{(\gamma(\alpha)\sqrt{\lambda_m}+e^{-\sqrt{\lambda_m}L})^{\frac{z}{L}}}.
\end{aligned} \tag{3.22}
$$

On other hand, it is easy to see that

$$
f(\nu) = \frac{1}{c\nu + e^{-\nu L}} \le \frac{L}{c\log(\frac{L}{c})},
$$

for $0 < c < Le$. Hence if $\gamma(\alpha) < Le$, then we obtain

$$
\frac{1}{\gamma(\alpha)\sqrt{\lambda_m}+e^{-\sqrt{\lambda_m}L}} \le \frac{L}{\gamma(\alpha)\log(\frac{L}{\gamma(\alpha)})}. \tag{3.23}
$$

It follows from (3.22) that

$$
\frac{1}{(\gamma(\alpha)\sqrt{\lambda_m}+e^{-\sqrt{\lambda_m}L})^{\frac{z}{L}}} \le \left[\frac{L}{\gamma(\alpha)\log(\frac{L}{\gamma(\alpha)})}\right]^{\frac{z}{L}}. \tag{3.24}
$$

The proof of (b) is similar. This completes the proof of the lemma. □

We are now in a position to prove the theorem.

Proof of Theorem 3.1 Using the triangle inequality, we have

$$
\begin{aligned}
&\left\|u^{\gamma(\alpha)}(y,\cdot) - u(y,\cdot)\right\|_{\mathcal{L}^2(\Omega)} \\
&\quad \le \underbrace{\left\|u^{\gamma(\alpha)}(y,\cdot) - \widehat{\mathbf{Q}}^{\gamma(\alpha)}u(y,\cdot)\right\|_{\mathcal{L}^2(\Omega)}}_{\widehat{\mathcal{A}}^{\alpha}} + \underbrace{\left\|\widehat{\mathbf{Q}}^{\gamma(\alpha)}u(y,\cdot) - u(\cdot,y)\right\|_{\mathcal{L}^2(\Omega)}}_{\widehat{\mathcal{B}}^{\alpha}}.
\end{aligned} \tag{3.25}
$$

We observe that

$$
\begin{aligned}
\widehat{\mathbf{Q}}^{\gamma(\alpha)}u(y,x) &= \sum_{m=1}^{\infty}\left[\cosh^{\gamma(\alpha)}\left(\sqrt{\lambda_m}(L-y)\right)g_m\right]\xi_m(x) \\
&\quad + \sum_{m=1}^{\infty}\left[\frac{(L-y)\sinh^{\gamma(\alpha)}(\sqrt{\lambda_m}(L-y))}{2\sqrt{\lambda_m}}h_m\right]\xi_m(x) \\
&\quad + \sum_{m=1}^{\infty}\left[\int_y^L \frac{(\sigma-y)\cosh^{\gamma(\alpha)}(\sqrt{\lambda_m}(\sigma-y))}{2\lambda_m}\rho_m(\sigma)\,d\sigma\right]\xi_m(x) \\
&\quad + \sum_{m=1}^{\infty}\left[\int_y^L \frac{\sinh^{\gamma(\alpha)}(\sqrt{\lambda_m}(\sigma-y))}{2\lambda_m\sqrt{\lambda_m}}\rho_m(\sigma)\,d\sigma\right]\xi_m(x).
\end{aligned} \tag{3.26}
$$

We first estimate the term $\widetilde{\mathcal{A}^\alpha}$. Combining with (2.7) and (3.17) we obtain

$$\begin{aligned}
&u^{\gamma(\alpha)}(y,x) - \widehat{\mathbf{Q}}^{\gamma(\alpha)}u(y,x) \\
&= \sum_{m=1}^{\infty}\bigl[\cosh^{\gamma(\alpha)}\bigl(\sqrt{\lambda_m}(L-y)\bigr)\bigl(g_m^\alpha - g_m\bigr)\bigr]\xi_m(x) \\
&\quad + \sum_{m=1}^{\infty}\biggl[\frac{(L-y)\sinh^{\gamma(\alpha)}(\sqrt{\lambda_m}(L-y))}{2\sqrt{\lambda_m}}\bigl(h_m^\alpha - h_m\bigr)\biggr]\xi_m(x) \\
&\quad + \sum_{m=1}^{\infty}\biggl[\int_y^L \frac{(\sigma-y)\cosh^{\gamma(\alpha)}(\sqrt{\lambda_m}(\sigma-y))}{2\lambda_m}\bigl(\rho_m^\alpha(\sigma) - \rho_m(\sigma)\bigr)\,d\sigma\biggr]\xi_m(x) \\
&\quad + \sum_{m=1}^{\infty}\biggl[\int_y^L \frac{\sinh^{\gamma(\alpha)}(\sqrt{\lambda_m}(\sigma-y))}{2\lambda_m\sqrt{\lambda_m}}\bigl(\rho_m^\alpha(\sigma) - \rho_m(\sigma)\bigr)\,d\sigma\biggr]\xi_m(x).
\end{aligned} \tag{3.27}$$

