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Bivariate Bernstein–Schurer–Stancu type GBS operators in (p,q)-analogue

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Abstract

The purpose of this paper is to construct a (p,q)-analogue of Bernstein–Schurer–Stancu type GBS (generalized Boolean sum) operators for approximating B-continuous and B-differentiable functions. We also establish uniform convergence theorem and estimate the degree of approximation of B-continuous and B-differentiable functions.

MSC: Primary 41A10; 41A25; secondary 41A36

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1 Introduction

Badea et al. [5] introduced the following operators known as GBS operators associated with L.

Let $I_1, I_2 \subseteq \mathbb{R}$ be nonempty intervals, and let $L : \mathbb{R}^{I_1\times I_2} \to \mathbb{R}^{I_1\times I_2}$ be a positive linear bivariate operator, where $\mathbb{R}^{I_1\times I_2} = \{f \mid f : I_1 \times I_2 \to \mathbb{R}\}$. If $f(\circ,*) \in \mathbb{R}^{I_1\times I_2}$, then the bivariate operators $U : \mathbb{R}^{I_1\times I_2} \to \mathbb{R}^{I_1\times I_2}$ are defined by

$$Uf(x,y) = L\big(f(\circ,y) + f(x,*) - f(\circ,*)\big)(x,y), \quad \text{for } (x,y) \in I_1 \times I_2. \tag{1.1}$$

In 1934, Karl Bögel [13] introduced the notion of B-continuity and B-differentiability. B-continuity by means of bivariate mixed difference operator $\Delta_2 : \mathbb{R}^{I_1\times I_2} \to \mathbb{R}^{I_1\times I_2}$ is defined in [12].

We now recall some definitions and results based on B-continuity as follows.

Definition 1.1 A function $f \in \mathbb{R}^{I_1\times I_2}$ is B-continuous if, for each $(x,y) \in I_1 \times I_2$,

$$\lim_{(u,v)\to(x,y)} \Delta_{u,v}[f : x,y] = 0, \tag{1.2}$$

where $\Delta_{u,v}[f : x,y]$ is the mixed difference defined by

$$\Delta_{u,v}[f : x,y] = f(u,v) - f(u,y) - f(x,v) + f(x,y).$$

If the function f is B-continuous at any point $(x,y) \in I_1 \times I_2$, then it is B-continuous on the interval $I_1 \times I_2$.

For any $(x,y),(u,v) \in I_1 \times I_2$, if there exists $M > 0$ such that

$$\left|\Delta_{u,v}[f:x,y]\right| \le M$$

holds, then f is B-bounded.

Definition 1.2 A function $f \in \mathbb{R}^{I_1 \times I_2}$ is said to be uniformly B-continuous if, for any $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that, for every $(x,y),(u,v) \in I_1 \times I_2$ with $|x-u| < \delta(\epsilon)$, $|y-v| < \delta(\epsilon)$, we have

$$\left|\Delta_{u,v}[f:x,y]\right| < \epsilon. \tag{1.3}$$

If $f \in C_b(I_1 \times I_2)$ and $I_1 \times I_2 \subseteq \mathbb{R}$ are compact intervals of $\mathbb{R}$, then f is uniform B-continuous on $I_1 \times I_2$, where $C_b(I_1 \times I_2)$ is the set of B-continuous functions. For more information, we refer to [35].

Badea et al. [6] proved the following Korovkin type theorem to approximate bivariate function in the space of Bögel-continuous (B-continuous) functions.

Theorem 1.3 *Let $\{L_{m,n}\}$ be a sequence of positive linear operators which maps $\mathbb{R}^{I_1 \times I_2}$ to $\mathbb{R}^{I_1 \times I_2}$ such that, for all $(x,y) \in I_1 \times I_2$,*

(i) $L_{m,n}(e_{00};x,y) = L(1,x,y) = 1$,

(ii) $L_{m,n}(e_{10};x,y) = L(u,x,y) = x + u_{m,n}(x,y)$,

(iii) $L_{m,n}(e_{01};x,y) = L(v,x,y) = y + v_{m,n}(x,y)$,

(iv) $L_{m,n}(e_{02}+e_{20};x,y) = L(u^2+v^2,x,y) = x^2 + y^2 + w_{m,n}(x,y)$,

where $u_{m,n}(x,y)$, $v_{m,n}(x,y)$, and $w_{m,n}(x,y)$ converge uniformly to zero as $m,n \to \infty$. Then the sequence $\{U_{m,n}f\}$ converges uniformly to f on $I_1 \times I_2$ for any $f \in C_b(I_1 \times I_2)$, where I_1, I_2 are compact intervals of $\mathbb{R}$; and $U_{m,n}$ is a GBS operator associated with $L_{m,n}$.

The mixed modulus of continuity is an important tool to approximate degree of B-continuous functions introduced by Marchaud [25]. Let $f \in \mathbb{R}^{I_1 \times I_2}$ and I_1, I_2 be compact intervals of $\mathbb{R}$. Then $\omega_{\text{mixed}} : [0,\infty) \times [0,\infty) \to [0,\infty)$ is defined by

$$\omega_{\text{mixed}}(\delta_1,\delta_2) = \sup\left\{\left|\Delta_{u,v}[f:x,y]\right| : |u-x| < \delta_1, |v-y| < \delta_2\right\}, \tag{1.4}$$

for any $\delta_1,\delta_2 \in (0,\infty) \times (0,\infty)$ and $(x,y),(u,v) \in I_1 \times I_2$.

Badea et al. [5] proved the following Shisha–Mond type theorem (introduced by Mahmedov [24]) to evaluate the degree of approximation of Bögel-continuous (continuous in Bögel sense) functions using GBS operators.

Theorem 1.4 *Let $L : C_b(I_1 \times I_2) \to C_b(I_1 \times I_2)$ be a positive linear operator and $Uf(x,y)$ be the associated GBS operator. Then the following inequality holds for any $f \in C_b(I_1 \times I_2)$,*

$(x,y) \in I_1 \times I_2$, *and* $\delta_1, \delta_2 \geq 0$:

$$\begin{aligned} \big|f(x,y) - Uf(x,y)\big| &\leq \big|f(x,y)\big|\big|1 - L(e_{00};x,y)\big| + \big\{L(e_{00};x,y) \\ &\quad + \delta_1^{-1}\sqrt{L\big((e_{10}-x)^2;x,y\big)} + \delta_2^{-1}\sqrt{L\big((e_{01}-y)^2;x,y\big)} \\ &\quad + (\delta_1\delta_2)^{-1}\sqrt{L\big((e_{10}-x)^2;x,y\big)L\big((e_{01}-y)^2;x,y\big)}\big\}\omega_{\text{mixed}}(\delta_1,\delta_2). \end{aligned}$$

Note that $\omega_{\text{mixed}}(\delta_1,\delta_2)$ is a B-continuous function and $\omega_{\text{mixed}}(0,0) = 0$. By the inequality defined in Theorem 1.4 and the properties of $\omega_{\text{mixed}}(\delta_1,\delta_2)$, it is possible to obtain the uniform convergence for the sequence introduced by GBS operators.