From Parseval's relation we obtain

$$\begin{aligned}
\bigl|\widetilde{\mathcal{A}^\alpha}\bigr|^2 &= 4\sum_{m=1}^{\infty}\bigl|\cosh^{\gamma(\alpha)}\bigl(\sqrt{\lambda_m}(L-y)\bigr)\bigr|^2\bigl|g_m^\alpha - g_m\bigr|^2 \\
&\quad + 4\sum_{m=1}^{\infty}\biggl|\frac{(L-y)\sinh^{\gamma(\alpha)}(\sqrt{\lambda_m}(L-y))}{2\sqrt{\lambda_m}}\biggr|^2\bigl|h_m^\alpha - h_m\bigr|^2 \\
&\quad + 4\sum_{m=1}^{\infty}\biggl|\int_y^L \frac{(\sigma-y)\cosh^{\gamma(\alpha)}(\sqrt{\lambda_m}(\sigma-y))}{2\lambda_m}\bigl(\rho_m^\alpha(\sigma) - \rho_m(\sigma)\bigr)\,d\sigma\biggr|^2 \\
&\quad + 4\sum_{m=1}^{\infty}\biggl|\int_y^L \frac{\sinh^{\gamma(\alpha)}(\sqrt{\lambda_m}(\sigma-y))}{2\lambda_m\sqrt{\lambda_m}}\bigl(\rho_m^\alpha(\sigma) - \rho_m(\sigma)\bigr)\,d\sigma\biggr|^2 \\
&= A_1^\alpha + A_2^\alpha + A_3^\alpha + A_4^\alpha.
\end{aligned} \tag{3.28}$$

Using (3.21a) we have

$$\begin{aligned}
\bigl|A_1^\alpha\bigr| &= 4\sum_{m=1}^{\infty}\bigl|\cosh^{\gamma(\alpha)}\bigl(\sqrt{\lambda_m}(L-y)\bigr)\bigr|^2\bigl|g_m^\alpha - g_m\bigr|^2 \\
&\le 4\biggl[\frac{L}{\gamma(\alpha)\log(\frac{L}{\gamma(\alpha)})}\biggr]^{\frac{2L-2y}{L}}\sum_{m=1}^{\infty}\bigl|g_m^\alpha - g_m\bigr|^2 \\
&\le 4\biggl[\frac{L}{\gamma(\alpha)\log(\frac{L}{\gamma(\alpha)})}\biggr]^{\frac{2L-2y}{L}}\bigl\|g^\alpha - g\bigr\|_{\mathcal{L}^2(\Omega)}^2 \\
&\le 4\biggl[\frac{L}{\gamma(\alpha)\log(\frac{L}{\gamma(\alpha)})}\biggr]^2\alpha^2,
\end{aligned} \tag{3.29}$$

where we have used the elementary inequality $e^z \ge z$, for $z > 0$ which leads to

$$\frac{L}{\gamma(\alpha)\log(\frac{L}{\gamma(\alpha)})} > 1,$$

and thus it follows that

$$\left[\frac{L}{\gamma(\alpha)\log(\frac{L}{\gamma(\alpha)})}\right]^{\frac{L-y}{L}} \le \frac{L}{\gamma(\alpha)\log(\frac{L}{\gamma(\alpha)})}, \quad 0 \le y < L.$$

It follows from (3.21b) that

$$\begin{aligned}
|A_2^\alpha| &= 4\sum_{m=1}^{\infty}\left|\frac{(L-y)\sinh^{\gamma(\alpha)}(\sqrt{\lambda_m}(L-y))}{2\sqrt{\lambda_m}}\right|^2 |h_m^\alpha - h_m|^2 \\
&\le \frac{L^2}{\lambda_1}\left[\frac{L}{\gamma(\alpha)\log(\frac{L}{\gamma(\alpha)})}\right]^{\frac{2L-2y}{L}} \sum_{m=1}^{\infty}|h_m^\alpha - h_m|^2 \\
&\le \frac{L^2}{\lambda_1}\left[\frac{L}{\gamma(\alpha)\log(\frac{L}{\gamma(\alpha)})}\right]^{\frac{2L-2y}{L}} \|h^\alpha - h\|^2_{\mathcal{L}^2(\Omega)} \\
&\le \frac{L^2}{\lambda_1}\left[\frac{L}{\gamma(\alpha)\log(\frac{L}{\gamma(\alpha)})}\right]^2 \alpha^2.
\end{aligned} \tag{3.30}$$

Using Hölder's inequality, (3.21a) and (2.4), one has

$$\begin{aligned}
|A_3^\alpha| &= 4\sum_{m=1}^{\infty}\left|\int_y^L \frac{(\sigma-y)\cosh^{\gamma(\alpha)}(\sqrt{\lambda_m}(\sigma-y))}{2\lambda_m}\big(\rho_m^\alpha(\sigma) - \rho_m(\sigma)\big)\,d\sigma\right|^2 \\
&\le 4\sum_{m=1}^{\infty}(L-y)\int_y^L \left|\frac{(\sigma-y)\cosh^{\gamma(\alpha)}(\sqrt{\lambda_m}(\sigma-y))}{2\lambda_m}\right|^2 |\rho_m^\alpha(\sigma) - \rho_m(\sigma)|^2\,d\sigma \\
&\le \frac{L^3}{\lambda_1^2}\int_y^L \left[\frac{L}{\gamma(\alpha)\log(\frac{L}{\gamma(\alpha)})}\right]^{\frac{2\sigma-2y}{L}} \sum_{m=1}^{\infty}|\rho_m^\alpha(\sigma) - \rho_m(\sigma)|^2\,d\sigma \\
&\le \frac{L^4}{\lambda_1^2}\left[\frac{L}{\gamma(\alpha)\log(\frac{L}{\gamma(\alpha)})}\right]^2 \|\rho^\alpha - \rho\|^2_{\mathcal{L}^\infty(0,L;\mathcal{L}^2(\Omega))} \\
&\le \frac{L^4}{\lambda_1^2}\left[\frac{L}{\gamma(\alpha)\log(\frac{L}{\gamma(\alpha)})}\right]^2 \alpha^2.
\end{aligned} \tag{3.31}$$