In [10], Bărbosu defined Schurer–Stancu type GBS operators

$$\tilde{U}_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2} : C[0,1+r_1] \times C[0,1+r_2] \to C[0,1] \times C[0,1]$$

as follows:

$$\begin{aligned} &\tilde{U}_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2} f(x,y) \\ &\quad = \sum_{k=0}^{m+r_1}\sum_{l=0}^{n+r_2}\tilde{r}_{m,k}\tilde{r}_{n,l}\left\{f\left(\frac{k+\alpha_1}{m+\beta_1},y\right) + f\left(x,\frac{l+\alpha_2}{n+\beta_2}\right) - f\left(\frac{k+\alpha_1}{m+\beta_1},\frac{l+\alpha_2}{n+\beta_2}\right)\right\}, \end{aligned}$$

where r_1, r_2 are nonnegative integers and α_i, β_i are real parameters with $0 \leq \alpha_i \leq \beta_i$ ($i = 1,2$). For $\alpha_i = \beta_i = 0$ ($i = 1,2$), the above operators reduce to the first GBS operators which were introduced by Dobrescu and Matei [15]. For detailed study, one can refer to [7, 8], and [19].

2 q-Bernstein–Schurer–Stancu GBS operators

Quantum calculus (q-calculus) plays an important role in approximation theory. First of all, the q-calculus was applied by Lupaş on Bernstein polynomials. Then, focusing on bivariate case, Bărbosu [9] introduced the generalized bivariate Stancu operators, and many researchers have worked on different operators: Örkcü [32] established the q-Szász–Mirakjan–Kantorovich bivariate operators; Mursaleen and Ahasan [28] introduced the Dunkl generalization of Stancu type q-Szász–Mirakjan–Kantorovich operators; Ostrovska [33] determined the relation between the theory of q-Bernstein polynomials and limit q-Bernstein operators. For detailed study, we refer to [3, 6, 11, 14, 20, 34], and [37].

Agrawal et al. [4] introduced the q-Bernstein–Schurer–Stancu operators

$$S_{n,r}^{\alpha,\beta} : C[0,1+r] \to C[0,1]$$

as follows:

$$S_{n,r}^{\alpha,\beta}(f;q;x) = \sum_{k=0}^{n+r}\begin{bmatrix} n+r \\ k \end{bmatrix} x^k \prod_{j=0}^{n+r-k-1}\big(1-q^j x\big) f\left(\frac{[k]_q+\alpha}{[n]_q+\beta}\right), \quad q \in (0,1), x \in [0,1+r].$$

Recently Bărbosu et al. [12] introduced Bernstein–Schurer–Stancu type GBS operators based on q-integers.

For any $(x,y)\in I=[0,1+r_1]\times[0,1+r_2]$, the operators

$$U_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}: C_b\bigl([0,1+r_1]\times[0,1+r_2]\bigr)\to C_b\bigl([0,1]\times[0,1]\bigr)$$

associated with $S_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}$ are defined as follows:

$$\begin{aligned}U_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}(f;q_1,q_2;x,y)&=\sum_{k_1=0}^{m+r_1}\sum_{k_2=0}^{n+r_2}\begin{bmatrix}m+r_1\\k_1\end{bmatrix}\begin{bmatrix}n+r_2\\k_2\end{bmatrix}\prod_{s=0}^{m+r_1-k_1-1}\bigl(1-q_1^s x\bigr)\\&\quad\times\prod_{t=0}^{n+r_2-k_2-1}\bigl(1-q_2^t y\bigr)x^{k_1}y^{k_2}\{f_{k_1}+f_{k_2}-f_{k_1}f_{k_2}\},\end{aligned}\tag{2.1}$$

where

$$f_{k_1}(y)=f\left(\frac{[k_1]_{q_1}+\alpha_1}{[m]_{q_1}+\beta_1},y\right),\qquad f_{k_2}(x)=f\left(x,\frac{[k_2]_{q_2}+\alpha_2}{[n]_{q_2}+\beta_2}\right)$$

and

$$f_{k_1}f_{k_2}(x,y)=f\left(\frac{[k_1]_{q_1}+\alpha_1}{[m]_{q_1}+\beta_1},\frac{[k_2]_{q_2}+\alpha_2}{[n]_{q_2}+\beta_2}\right).$$

The operators (2.1) satisfy the following properties as proved in [12].

Lemma 2.1 *Let $e_{i,j}: I\to I$, where $I=[0,1+r_1]\times[0,1+r_2]$ is the test functions defined by $e_{i,j}(x,y)=x^iy^j$ (i,j are nonnegative integers). Then the following equalities hold*:

(i) $S_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}(e_{0,0};q_1,q_2;x,y)=e_{0,0}(x,y)$,

(ii) $S_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}(e_{1,0};q_1,q_2;x,y)=\frac{[m+r_1]_{q_1}x+\alpha_1}{[m]_{q_1}+\beta_1}$,

(iii) $S_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}(e_{0,1};q_1,q_2;x,y)=\frac{[n+r_2]_{q_2}y+\alpha_2}{[n]_{q_2}+\beta_2}$,

(iv) $S_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}(e_{2,0};q_1,q_2;x,y)=\frac{([m+r_1]_{q_1}^2x^2+[m+r_1]_{q_1}x(1-x)+2\alpha_1[m+r_1]_{q_1}x+\alpha_1^2)}{([m]_{q_1}+\beta_1)^2}$,

(v) $S_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}(e_{0,2};q_1,q_2;x,y)=\frac{([n+r_2]_{q_2}^2y^2+[n+r_2]_{q_2}y(1-y)+2\alpha_2[n+r_2]_{q_2}y+\alpha_2^2)}{([n]_{q_2}+\beta_2)^2}$.

3 (p,q)-Bernstein–Schurer–Stancu type GBS operators

In 2015 Mursaleen et al. [29] used (p,q)-calculus in approximation theory and defined first (p,q)-analogue of Bernstein polynomials. Later on this idea was used to generalize several operators, e.g., [1, 2, 16–18, 21, 26, 27, 30, 31]; for its applications, see [22] and [23].

We now recall some notations on (p,q)-calculus.

For any $p>0$ and $q>0$, the (p,q) integers $[k]_{p,q}$ are defined as follows:

$$[k]_{p,q}=p^{k-1}+p^{k-2}q+p^{k-3}q^2+\cdots+pq^{k-2}+q^{k-1}=\begin{cases}\frac{p^k-q^k}{p-q}, & \text{when } p\neq q\neq 1\\ kp^{k-1}, & \text{when } p=q\neq 1\\ [k]_q, & \text{when } p=1\\ k, & \text{when } p=q=1\end{cases}\tag{3.1}$$

$k=0,1,2,3,4,\ldots$.