Thus from (2.4) and (3.21b), by the Hölder inequality, we have

$$\begin{aligned}
|A_4^\alpha| &= 4\sum_{m=1}^{\infty}\left|\int_y^L \frac{\sinh^{\gamma(\alpha)}(\sqrt{\lambda_m}(\sigma-y))}{2\lambda_m\sqrt{\lambda_m}}\big(\rho_m^\alpha(\sigma) - \rho_m(\sigma)\big)\,d\sigma\right|^2 \\
&\le 4\sum_{m=1}^{\infty}(L-y)\int_y^L \left|\frac{\sinh^{\gamma(\alpha)}(\sqrt{\lambda_m}(\sigma-y))}{2\lambda_m\sqrt{\lambda_m}}\right|^2 |\rho_m^\alpha(\sigma) - \rho_m(\sigma)|^2\,d\sigma \\
&\le \frac{L}{\lambda_1^3}\left[\frac{L}{\gamma(\alpha)\log(\frac{L}{\gamma(\alpha)})}\right]^2 \|\rho^\alpha - \rho\|^2_{\mathcal{L}^\infty(0,L;\mathcal{L}^2(\Omega))} \\
&\le \frac{L^2}{\lambda_1^3}\left[\frac{L}{\gamma(\alpha)\log(\frac{L}{\gamma(\alpha)})}\right]^2 \alpha^2.
\end{aligned} \tag{3.32}$$

Combining (3.28)–(3.32) yields

$$\big|\widetilde{\mathcal{A}^{\alpha}}\big| = \frac{C(\lambda_1,L)}{\log(\frac{L}{\gamma(\alpha)})}\frac{\alpha}{\gamma(\alpha)}, \tag{3.33}$$

where $C(\lambda_1,L)$ is a positive constant that depends on λ_1, L but it is independent of y and m. Next we have

$$\widehat{\mathbf{Q}}^{\gamma(\alpha)}u(y,x) - u(y,x) = \sum_{m=1}^{\infty}\big[\mathcal{I}_{L,m}^{\gamma(\alpha)}u_m(y) - u_m(y)\big]\xi_m(x).$$

It follows from Parseval's relation that

$$\begin{aligned}
\big|\widetilde{\mathcal{B}^{\alpha}}\big|^2 &= \big\|\widehat{\mathbf{Q}}^{\gamma(\alpha)}u(y,\cdot) - u(y,\cdot)\big\|_{\mathcal{L}^2(\Omega)}^2 = \sum_{m=1}^{\infty}\big|\mathcal{I}_{L,m}^{\gamma(\alpha)}u_m(y) - u_m(y)\big|^2 \\
&= \sum_{m=1}^{\infty}\big|1-\mathcal{I}_{L,m}^{\gamma(\alpha)}\big|^2\big|u_m(y)\big|^2 \\
&= \sum_{m=1}^{\infty}\bigg|\frac{\gamma(\alpha)\sqrt{\lambda_m}e^{\sqrt{\lambda_m}L}}{1+\gamma(\alpha)\sqrt{\lambda_m}e^{\sqrt{\lambda_m}L}}\bigg|^2\big|u_m(y)\big|^2 \\
&= \sum_{m=1}^{\infty}\gamma^2(\alpha)\bigg|\frac{1}{\gamma(\alpha)\sqrt{\lambda_m}+e^{-\sqrt{\lambda_m}L}}\bigg|^2\lambda_m\big|u_m(y)\big|^2.
\end{aligned}$$

Using inequality (3.23), we get

$$\begin{aligned}
\big|\widetilde{\mathcal{B}^{\alpha}}\big|^2 &\le \frac{\gamma^2(\alpha)}{\lambda_1^{1+2p}}\bigg[\frac{L}{\gamma(\alpha)\log(\frac{L}{\gamma(\alpha)})}\bigg]^2\sum_{m=1}^{\infty}\lambda_m^{2(1+p)}\big|u_m(y)\big|^2 \\
&\le \frac{\gamma^2(\alpha)}{\lambda_1^{1+2p}}\bigg[\frac{L}{\gamma(\alpha)\log(\frac{L}{\gamma(\alpha)})}\bigg]^2\|u\|_{\mathcal{L}^{\infty}(0,L;\mathbb{H}^{p+1}(\Omega))}^2 \\
&\le \bigg[\frac{C(\lambda_1,L)E_1}{\log(\frac{L}{\gamma(\alpha)})}\bigg]^2,
\end{aligned} \tag{3.34}$$

for $C(\lambda_1,L)$ a positive constant which depends on L and λ_1. Hence, we get

$$\big|\widetilde{\mathcal{B}^{\alpha}}\big| \le \frac{C(\lambda_1,L)E_1}{\log(\frac{L}{\gamma(\alpha)})}. \tag{3.35}$$

Combining (3.25), (3.33) and (3.35), we deduce that

$$\big\|u^{\gamma(\alpha)}(y,\cdot) - u(y,\cdot)\big\|_{\mathcal{L}^2(\Omega)} \le \frac{C(\lambda_1,L)}{\log(\frac{L}{\gamma(\alpha)})}\frac{\alpha}{\gamma(\alpha)} + \frac{C(\lambda_1,L)E_1}{\log(\frac{L}{\gamma(\alpha)})}, \tag{3.36}$$

which leads to (3.8). The proof of Theorem 3.1 is completed. □

3.3 Proof of Theorem 3.2

It is easy to verify the following result.

Lemma 3.3 *For $z \geq 0$ and $\lambda_m \leq M^\alpha$, we have*

$$\text{(a)} \quad \left|\cosh(\sqrt{\lambda_m} z)\right| \leq e^{\sqrt{M^\alpha} z}, \tag{3.37a}$$

$$\text{(b)} \quad \left|\sinh(\sqrt{\lambda_m} z)\right| \leq e^{\sqrt{M^\alpha} z}. \tag{3.37b}$$

The solution of the regularized problem (3.9) is given by

$$\begin{aligned} u^\alpha(y,x) &= \sum_{m \in \mathbb{T}_\alpha^\dagger} \left[\cosh\big(\sqrt{\lambda_m}(L-y)\big) g_m^\alpha\right] \xi_m(x) \\ &\quad + \sum_{m \in \mathbb{T}_\alpha^\dagger} \left[\frac{(L-y)\sinh(\sqrt{\lambda_m}(L-y))}{2\sqrt{\lambda_m}} h_m^\alpha\right] \xi_m(x) \\ &\quad + \sum_{m \in \mathbb{T}_\alpha^\dagger} \left[\int_y^L \frac{(\sigma-y)\cosh(\sqrt{\lambda_m}(\sigma-y))}{2\lambda_m} \rho_m^\alpha(\sigma)\, d\sigma\right] \xi_m(x) \\ &\quad + \sum_{m \in \mathbb{T}_\alpha^\dagger} \left[\int_y^L \frac{\sinh(\sqrt{\lambda_m}(\sigma-y))}{2\lambda_m\sqrt{\lambda_m}} \rho_m^\alpha(\sigma)\, d\sigma\right] \xi_m(x). \end{aligned} \tag{3.38}$$

By the triangle inequality, one has

$$\begin{aligned} &\left\| u^\alpha(y,\cdot) - u(y,\cdot) \right\|_{\mathcal{L}^2(\Omega)} \\ &\quad \leq \underbrace{\left\| u^\alpha(y,\cdot) - \widehat{\mathbf{B}}^{M_\alpha} u(y,\cdot) \right\|_{\mathcal{L}^2(\Omega)}}_{\widetilde{\mathcal{J}^\alpha}} + \underbrace{\left\| \widehat{\mathbf{B}}^{M_\alpha} u(y,\cdot) - u(y,\cdot) \right\|_{\mathcal{L}^2(\Omega)}}_{\widetilde{\mathcal{K}^\alpha}}. \end{aligned} \tag{3.39}$$

It is straightforward to see that

$$\begin{aligned} \widehat{\mathbf{B}}^{M_\alpha} u(y,x) &= \sum_{m \in \mathbb{T}_\alpha^\dagger} \left[\cosh\big(\sqrt{\lambda_m}(L-y)\big) g_m\right] \xi_m(x) \\ &\quad + \sum_{m \in \mathbb{T}_\alpha^\dagger} \left[\frac{(L-y)\sinh(\sqrt{\lambda_m}(L-y))}{2\sqrt{\lambda_m}} h_m\right] \xi_m(x) \\ &\quad + \sum_{m \in \mathbb{T}_\alpha^\dagger} \left[\int_y^L \frac{(\sigma-y)\cosh(\sqrt{\lambda_m}(\sigma-y))}{2\lambda_m} \rho_m(\sigma)\, d\sigma\right] \xi_m(x) \\ &\quad + \sum_{m \in \mathbb{T}_\alpha^\dagger} \left[\int_y^L \frac{\sinh(\sqrt{\lambda_m}(\sigma-y))}{2\lambda_m\sqrt{\lambda_m}} \rho_m(\sigma)\, d\sigma\right] \xi_m(x). \end{aligned} \tag{3.40}$$

Observe that, from (2.7) and (3.2), we get

$$\begin{aligned} &u^\alpha(y,x) - \widehat{\mathbf{B}}^{M_\alpha} u(y,x) \\ &\quad = \sum_{m \in \mathbb{T}_\alpha^\dagger} \left[\cosh\big(\sqrt{\lambda_m}(L-y)\big)\big(g_m^\alpha - g_m\big)\right] \xi_m(x) \\ &\qquad + \sum_{m \in \mathbb{T}_\alpha^\dagger} \left[\frac{(L-y)\sinh(\sqrt{\lambda_m}(L-y))}{2\sqrt{\lambda_m}} \big(h_m^\alpha - h_m\big)\right] \xi_m(x) \end{aligned}$$

$$+\sum_{m\in\mathbb{T}_\alpha^\dagger}\left[\int_y^L \frac{(\sigma-y)\cosh(\sqrt{\lambda_m}(\sigma-y))}{2\lambda_m}\big(\rho_m^\alpha(\sigma)-\rho_m(\sigma)\big)\,d\sigma\right]\xi_m(x)$$
$$+\sum_{m\in\mathbb{T}_\alpha^\dagger}\left[\int_y^L \frac{\sinh(\sqrt{\lambda_m}(\sigma-y))}{2\lambda_m\sqrt{\lambda_m}}\big(\rho_m^\alpha(\sigma)-\rho_m(\sigma)\big)\,d\sigma\right]\xi_m(x). \tag{3.41}$$

Using Parseval's relation coupled with the basic inequality $(a+b+c+d)^4 \le 4(a^2+b^2+c^2+c^2)$, we have

$$\begin{aligned}\big|\widetilde{\mathcal{J}}^\alpha\big|^2 &\le 4\sum_{m\in\mathbb{T}_\alpha^\dagger}\big|\cosh\big(\sqrt{\lambda_m}(L-y)\big)\big(g_m^\alpha-g_m\big)\big|^2\\ &\quad+4\sum_{m\in\mathbb{T}_\alpha^\dagger}\left|\frac{(L-y)\sinh(\sqrt{\lambda_m}(L-y))}{2\sqrt{\lambda_m}}\big(h_m^\alpha-h_m\big)\right|^2\\ &\quad+4\sum_{m\in\mathbb{T}_\alpha^\dagger}\left|\int_y^L\frac{(\sigma-y)\cosh(\sqrt{\lambda_m}(\sigma-y))}{2\lambda_m}\big(\rho_m^\alpha(\sigma)-\rho_m(\sigma)\big)\,d\sigma\right|^2\\ &\quad+4\sum_{m\in\mathbb{T}_\alpha^\dagger}\left|\int_y^L\frac{\sinh(\sqrt{\lambda_m}(\sigma-y))}{2\lambda_m\sqrt{\lambda_m}}\big(\rho_m^\alpha(\sigma)-\rho_m(\sigma)\big)\,d\sigma\right|^2\\ &=J_1^\alpha+J_2^\alpha+J_3^\alpha+J_4^\alpha.\end{aligned} \tag{3.42}$$