Also,

$$[k]_{p,q}! = \prod_{j=1}^{k}[J]_{p,q} = [k]_{p,q}[k-1]_{p,q}\cdots[1]_{p,q}, \quad k = 1,2,3,\ldots, \tag{3.2}$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}, \quad \text{for } k = 1,2,3,\ldots, \tag{3.3}$$

$$(ax+by)_{p,q}^{n} = \sum_{i=0}^{n} p^{\frac{(n-i)(n-i-1)}{2}} q^{\frac{i(i-1)}{2}} \begin{bmatrix} n \\ i \end{bmatrix}_{p,q} (ax)^{n-i}(by)^{i}, \tag{3.4}$$

$$(x+y)_{p,q}^{n} = \prod_{i=0}^{n-1}\left(p^{i}x+q^{i}y\right) = (x+y)(px+qy)\left(p^{2}x+q^{2}y\right)\cdots\left(p^{n-1}x+q^{n-1}y\right). \tag{3.5}$$

For $y = 0$, formula (3.5) becomes

$$\begin{aligned} &(x+0)_{p,q}^{n} = p^{\frac{n(n-1)}{2}}x^{n}, \\ &(1-x)_{p,q}^{n} = (1-x)(p-qx)\left(p^{2}-q^{2}x\right)\cdots\left(p^{n-1}-q^{n-1}x\right). \end{aligned} \tag{3.6}$$

For $x = 0$, formula (3.6) becomes

$$(1-0)_{p,q}^{n} = p^{\frac{n(n-1)}{2}}.$$

Now we define some useful notations which are used in this paper. For any nonnegative integer k, we have

$$[n+k]_{p,q} = p^{n}[k]_{p,q} + q^{k}[n]_{p,q}, \tag{3.7}$$

$$[k]_{p,q}^{2} = p^{k-1}[k]_{p,q} + q[k]_{p,q}[k-1]_{p,q}, \tag{3.8}$$

$$[k]_{p,q}^{3} = p^{k-1}q^{k-1}[k]_{p,q} + p[k]_{p,q}^{2}[k-1]_{p,q} + q^{k}[k]_{p,q}[k-1]_{p,q}, \tag{3.9}$$

$$\begin{aligned} [k]_{p,q}^{4} = {}& p^{2k-2}q^{k-1}[k]_{p,q} + p[k]_{p,q}^{3}[k-1]_{p,q} \\ &+ q^{k}[k]_{p,q}^{2}[k-1]_{p,q} + p^{k-1}q^{k}[k]_{p,q}[k-1]_{p,q}. \end{aligned} \tag{3.10}$$

For $p = 1$ in (3.1)–(3.10) all these reduce to q-analogues.

Now first of all, we construct a (p,q)-analogue of Bernstein–Schurer–Stancu operators as follows:

$$\begin{aligned} S_{n,r}^{\alpha,\beta}(f;p,q;x) = {}& \frac{1}{p^{\frac{(n+r)(n+r-1)}{2}}} \sum_{k=0}^{n+r} \begin{bmatrix} n+r \\ k \end{bmatrix} p^{\frac{k(k-1)}{2}} x^{k} \\ &\times \prod_{j=0}^{n+r-k-1}\left(p^{j}-q^{j}x\right) f\left(\frac{p^{n+r-k}[k]+\alpha}{[n]+\beta}\right) \end{aligned} \tag{3.11}$$

for any $x \in [0, 1+r]$ and $0 < q < p \leq 1$, where r is a nonnegative integer.

Lemma 3.1 *The operators* (3.11) *satisfy the following properties for the test functions* $e_i = x^i$ $(i = 0, 1, 2, 3, 4)$:

(i) $S_{n,r}^{\alpha,\beta}(e_0;p,q;x) = 1$,

(ii) $S_{n,r}^{\alpha,\beta}(e_1;p,q;x) = \frac{[n+r]_{p,q}x+\alpha}{[n]_{p,q}+\beta}$,

(iii) $S_{n,r}^{\alpha,\beta}(e_2;p,q;x) = \frac{[n+r]_{p,q}^2x^2+p^{n+r-1}[n+r]_{p,q}x(1-x)+2\alpha[n+r]_{p,q}x+\alpha^2}{([n]_{p,q}+\beta)^2}$,

(iv) $S_{n,r}^{\alpha,\beta}(e_3;p,q;x) = \frac{[n+r]_{p,q}p^{n+r}(p^{-2}-[n+r-1]_{p,q}+3\alpha+3\alpha^2)x}{([n]_{p,q}+\beta)^3} + \frac{[n+r]_{p,q}[n+r-1]_{p,q}\{p^2q+p^{n+r-3}[2]_{p,q}+p^{n+r}-p^{n+r-1}[2]_{p,q}+p^{n+r-2}q[2]_{p,q}+3\alpha q\}x^2}{([n]_{p,q}+\beta)^3} + \frac{[n+r]_{p,q}[n+r-1]_{p,q}[n+r-2]_{p,q}\{p^2q^2-pq^2+q^3\}x^3+\alpha^3}{([n]_{p,q}+\beta)^3}$.

(v) $S_{n,r}^{\alpha,\beta}(e_4;p,q;x) = \frac{[n+r]_{p,q}\{4\alpha^3p^{3(n+r-1)}+4\alpha p^{2(n+r-1)}+6\alpha^2p^{n+r-1}\}x}{([n]_{p,q}+\beta)^4} + \frac{[n+r]_{p,q}[n+r-1]_{p,q}\{6\alpha^2q+p^{n+r-1}+2qp^{2(n+r-1)}+q[2]_{p,q}p^{2(n+r)-3}+4\alpha q(p^{n+r-1}+[2]_{p,q}p^{n+r-2})\}x^2}{([n]_{p,q}+\beta)^4} + \frac{[n+r]_{p,q}[n+r-1]_{p,q}[n+r-2]_{p,q}\{4\alpha pq^2(1-p)+4\alpha q^3+pq+q([2]_{p,q}+q)p^{n+r}-qp^{n+r+1}+2q^3p^{n+r-1}\}x^3}{([n]_{p,q}+\beta)^4} + \frac{[n+r]_{p,q}[n+r-1]_{p,q}[n+r-2]_{p,q}[n+r-3]_{p,q}p^4q^4x^4+\alpha^4}{([n]_{p,q}+\beta)^4}$.

Proof

$$S_{n,r}^{\alpha,\beta}(e_0;p,q;x) = \frac{1}{p^{\frac{(n+r)(n+r-1)}{2}}}\sum_{k=0}^{n+r}\begin{bmatrix} n+r \\ k \end{bmatrix} p^{\frac{k(k-1)}{2}}x^k\prod_{j=0}^{n+r-k-1}(p^j-q^jx) = 1, \tag{3.12}$$

$$\begin{aligned}
S_{n,r}^{\alpha,\beta}(e_1;p,q;x) &= \frac{1}{p^{\frac{(n+r)(n+r-1)}{2}}}\sum_{k=0}^{n+r}\frac{[n+r]_{p,q}!}{[k]_{p,q}![n+r-k]_{p,q}!}p^{\frac{k(k-1)}{2}}x^k\left(\frac{p^{n+r-k}[k]_{p,q}+\alpha}{[n]_{p,q}+\beta}\right) \\
&\quad\times\prod_{j=0}^{n+r-k-1}(p^j-q^jx) \\
&= \frac{1}{p^{\frac{(n+r)(n+r-1)}{2}}}\sum_{k=1}^{n+r}\frac{[n+r]_{p,q}!}{[k-1]_{p,q}![n+r-k]_{p,q}!}p^{\frac{k(k-1)}{2}}x^k\frac{p^{n+r-k}}{[n]_{p,q}+\beta} \\
&\quad\times\prod_{j=0}^{n+r-k-1}(p^j-q^jx) + \frac{\alpha}{[n]_{p,q}+\beta} \\
&= \frac{[n+r]_{p,q}}{p^{\frac{(n+r)(n+r-3)}{2}}}\sum_{k=0}^{n+r}\frac{[n+r-1]_{p,q}!}{[k]_{p,q}![n+r-k-1]_{p,q}!}p^{\frac{k^2-k-2}{2}}x^{k+1}\frac{1}{[n]_{p,q}+\beta} \\
&\quad\times\prod_{j=0}^{n+r-k-1}(p^j-q^jx) + \frac{\alpha}{[n]_{p,q}+\beta} \\
&= \frac{[n+r]_{p,q}x}{[n]_{p,q}+\beta} + \frac{\alpha}{[n]_{p,q}+\beta},
\end{aligned}$$

$$\begin{aligned}
S_{n,r}^{\alpha,\beta}(e_2;p,q;x) &= \frac{1}{p^{\frac{(n+r)(n+r-1)}{2}}}\sum_{k=0}^{n+r}\frac{[n+r]_{p,q}!}{[k]_{p,q}![n+r-k]_{p,q}!}p^{\frac{k(k-1)}{2}}x^k\prod_{j=0}^{n+r-k-1}(p^j-q^jx) \\
&\quad\times\left(\frac{p^{n+r-k}[k]_{p,q}+\alpha}{[n]_{p,q}+\beta}\right)^2 \\
&= \frac{1}{p^{\frac{(n+r)(n+r-1)}{2}}}\sum_{k=0}^{n+r}\frac{[n+r]_{p,q}!}{[k]_{p,q}![n+r-k]_{p,q}!}p^{\frac{k(k-1)}{2}}x^k\prod_{j=0}^{n+r-k-1}(p^j-q^jx)
\end{aligned}$$

$$\times \frac{p^{2(n+r-k)}[k]_{p,q}^2 + \alpha^2 + 2\alpha p^{n+r-k}[k]_{p,q}}{([n]_{p,q}+\beta)^2}$$
$$= \frac{[n+r]_{p,q}^2 x^2 + p^{n+r-1}[n+r]_{p,q}x(1-x) + 2\alpha[n+r]_{p,q}x + \alpha^2}{([n]_{p,q}+\beta)^2}, \quad (3.13)$$

$$S_{n,r}^{\alpha,\beta}(e_3;p,q;x) = \frac{1}{p^{\frac{(n+r)(n+r-1)}{2}}} \sum_{k=0}^{n+r} \frac{[n+r]_{p,q}!}{[k]_{p,q}![n+r-k]_{p,q}!} p^{\frac{k(k-1)}{2}} x^k \prod_{j=0}^{n+r-k-1} (p^j - q^j x)$$
$$\times \frac{p^{3(n+r-k)}[k]_{p,q}^3 + \alpha^3 + 3\alpha p^{2(n+r-k)}[k]_{p,q}^2 + 3\alpha^2 p^{n+r-k}[k]_{p,q}}{([n]_{p,q}+\beta)^3}. \quad (3.14)$$

After solving, we get

$$= \frac{[n+r]_{p,q}p^{n+r}(p^{-2} - [n+r-1]_{p,q} + 3\alpha + 3\alpha^2)x}{([n]_{p,q}+\beta)^3}$$
$$+ \Big([n+r]_{p,q}[n+r-1]_{p,q}\big\{p^2 q + p^{n+r-3}[2]_{p,q} + p^{n+r}$$
$$- p^{n+r-1}[2]_{p,q} + p^{n+r-2}q[2]_{p,q} + 3\alpha q\big\}x^2\Big)$$
$$/\Big(\big([n]_{p,q}+\beta\big)^3\Big)$$
$$+ \frac{[n+r]_{p,q}[n+r-1]_{p,q}[n+r-2]_{p,q}\{p^2q^2 - pq^2 + q^3\}x^3 + \alpha^3}{([n]_{p,q}+\beta)^3}.$$

Finally, we have

$$S_{n,r}^{\alpha,\beta}(e_4;p,q;x)$$
$$= \frac{1}{p^{\frac{(n+r)(n+r-1)}{2}}} \sum_{k=0}^{n+r} \frac{[n+r]_{p,q}!}{[k]_{p,q}![n+r-k]_{p,q}!} p^{\frac{k(k-1)}{2}} x^k \prod_{j=0}^{n+r-k-1} (p^j - q^j x)$$
$$\times \frac{p^{4(n+r-k)}[k]_{p,q}^4 + \alpha^4 + 6\alpha^2 p^{2(n+r-k)}[k]_{p,q}^2 + 4\alpha p^{3(n+r-k)}[k]_{p,q}^3 + 4\alpha^3 p^{(n+r-k)}[k]_{p,q}}{([n]_{p,q}+\beta)^4}.$$

By using (3.8), (3.9), and (3.10), we obtain (v). □

Rao and Wafi [36] introduced a (p,q)-analogue of Bivariate–Schurer–Stancu operators in the following form:

Let $I_1 \times I_2 = [0, 1+r_1] \times [0, 1+r_2]$, $0 < q_1 < p_1 \leq 1$, $0 < q_2 < p_2 \leq 1$, and $m, n \in \mathbb{N} \times \mathbb{N}$. Then, for any $f \in C(I_1 \times I_2)$, the operators $S_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2} : C(I_1 \times I_2) \to C([0,1] \times [0,1])$ are defined by

$$S_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}(f;p_1,p_2,q_1,q_2;x,y) = \sum_{k_1=0}^{m+r_1} \sum_{k_2=0}^{n+r_2} s_{m,r_1,k_1}^{p_1,q_1}(x) s_{n,r_2,k_2}^{p_2,q_2}(y)$$
$$\times f\left(\frac{p^{m-k_1}[k_1]_{p_1,q_1} + \alpha_1}{[m]_{p_1,q_1} + \beta_1}, \frac{p^{n-k_2}[k_2]_{p_2,q_2} + \alpha_2}{[n]_{p_2,q_2} + \beta_2}\right), \quad (3.15)$$

where

$$s^{p_1,q_1}_{m,r_1,k_1}(x) = \frac{1}{p_1^{\frac{(m+r_1)(m+r_1-1)}{2}}} \begin{bmatrix} m+r_1 \\ k_1 \end{bmatrix}_{p_1,q_1} p_1^{\frac{k_1(k_1-1)}{2}} x^{k_1} \prod_{s=0}^{m+r_1-k_1-1} \left(p_1^s - q_1^s x\right),$$

$$s^{p_2,q_2}_{n,r_2,k_2}(y) = \frac{1}{p_2^{\frac{(n+r_2)(n+r_2-1)}{2}}} \begin{bmatrix} n+r_2 \\ k_2 \end{bmatrix}_{p_2,q_2} p_2^{\frac{k_2(k_2-1)}{2}} y^{k_2} \prod_{s=0}^{n+r_2-k_2-1} \left(p_2^s - q_2^s y\right).$$

In the following, we define a (p,q)-analogue of the bivariate Schurer–Stancu operators as follows:

Let $I_1 \times I_2 = [0, 1 + r_1] \times [0, 1 + r_2]$, $0 < q_1 < p_1 \le 1$, $0 < q_2 < p_2 \le 1$, and $m, n \in \mathbb{N} \times \mathbb{N}$. Then, for any $f \in C(I_1 \times I_2)$, the operators $S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{m,n,r_1,r_2} : C(I_1 \times I_2) \to C([0,1] \times [0,1])$ are defined by

$$\begin{aligned} &S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{m,n,r_1,r_2}(f; p_1, p_2, q_1, q_2; x, y) \\ &\quad = \sum_{k_1=0}^{m+r_1} \sum_{k_2=0}^{n+r_2} s^{p_1,q_1}_{m,r_1,k_1}(x) s^{p_2,q_2}_{n,r_2,k_2}(y) \\ &\qquad \times f\left(\frac{p^{m+r_1-k_1}[k_1]_{p_1,q_1} + \alpha_1}{[m]_{p_1,q_1} + \beta_1}, \frac{p^{n+r_2-k_2}[k_2]_{p_2,q_2} + \alpha_2}{[n]_{p_2,q_2} + \beta_2}\right), \end{aligned} \tag{3.16}$$

where

$$s^{p_1,q_1}_{m,r_1,k_1}(x) = \frac{1}{p_1^{\frac{(m+r_1)(m+r_1-1)}{2}}} \begin{bmatrix} m+r_1 \\ k_1 \end{bmatrix}_{p_1,q_1} p_1^{\frac{k_1(k_1-1)}{2}} x^{k_1} \prod_{s=0}^{m+r_1-k_1-1} \left(p_1^s - q_1^s x\right),$$

$$s^{p_2,q_2}_{n,r_2,k_2}(y) = \frac{1}{p_2^{\frac{(n+r_2)(n+r_2-1)}{2}}} \begin{bmatrix} n+r_2 \\ k_2 \end{bmatrix}_{p_2,q_2} p_2^{\frac{k_2(k_2-1)}{2}} y^{k_2} \prod_{s=0}^{n+r_2-k_2-1} \left(p_2^s - q_2^s y\right).$$

The operators (3.16) satisfy the following properties.

Lemma 3.2 *Let $e_{i,j}(x,y) = x^i y^j$, $0 \le i, j \le 2$, be two-dimensional test functions. Then*

(i) $S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{m,n,r_1,r_2}(e_{0,0}; p_1, p_2, q_1, q_2; x, y) = 1$,

(ii) $S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{m,n,r_1,r_2}(e_{1,0}; p_1, p_2, q_1, q_2; x, y) = \frac{[m+r_1]_{p_1,q_1} x + \alpha_1}{[m]_{p_1,q_1} + \beta_1}$,

(iii) $S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{m,n,r_1,r_2}(e_{0,1}; p_1, p_2, q_1, q_2; x, y) = \frac{[n+r_2]_{p_2,q_2} y + \alpha_2}{[n]_{p_2,q_2} + \beta_2}$,

(iv) $S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{m,n,r_1,r_2}(e_{2,0}; p_1, p_2, q_1, q_2; x, y) = \frac{[m+r_1]_{p_1,q_1}(p_1^{m+r_1-1} + 2\alpha_1)x}{([m]_{p_1,q_1} + \beta_1)^2} + \frac{q_1[m+r_1]_{p_1,q_1}[m+r_1-1]_{p_1,q_1} x^2}{([m]_{p_1,q_1} + \beta_1)^2}$
$+ \frac{\alpha_1^2}{([m]_{p_1,q_1} + \beta_1)^2}$,

(v) $S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{m,n,r_1,r_2}(e_{0,2}; p_1, p_2, q_1, q_2; x, y) = \frac{[n+r_2]_{p_2,q_2}(p_2^{n+r_2-1} + 2\alpha_2)y}{([n]_{p_2,q_2} + \beta_2)^2} + \frac{q_2[n+r_2]_{p_2,q_2}[n+r_2-1]_{p_2,q_2} y^2}{([n]_{p_2,q_2} + \beta_2)^2}$
$+ \frac{\alpha_2^2}{([n]_{p_2,q_2} + \beta_2)^2}$.

Proof From Lemma 3.1, (i) follows immediately.

$$\begin{aligned} &S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{m,n,r_1,r_2}(e_{0,0}; p_1, p_2, q_1, q_2; x, y) \\ &\quad = S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{m,n,r_1,r_2}(e_0; p_1, q_1; x, y) S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{m,n,r_1,r_2}(e_0; p_2, q_2; x, y) = 1. \end{aligned}$$

Now for (*ii*), again from Lemma 3.1, we have

$$\begin{aligned} S_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}(e_{1,0};p_1,p_2,q_1,q_2;x,y) &= S_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}(e_1;p_1,q_1;x,y)S_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}(e_0;p_2,q_2;x,y) \\ &= \frac{[m+r_1]_{p_1,q_1}x+\alpha_1}{[m]_{p_1,q_1}+\beta_1}. \end{aligned}$$

Similarly, we obtain (*iii*).