We first estimate the term J_1^α. Using (3.37a) and (1.2), one has

$$\begin{aligned}\big|J_1^\alpha\big| &\le 4\sum_{m\in\mathbb{T}_\alpha^\dagger}e^{2\sqrt{M_\alpha}(L-y)}\big|g_m^\alpha-g_m\big|^2\\ &\le 4e^{2\sqrt{M_\alpha}L}\sum_{m=1}^{\infty}\big|g_m^\alpha-g_m\big|^2\\ &\le 4e^{2\sqrt{M^\alpha}L}\big\|g^\alpha-g\big\|^2_{\mathcal{L}^2(\Omega)}\le 4e^{2\sqrt{M_\alpha}L}\alpha^2.\end{aligned} \tag{3.43}$$

It follows from (3.37b) and (1.2) that

$$\begin{aligned}\big|J_2^\alpha\big| &\le 4\sum_{m\in\mathbb{T}_\alpha^\dagger}\left|\frac{(L-y)\sinh(\sqrt{\lambda_m}(L-y))}{2\sqrt{\lambda_m}}\right|^2\big|h_m^\alpha-h_m\big|^2\\ &\le L^2\sum_{m\in\mathbb{T}_\alpha^\dagger}\frac{e^{2\sqrt{M_\alpha}(L-y)}}{\lambda_1}\big|h_m^\alpha-h_m\big|^2\le\frac{L^2e^{2\sqrt{M_\alpha}L}}{\lambda_1}\sum_{m=1}^{\infty}\big|h_m^\alpha-h_m\big|^2\\ &\le\frac{L^2e^{2\sqrt{M_\alpha}L}}{\lambda_1}\big\|h^\alpha-h\big\|^2_{\mathcal{L}^2(\Omega)}\le\frac{L^2e^{2\sqrt{M_\alpha}L}\alpha^2}{\lambda_1}.\end{aligned} \tag{3.44}$$

For J_3^α, applying Hölder's inequality and using (3.37a) coupled with (2.4) we have

$$\big|J_3^\alpha\big|\le 4\sum_{m\in\mathbb{T}_\alpha^\dagger}(L-y)\int_y^L\left|\frac{(\sigma-y)\cosh(\sqrt{\lambda_m}(\sigma-y))}{2\lambda_m}\right|^2\big|\rho_m^\alpha(\sigma)-\rho_m(\sigma)\big|^2\,d\sigma$$

$$\le L \sum_{m\in\mathbb{T}_\alpha^\dagger} \int_y^L \frac{(\sigma-y)^2 e^{2\sqrt{M_\alpha}(\sigma-y)}}{\lambda_1^2} \left|\rho_m^\alpha(\sigma)-\rho_m(\sigma)\right|^2 d\sigma$$

$$\le \frac{L^3}{\lambda_1^2} e^{2\sqrt{M_\alpha}L} \int_y^L \sum_{m=1}^\infty \left|\rho_m^\alpha(\sigma)-\rho_m(\sigma)\right|^2 d\sigma$$

$$\le \frac{L^3}{\lambda_1^2} e^{2\sqrt{M_\alpha}L} \int_y^L \left\|\rho^\alpha(\cdot,\sigma)-\rho(\cdot,\sigma)\right\|^2_{\mathcal{L}^2(\Omega)} d\sigma$$

$$\le \frac{L^4}{\lambda_1^2} e^{2\sqrt{M_\alpha}L} \left\|\rho^\alpha-\rho\right\|^2_{\mathcal{L}^\infty(0,L;\mathcal{L}^2(\Omega))} \le \frac{L^4}{\lambda_1^2} e^{2\sqrt{M_\alpha}L}\alpha^2. \tag{3.45}$$

Similarly, from (3.37b), (2.4) and Hölder's inequality, we deduce that

$$\left|J_4^\alpha\right| = 4 \sum_{m\in\mathbb{T}_\alpha^\dagger} \left|\int_y^L \frac{\sinh(\sqrt{\lambda_m}(\sigma-y))}{2\lambda_m\sqrt{\lambda_m}} \left(\rho_m^\alpha(\sigma)-\rho_m(\sigma)\right) d\sigma\right|^2$$

$$\le \sum_{m\in\mathbb{T}_\alpha^\dagger} L \int_y^L \frac{e^{2\sqrt{M_\alpha}(\sigma-y)}}{\lambda_1^3} \left|\rho_m^\alpha(\sigma)-\rho_m(\sigma)\right|^2 d\sigma$$

$$\le \frac{Le^{2\sqrt{M_\alpha}L}}{\lambda_1^3} \int_y^L \left\|\rho^\alpha(\cdot,\sigma)-\rho(\cdot,\sigma)\right\|^2_{\mathcal{L}^2(\Omega)} d\sigma$$

$$\le \frac{L^2e^{2\sqrt{M_\alpha}L}}{\lambda_1^3} \left\|\rho^\alpha-\rho\right\|^2_{\mathcal{L}^\infty(0,L;\mathcal{L}^2(\Omega))} \le \frac{L^2e^{2\sqrt{M_\alpha}L}}{\lambda_1^3}\alpha^2. \tag{3.46}$$