Further, for (*iv*), we have

$$\begin{aligned} S_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}(e_{2,0};p_1,p_2,q_1,q_2;x,y) &= S_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}(e_2;p_1,q_1;x,y)S_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}(e_0;p_2,q_2;x,y) \\ &= \frac{[m+r_1]_{p_1,q_1}(p_1^{m+r_1-1}+2\alpha_1)x}{([m]_{p_1,q_1}+\beta_1)^2}+\frac{\alpha_1^2}{([m]_{p_1,q_1}+\beta_1)^2} \\ &\quad+\frac{q_1[m+r_1]_{p_1,q_1}[m+r_1-1]_{p_1,q_1}x^2}{([m]_{p_1,q_1}+\beta_1)^2}. \end{aligned}$$

In a similar way, we get (v). □

Now, motivated by q-Bernstein–Schurer–Stancu type GBS operators (2.1), we construct (p,q)-Bernstein–Schurer–Stancu type GBS operators as follows.

For any $(x,y)\in I=[0,1+r_1]\times[0,1+r_2]$, the (p,q)-Bernstein–Schurer–Stancu type GBS operators $U_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}: C_b([0,1+r_1]\times[0,1+r_2])\to C_b([0,1]\times[0,1])$ associated with $S_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}$ are defined by

$$\begin{aligned} U_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}(f;p_1,p_2,q_1,q_2;x,y) &= \frac{1}{p_1^{\frac{(m+r_1)(m+r_1-1)}{2}}}\frac{1}{p_2^{\frac{(n+r_2)(n+r_2-1)}{2}}}\sum_{k_1=0}^{m+r_1}\sum_{k_2=0}^{n+r_2}\begin{bmatrix} m+r_1 \\ k_1 \end{bmatrix} \\ &\quad\times\begin{bmatrix} n+r_2 \\ k_2 \end{bmatrix}p_1^{\frac{k_1(k_1-1)}{2}}p_2^{\frac{k_2(k_2-1)}{2}}\prod_{s=0}^{m+r_1-k_1-1}\left(p_1^s-q_1^sx\right) \\ &\quad\times\prod_{t=0}^{n+r_2-k_2-1}\left(p_2^t-q_2^ty\right)x^{k_1}y^{k_2}\{f_{k_1}+f_{k_2}-f_{k_1}f_{k_2}\}, \end{aligned} \tag{3.17}$$

where

$$\begin{aligned} &f_{k_1}(y)=f\left(\frac{p_1^{m+r_1-k_1}[k_1]_{p_1,q_1}+\alpha_1}{[m]_{p_1,q_1}+\beta_1},y\right),\qquad f_{k_2}(x)=f\left(x,\frac{p_2^{n+r_2-k_2}[k_2]_{p_2,q_2}+\alpha_2}{[n]_{p_2,q_2}+\beta_2}\right), \\ &f_{k_1}f_{k_2}(x,y)=f\left(\frac{p_1^{m+r_1-k_1}[k_1]_{p_1,q_1}+\alpha_1}{[m]_{p_1,q_1}+\beta_1},\frac{p_2^{n+r_2-k_2}[k_2]_{p_2,q_2}+\alpha_2}{[n]_{p_2,q_2}+\beta_2}\right). \end{aligned}$$

Lemma 3.3 *Let* $\psi_x,\psi_y: I\to\mathbb{R}$ *be defined as*

$$\psi_x(u,v)=|u-x|,\qquad \psi_y(u,v)=|v-y|,\quad \textit{for any } (u,v)\in I \textit{ and } (x,y)\in I,$$

where $I=[0,1+r_1]\times[0,1+r_2]$. *Then the following equalities hold*:

$$\begin{aligned} &S_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}\left(\psi_x^2;p_1,p_2,q_1,q_2;x,y\right) \\ &\quad=\frac{(((p_1^{r_1}-1)[m]_{p_1,q_1}+q_1^m[r_1]_{p_1,q_1}-\beta_1)x+\alpha_1)^2+p_1^{m+r_1-1}[m+r_1]_{p_1,q_1}x(1-x)}{([m]_{p_1,q_1}+\beta_1)^2} \end{aligned} \tag{3.18}$$