Combining (3.42)–(3.46), we conclude that

$$\left|\widetilde{\mathcal{J}}^\alpha\right| \le C(\lambda_1,L)e^{\sqrt{M_\alpha}L}\alpha. \tag{3.47}$$

Also we have

$$\left|\widetilde{\mathcal{K}}^\alpha\right|^2 = \sum_{m\in\mathbb{N}^*\setminus\mathbb{T}_\alpha^\dagger} \left|\left\langle u(y,x),\xi_m(x)\right\rangle_{\mathcal{L}^2(\Omega)}\right|^2$$

$$\le \sum_{m\in\mathbb{N}^*\setminus\mathbb{T}_\alpha^\dagger} \lambda_m^{-2p}\left[\lambda_m^{2p}\left|\left\langle u(y,x),\xi_m(x)\right\rangle_{\mathcal{L}^2(\Omega)}\right|^2\right]$$

$$\le M_\alpha^{-2p} \sum_{m\in\mathbb{N}^*\setminus\mathbb{T}_\alpha^\dagger} \lambda_m^{2p}\left|\left\langle u(y,x),\xi_m(x)\right\rangle_{\mathcal{L}^2(\Omega)}\right|^2$$

$$\le M_\alpha^{-2p}\|u\|^2_{\mathcal{L}^\infty(0,L;\mathbb{H}^p(\Omega))}. \tag{3.48}$$

Combining (3.39), (3.47) and (3.48), we get

$$\left\|u^\alpha(y,\cdot)-u(y,\cdot)\right\|_{\mathcal{L}^2(\Omega)} \le C(\lambda_1,L)e^{\sqrt{M_\alpha}L}\alpha + \frac{E_2}{M_\alpha^p}. \tag{3.49}$$

This completes the proof of the theorem.

3.4 Proof of Theorem 3.3

Proof Using the triangle inequality, we deduce that

$$\begin{aligned}&\big\|u^\alpha(y,\cdot)-u(y,\cdot)\big\|_{\mathbb{H}^p(\Omega)}\\&\quad\le\big\|u^\alpha(y,\cdot)-\widehat{\mathbf{B}}^{M_\alpha}u(y,\cdot)\big\|_{\mathbb{H}^p(\Omega)}+\big\|u(y,\cdot)-\widehat{\mathbf{B}}^{M_\alpha}u(y,\cdot)\big\|_{\mathbb{H}^p(\Omega)}.\end{aligned} \tag{3.50}$$

From (3.41), we have

$$\begin{aligned}&\big\|u^\alpha(y,\cdot)-\widehat{\mathbf{B}}^{M_\alpha}u(y,\cdot)\big\|^2_{\mathbb{H}^p(\Omega)}\\&\quad\le 4\sum_{m\in\mathbb{T}_\alpha^\dagger}\lambda_m^{2p}\big|\cosh\big(\sqrt{\lambda_m}(L-y)\big)\big(g_m^\alpha-g_m\big)\big|^2\\&\qquad+4\sum_{m\in\mathbb{T}_\alpha^\dagger}\lambda_m^{2p}\bigg|\frac{(L-y)\sinh(\sqrt{\lambda_m}(L-y))}{2\sqrt{\lambda_m}}\big(h_m^\alpha-h_m\big)\bigg|^2\\&\qquad+4\sum_{m\in\mathbb{T}_\alpha^\dagger}\lambda_m^{2p}\bigg|\int_y^L\frac{(\sigma-y)\cosh(\sqrt{\lambda_m}(\sigma-y))}{2\lambda_m}\big(\rho_m^\alpha(\sigma)-\rho_m(\sigma)\big)\,d\sigma\bigg|^2\\&\qquad+4\sum_{m\in\mathbb{T}_\alpha^\dagger}\lambda_m^{2p}\bigg|\int_y^L\frac{\sinh(\sqrt{\lambda_m}(\sigma-y))}{2\lambda_m\sqrt{\lambda_m}}\big(\rho_m^\alpha(\sigma)-\rho_m(\sigma)\big)\,d\sigma\bigg|^2.\end{aligned} \tag{3.51}$$

Using similar arguments to obtaining (3.47), we deduce that

$$\big\|u^\alpha(y,\cdot)-\widehat{\mathbf{B}}^{M_\alpha}u(y,\cdot)\big\|_{\mathbb{H}^p(\Omega)}\le C(\lambda_1,L)M_\alpha^p e^{\sqrt{M_\alpha}L}\alpha. \tag{3.52}$$

Similarly, we infer from (3.48) that

$$\begin{aligned}&\big\|u(y,\cdot)-\widehat{\mathbf{B}}^{M_\alpha}u(y,\cdot)\big\|_{\mathbb{H}^p(\Omega)}\\&\quad=\sum_{m\in\mathbb{N}^*\setminus\mathbb{T}_\alpha^\dagger}\lambda_m^{2p}\big|\big\langle u(y,x),\xi_m(x)\big\rangle_{\mathcal{L}^2(\Omega)}\big|^2\\&\quad\le M_\alpha^{-2q}\sum_{m\in\mathbb{N}^*\setminus\mathbb{T}_\alpha^\dagger}\lambda_m^{2p+2q}\big|\big\langle u(y,x),\xi_m(x)\big\rangle_{\mathcal{L}^2(\Omega)}\big|^2\\&\quad\le M_\alpha^{-2q}\|u\|^2_{\mathcal{L}^\infty(0,L;\mathbb{H}^{p+q}(\Omega))}.\end{aligned} \tag{3.53}$$

Combining (3.50), (3.52) and (3.53), we get

$$\begin{aligned}&\big\|u^\alpha(y,\cdot)-u(y,\cdot)\big\|_{\mathbb{H}^p(\Omega)}\\&\quad\le C(\lambda_1,L)M_\alpha^p e^{\sqrt{M_\alpha}L}\alpha+M_\alpha^{-q}\|u\|_{\mathcal{L}^\infty(0,L;\mathbb{H}^{p+q}(\Omega))},\end{aligned} \tag{3.54}$$

leading as a result to (3.15). □

4 Numerical results

In this section, we provide an example to illustrate how the proposed regularized solution approximates the exact solution for the biharmonic elliptic problem. Let $Q_L := (0,1)\times$

$(0,\pi)$, and the problem has the following form:

$$\begin{aligned}\Delta^2 u = {}& -\sin(x)\bigl(-2\sin(x)\cosh(y-1)y^2 + 2\sin(x)\cosh(y-1)^2 \\ & + 4\sin(x)\cosh(y-1)y + 4\sin(x)y^2 - 6\sin(x)\cosh(y-1) \\ & - 8\sin(x)y - y^2 + 4\sin(x) + 2\cosh(y-1) + 2y - 1\bigr),\end{aligned} \tag{4.1}$$

subject to the conditions given by

$$\begin{cases} u(y,0) = \Delta u(y,0) = 0, & y \in (0,1), \\ u(y,\pi) = \Delta u(y,\pi) = 0, & y \in (0,1), \\ u(1,x) = -\sin(x), \quad \frac{\partial u}{\partial y}u(1,x) = 0, & x \in (0,\pi), \\ \Delta u(1,x) = 2\sin(x), \quad \frac{\partial \Delta u}{\partial y}(1,x) = 0, & x \in (0,\pi). \end{cases} \tag{4.2}$$

The eigenvalues and eigenvectors of the operator $-\Delta$ depend on the specified boundary conditions. For the Dirichlet boundary conditions, the eigenvalues are $\lambda_m = m^2$ and the corresponding eigenelements $\xi_m(x) = \sqrt{\frac{2}{\pi}}\sin(mx)$ which form an orthonormal basis in $\mathcal{L}^2(0,\pi)$.

Then we have the exact solution

$$u(y,x) = \bigl[(y-1)^2 - \cosh(y-1)\bigr]\sin(x). \tag{4.3}$$

Next, we generate the final measurement data with noise by

$$\begin{cases} g^\alpha(x) = \alpha\,\mathrm{rand}(\mathrm{size}(g)) - \sin(x), \\ h^\alpha(x) = \alpha\,\mathrm{rand}(\mathrm{size}(h)) + 2\sin(x). \end{cases} \tag{4.4}$$

For the discretization, a uniform grid of mesh points (x_i, y_j) is used to discretize the space and time intervals for $i = \overline{1, N_x + 1}$, $j = \overline{1, N_y + 1}$,

$$\Delta x = \frac{\pi}{N_x}, \qquad \Delta y = \frac{1}{N_y}, \qquad x_i = \frac{(i-1)\pi}{N_x}, \qquad y_j = \frac{(j-1)\pi}{N_y}.$$

The inner product in $\mathcal{L}^2(0,\pi)$ can be approximated by the 1-D composite Simpson rule of numerical integration as

$$\int_0^\pi f(x)\,dx \approx \frac{\pi}{3(N_x+1)} \sum_{k=1}^{(N_x+1)/2} \bigl[f(x_{2k-2}) + 4f(x_{2k-1}) + f(x_{2k})\bigr], \tag{4.5}$$

where $x_k = \frac{k\pi}{N_x+1}$, $x_0 = 0$, $x_{N_x+1} = \pi$.

The regularized solution of the problem (4.1)–(4.2) is as follows:

$$\begin{aligned} & u^{\gamma(\alpha)}(y,x) \\ &\quad = \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} \bigl[\cosh^{\gamma(\alpha)}\bigl(\sqrt{\lambda_m}(1-y)\bigr) g_m^\alpha\bigr] \sin(mx) \end{aligned}$$

$$+\sqrt{\frac{2}{\pi}}\sum_{m=1}^{\infty}\left[\frac{(1-y)\sinh^{\gamma(\alpha)}(\sqrt{\lambda_m}(1-y))}{2\sqrt{\lambda_m}}h_m^{\alpha}\right]\sin(mx)$$
$$+\sqrt{\frac{2}{\pi}}\sum_{m=1}^{\infty}\left[\int_y^1\frac{(\sigma-y)\cosh^{\gamma(\alpha)}(\sqrt{\lambda_m}(\sigma-y))}{2\lambda_m}\rho_m^{\alpha}(\sigma)\,d\sigma\right]\sin(mx)$$
$$+\sqrt{\frac{2}{\pi}}\sum_{m=1}^{\infty}\left[\int_y^1\frac{\sinh^{\gamma(\alpha)}(\sqrt{\lambda_m}(\sigma-y))}{2\lambda_m\sqrt{\lambda_m}}\rho_m^{\alpha}(\sigma)\,d\sigma\right]\sin(mx), \tag{4.6}$$

where we define the operators for $z > 0$

$$\cosh^{\gamma(\alpha)}(\sqrt{\lambda_m}z) := \mathcal{I}_{L,m}^{\gamma(\alpha)}\cosh(\sqrt{\lambda_m}z), \tag{4.7}$$
$$\sinh^{\gamma(\alpha)}(\sqrt{\lambda_m}z) := \mathcal{I}_{L,m}^{\gamma(\alpha)}\sinh(\sqrt{\lambda_m}z), \tag{4.8}$$

and

$$g_m^{\alpha} = \left\langle g^{\alpha}(x), \sqrt{\frac{2}{\pi}}\sin(mx)\right\rangle_{\mathcal{L}^2(0,\pi)}, \qquad h_m^{\alpha} = \left\langle h^{\alpha}(x), \sqrt{\frac{2}{\pi}}\sin(mx)\right\rangle_{\mathcal{L}^2(0,\pi)},$$
$$\rho_m^{\alpha}(\sigma) = \left\langle \rho^{\alpha}(x,\sigma), \sqrt{\frac{2}{\pi}}\sin(mx)\right\rangle_{\mathcal{L}^2(0,\pi)}.$$