and

$$\begin{aligned}
&S_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}\bigl(\psi_y^2;p_1,p_2,q_1,q_2;x,y\bigr)\\
&\quad=\frac{(((p_2^{r_2}-1)[n]_{p_2,q_2}+q_2^n[r_2]_{p_2,q_2}-\beta_2)y+\alpha_2)^2+p_2^{n+r_2-1}[n+r_2]_{p_2,q_2}y(1-y)}{([n]_{p_2,q_2}+\beta_2)^2}. \qquad (3.19)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&S_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}\bigl(\psi_x^4;p_1,p_2,q_1,q_2;x,y\bigr)\\
&\quad=\frac{1}{([m]_{p_1,q_1}+\beta_1)^4}\biggl[4\alpha_1\biggl\{[m+r_1]_{p_1,q_1}p_1^{m+r_1-1}\biggl(\alpha_1^2p_1^{2(m+r_1-1)}+p_1^{m+r_1-1}+\frac{3}{2}\alpha_1\biggr)\\
&\qquad-\alpha_1^2\bigl([m]_{p_1,q_1}+\beta_1\bigr)\biggr\}x+\bigl\{[m+r_1]_{p_1,q_1}[m+r_1-1]_{p_1,q_1}\bigl(6\alpha_1^2q_1\\
&\qquad+p_1^{m+r_1-1}\bigl(1+2q_1p_1^{m+r_1-1}\bigr)+q_1[2]_{p_1,q_1}p_1^{2(m+r_1)-3}+4\alpha_1q_1p_1^{m+r_1-1}\bigl(1+[2]_{p_1,q_1}p_1^{-1}\bigr)\bigr)\\
&\qquad-4\bigl([m]_{p_1,q_1}+\beta_1\bigr)[m+r_1]_{p_1,q_1}p_1^{m+r_1}\bigl(p_1^{-2}-[m+r_1-1]_{p_1,q_1}+3\alpha_1(1+\alpha_1)\bigr)\\
&\qquad+6\alpha_1^2\bigl([m]_{p_1,q_1}+\beta_1\bigr)^2\bigr\}x^2+\bigl\{[m+r_1]_{p_1,q_1}q_1[m+r_1-1]_{p_1,q_1}[m+r_1-2]_{p_1,q_1}\\
&\qquad\times\bigl(4\alpha_1p_1q_1(1-p_1)+q_1^2\bigl(4\alpha_1-2p_1^{m+r_1-1}\bigr)+p_1\bigl(1-p_1^{m+r_1}\bigr)+p_1^{m+r_1}\bigl([2]_{p_1,q_1}+q_1\bigr)\bigr)\\
&\qquad-4[m+r_1]_{p_1,q_1}[m+r_1-1]_{p_1,q_1}\bigl([m]_{p_1,q_1}+\beta_1\bigr)\bigl(q_1\bigl(p_1^2+3\alpha_1\bigr)\\
&\qquad+p_1^{m+r_1}+[2]_{p_1,q_1}p_1^{m+r_1-3}\bigl(1-p_1^2+q_1p_1\bigr)\bigr)+12\alpha_1\bigl([m]_{p_1,q_1}+\beta_1\bigr)^2[m+r_1]_{p_1,q_1}\\
&\qquad+6p_1^{m+r_1-1}[m+r_1]_{p_1,q_1}\bigl([m]_{p_1,q_1}+\beta_1\bigr)^2-4\alpha_1\bigl([m]_{p_1,q_1}+\beta_1\bigr)^3\bigr\}x^3\\
&\qquad+[m+r_1]_{p_1,q_1}[m+r_1-1]_{p_1,q_1}[m+r_1-2]_{p_1,q_1}[m+r_1-3]_{p_1,q_1}p_1^4q_1^4x^4+\alpha_1^4\\
&\qquad+\bigl([m]_{p_1,q_1}+\beta_1\bigr)^4x^4+\bigl\{-6p^{m+r_1-1}[m+r_1]_{p_1,q_1}\bigl([m]_{p_1,q_1}+\beta_1\bigr)^2\\
&\qquad-4\bigl([m]_{p_1,q_1}+\beta_1\bigr)[m+r_1]_{p_1,q_1}[m+r_1-1]_{p_1,q_1}[m+r_1-2]_{p_1,q_1}q_1^2\bigl(p_1^2-p_1+q_1\bigr)\\
&\qquad-4\bigl([m]_{p_1,q_1}+\beta_1\bigr)^3[m+r_1]_{p_1,q_1}\bigr\}x^4\biggr]. \qquad (3.20)
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}
&S_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}\bigl(\psi_y^4;p_1,p_2,q_1,q_2;x,y\bigr)\\
&\quad=\frac{1}{([n]_{p_2,q_2}+\beta_2)^4}\biggl[4\alpha_2\biggl\{[n+r_2]_{p_2,q_2}p_2^{n+r_2-1}\biggl(\alpha_2^2p_2^{2(n+r_2-1)}+p_2^{n+r_2-1}+\frac{3}{2}\alpha_2\biggr)\\
&\qquad-\alpha_2^2\bigl([n]_{p,q}+\beta_2\bigr)\biggr\}y+\bigl\{[n+r_2]_{p_2,q_2}[n+r_2-1]_{p_2,q_2}\bigl(6\alpha_2^2q_2\\
&\qquad+p_2^{n+r_2-1}\bigl(1+2q_2p_2^{n+r_2-1}\bigr)+q_2[2]_{p,q}p_2^{2(n+r_2)-3}+4\alpha_2q_2p_2^{n+r_2-1}\bigl(1+[2]_{p_2,q_2}p_2^{-1}\bigr)\bigr)\\
&\qquad-4\bigl([n]_{p_2,q_2}+\beta_2\bigr)[n+r_2]_{p_2,q_2}p_2^{n+r_2}\bigl(p_2^{-2}-[n+r_2-1]_{p_2,q_2}+3\alpha_2(1+\alpha_2)\bigr)\\
&\qquad+6\alpha_2^2\bigl([n]_{p_2,q_2}+\beta_2\bigr)^2\bigr\}y^2+\bigl\{[n+r_2]_{p_2,q_2}[n+r_2-1]_{p_2,q_2}[n+r_2-2]_{p_2,q_2}\\
&\qquad\times q_2\bigl(4\alpha_2p_2q_2(1-p_2)+q_2^2\bigl(4\alpha_2-2p_2^{n+r_2-1}\bigr)+p_2\bigl(1-p_2^{n+r_2}\bigr)+p_2^{n+r_2}\bigl([2]_{p_2,q_2}+q_2\bigr)\bigr)\\
&\qquad-4[n+r_2]_{p_2,q_2}[n+r_2-1]_{p_2,q_2}\bigl([n]_{p_2,q_2}+\beta_2\bigr)\bigl(q_2\bigl(p_2^2+3\alpha_2\bigr)+p_2^{n+r_2}\\
&\qquad+[2]_{p_2,q_2}p_2^{n+r_2-3}\bigl(1-p_2^2+q_2p_2\bigr)\bigr)+12\alpha_2\bigl([n]_{p_2,q_2}+\beta_2\bigr)^2[n+r_2]_{p_2,q_2}
\end{aligned}$$

$$
\begin{aligned}
&+6p_2^{n+r_2-1}[n+r_2]_{p_2,q_2}\big([n]_{p_2,q_2}+\beta_2\big)^2-4\alpha\big([n]_{p_2,q_2}+\beta_2\big)^3\big\}y^3\\
&+[n+r_2]_{p_2,q_2}[n+r_2-1]_{p_2,q_2}[n+r_2-2]_{p_2,q_2}[n+r_2-3]_{p_2,q_2}p_2^4q_2^4y^4\\
&+\alpha_2^4+\big([n]_{p_2,q_2}+\beta_2\big)^4y^4+\big\{-6p^{n+r_2-1}[n+r_2]_{p,q}\big([n]_{p_2,q_2}+\beta_2\big)^2\\
&-4\big([n]_{p_2,q_2}+\beta_2\big)[n+r_2]_{p,q}[n+r_2-1]_{p,q}[n+r_2-2]_{p,q}\\
&\times q_2^2\big(p_2^2-p_2+q_2\big)-4\big([n]_{p_2,q_2}+\beta_2\big)^3[n+r_2]_{p_2,q_2}\big\}y^4\big].
\end{aligned}
\tag{3.21}
$$

Proof By definition of ψ_x, ψ_y and the linearity of $S_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}$, we obtain the results (3.18), (3.19), (3.20), and (3.21). □

Theorem 3.4 *Let $p_1=p_1^m$, $p_2=p_2^n$ and $q_1=q_1^m$, $q_2=q_2^n$ such that*

$$
\lim_{m\to\infty}p_1^m=\lim_{m\to\infty}q_1^m=\lim_{n\to\infty}p_2^n=\lim_{n\to\infty}q_2^n=1\quad\textit{with}\ \lim_{r_1\to\infty}p_1^{r_1}=\lim_{r_2\to\infty}p_2^{r_2}=1.
$$

Then the sequence $\{U_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}\}$ converges uniformly to f for any $f\in C_b(I)$ on the interval $[0,1]\times[0,1]$.

Proof The new operators are linear and positive in view of linearity and positivity of q-Bernstein–Schurer–Stancu operators on $[0,1]\times[0,1]$. Now we have to prove that the (p,q)-Bernstein–Schurer–Stancu operators satisfy the hypotheses of Theorem 1.3. By Lemma 3.2 (i), we have

$$
S_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}(e_{0,0};p_1,p_2,q_1,q_2;x,y)=1.
$$

That is, condition (i) of Theorem 1.3 verified.