The relative errors are evaluated by

$$\mathrm{Err} = \frac{\|u-u^{\gamma(\alpha)}\|_{\mathcal{L}^2(0,\pi)}}{\|u\|_{\mathcal{L}^2(0,\pi)}}. \tag{4.9}$$

Here, we present graphs illustrating the numerical example, which we are considering. Table 1 shows that the smaller α, the smaller the error between the exact solution and the regularized solution, and the errors are acceptable. Specifically, in Fig. 1, we can see the evaluation results at $y = 0.1$ and $y = 0.3$. Moreover, we also show 3D graphs of the exact and regularized solutions throughout the domain $(0,1) \times (0,\pi)$ in Fig. 2. Seen from that point of view, the result of the method of correction is effective.

5 Conclusions

Problem (1.1) was solved using two regularization methods based on problems (3.1) and (3.2). Convergence and stability estimates, as the noise level tends to zero, are formulated and proved. Numerical examples support the theoretical findings of the paper. In future work we hope to consider extending the current study to nonlinear sources to allow for an even wider range of physical applications in for example nonlinear elasticity.

Table 1 The errors between the exact solution and the regularized solution at $y = 0.1$ and $y = 0.3$ for $\alpha \in \{0.1; 0.01; 0.001\}$

Errors	$\alpha = 10^{-1}$	$\alpha = 10^{-2}$	$\alpha = 10^{-3}$
Err($y = 0.1$)	0.0575122776	0.0335725277	0.0267921267
Err($y = 0.3$)	0.1107632402	0.0724465656	0.0348216443

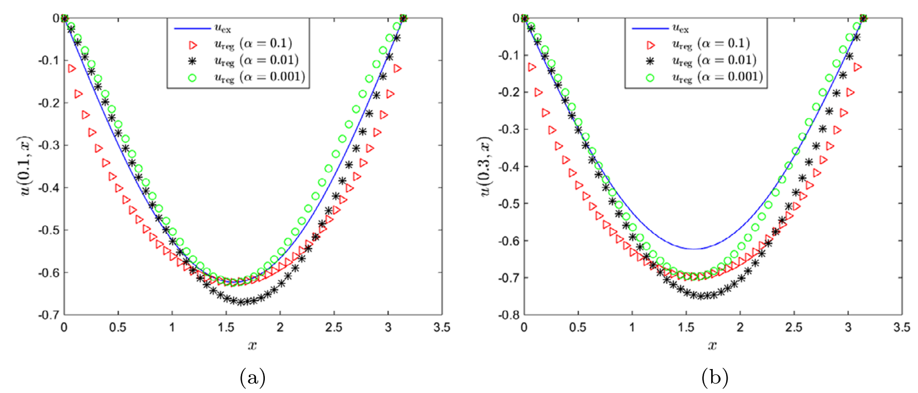

Figure 1 The exact and regularized solutions at $y = 0.1$ (**a**) and $y = 0.3$ (**b**) for $\alpha \in \{10^{-1}, 10^{-2}, 10^{-3}\}$

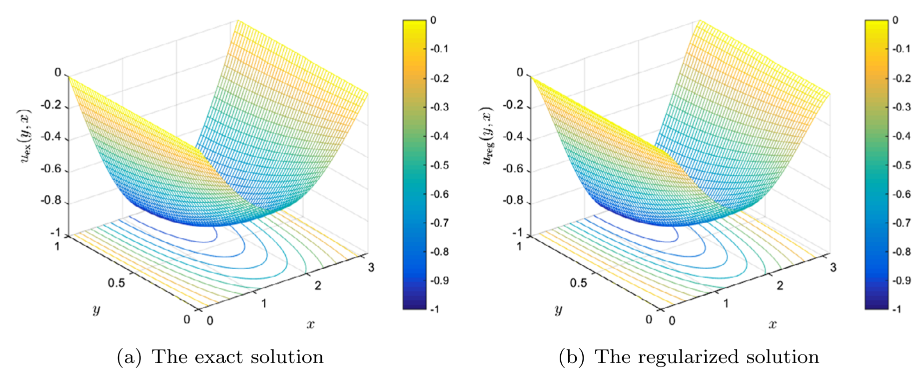

Figure 2 3D graphs of the exact and regularized solutions

Acknowledgements
Not applicable.

Funding
Not applicable.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interest.

Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

Author details
[1]Department of Research Management, Thu Dau Mot University, Binh Duong, Vietnam. [2]Department of Mathematics and Computer Science, VNUHCM—University of Science, HoChiMinh City, Vietnam. [3]School of Mathematics, Statistics and Applied Mathematics, National University of Ireland, Galway, Ireland. [4]Faculty of General Sciences, Can Tho University of Technology, Can Tho City, Viet Nam. [5]Institute of Research and Development, Duy Tan University, Da Nang, Vietnam. [6]Applied Analysis Research Group, Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam.

Publisher's Note

Received: 1 April 2019 Accepted: 12 June 2019 Published online: 27 June 2019

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