From the second condition (ii) of Lemma 3.2, we have

$$
\begin{aligned}
&S_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}(e_{1,0};p_1,p_2,q_1,q_2;x,y)\\
&\quad=x+\frac{((p_1^{r_1}-1)[m]_{p_1,q_1}+q_1^m[r_1]_{p_1,q_1}-\beta_1)x+\alpha_1}{[m]_{p_1,q_1}+\beta_1},
\end{aligned}
\tag{3.22}
$$

i.e., condition (ii) of Theorem 1.3 is verified with

$$
u_{m,n}(x,y)=\frac{((p_1^{r_1}-1)[m]_{p_1,q_1}+q_1^m[r_1]_{p_1,q_1}-\beta_1)x+\alpha_1}{[m]_{p_1,q_1}+\beta_1}.
$$

In a similar way, we get

$$
\begin{aligned}
&S_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}(e_{0,1};p_1,p_2,q_1,q_2;x,y)\\
&\quad=y+\frac{((p_2^{r_2}-1)[n]_{p_2,q_2}+q_2^n[r_2]_{p_2,q_2}-\beta_2)y+\alpha_2}{[n]_{p_2,q_2}+\beta_2},
\end{aligned}
\tag{3.23}
$$

where

$$
v_{m,n}(x,y)=\frac{((p_2^{r_2}-1)[n]_{p_2,q_2}+q_2^n[r_2]_{p_2,q_2}-\beta_2)y+\alpha_2}{[n]_{p_2,q_2}+\beta_2}.
$$

From statements (*iv*) and (*v*), again by applying Lemma 3.2, we obtain

$$S_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}(e_{2,0}+e_{0,2};p_1,p_2,q_1,q_2;x,y)=x^2+y^2+\omega_{m,n}(x,y), \tag{3.24}$$

where

$$\begin{aligned}
&\omega_{m,n}(x,y)\\
&\quad=\Big(\big([m+r_1]_{p_1,q_1}^2-\big([m]_{p_1,q_1}+\beta_1\big)^2\big)x^2+p_1^{m+r_1-1}[m+r_1]_{p_1,q_1}x(1-x)\\
&\qquad+2\alpha_1x[m+r_1]_{p_1,q_1}+\alpha_1^2\Big)/\Big(\big([m]_{p_1,q_1}+\beta_1\big)^2\Big)\\
&\qquad+\Big(\big([n+r_2]_{p_2,q_2}^2-\big([n]_{p_2,q_2}+\beta_2\big)^2\big)y^2+p_2^{n+r_2-1}[n+r_2]_{p_2,q_2}y(1-y)\\
&\qquad+2\alpha_2y[n+r_2]_{p_2,q_2}+\alpha_2^2\Big)/\Big(\big([n]_{p_2,q_2}+\beta_2\big)^2\Big).
\end{aligned}$$

From (3.22), (3.23), (3.24), and the hypotheses of Theorem 3.4, it follows that

$$\lim_{m,n\to\infty}u_{m,n}(x,y)=\lim_{m,n\to\infty}v_{m,n}(x,y)=\lim_{m,n\to\infty}\omega_{m,n}(x,y)=0$$

uniformly on $[0,1]\times[0,1]$.

The desired uniform convergence is verified as a consequence of Theorem 1.3. □

In the next result, we express the degree of approximation of B-continuous functions f by using (p,q)-GBS operators.

Theorem 3.5 *For any $f\in C_b(I)$ and $(x,y)\in[0,1]\times[0,1]$, the following estimation holds true*:

$$\big|f(x,y)-U_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}(f;p_1,p_2,q_1,q_2;x,y)\big|\le\frac{9}{4}\omega_{\text{mixed}}(\delta_1,\delta_2), \tag{3.25}$$

where

$$\begin{aligned}
\delta_1={}&\frac{1}{[m]_{p_1,q_1}+\beta_1}\\
&\times\sqrt{4\max_{x\in[0,1]}\Big(\Big(\big(p_1^{r_1}-1\big)[m]_{p_1,q_1}+q_1^m[r_1]_{p_1,q_1}-\beta_1\Big)x+\alpha_1\Big)^2+p_1^{m+r_1-1}[m+r_1]_{p_1,q_1}}
\end{aligned} \tag{3.26}$$

and

$$\begin{aligned}
\delta_2={}&\frac{1}{[n]_{p_2,q_2}+\beta_2}\\
&\times\sqrt{4\max_{y\in[0,1]}\Big(\Big(\big(p_2^{r_2}-1\big)[n]_{p_2,q_2}+q_2^n[r_2]_{p_2,q_2}-\beta_2\Big)y+\alpha_2\Big)^2+p_2^{n+r_2-1}[n+r_2]_{p_2,q_2}}.
\end{aligned} \tag{3.27}$$

Proof By applying Theorem 1.3, since $S_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}$ is a constant reproducing operator, we have

$$\begin{aligned}
&\left|f(x,y) - U_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}(f;p_1,p_2,q_1,q_2;x,y)\right| \\
&\quad \leq \Big\{1 + \delta_1^{-1}\sqrt{S_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}\big(\psi_x^2;p_1,p_2,q_1,q_2;x,y\big)} \\
&\qquad + \delta_2^{-1}\sqrt{S_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}\big(\psi_y^2;p_1,p_2,q_1,q_2;x,y\big)} \\
&\qquad + \delta_1^{-1}\delta_2^{-1}\sqrt{S_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}\big(\psi_x^2;p_1,p_2,q_1,q_2;x,y\big)S_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}\big(\psi_y^2;p_1,p_2,q_1,q_2;x,y\big)}\Big\} \\
&\qquad \times \omega_{\text{mixed}}(\delta_1,\delta_2)
\end{aligned} \tag{3.28}$$

for any $\delta_1, \delta_2 \geq 0$.

Since, for any $(x,y) \in [0,1] \times [0,1]$,

$$x(1-x) \leq \frac{1}{4} \quad \text{and} \quad y(1-y) \leq \frac{1}{4}.$$

Then

$$S_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}\big(\psi_x^2;p_1,p_2,q_1,q_2;x,y\big) \leq \frac{1}{2}\delta_1, \tag{3.29}$$

$$S_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}\big(\psi_y^2;p_1,p_2,q_1,q_2;x,y\big) \leq \frac{1}{2}\delta_2, \tag{3.30}$$

where δ_1, δ_2 are defined in (3.26) and (3.27), respectively. Thus, from (3.28), (3.29), and (3.30), we obtain the desired result. □

Remark 3.6 Suppose $p_1 = p_1^m$, $q_1 = q_1^m$ and $p_2 = p_2^n$, $q_2 = q_2^n$ such that

$$\lim_{m\to\infty} p_1^m = \lim_{m\to\infty} q_1^m = 1, \qquad \lim_{n\to\infty} p_2^n = \lim_{n\to\infty} p_2^n = 1 \quad \text{and} \lim_{r_1\to\infty} p_1^{r_1} = \lim_{r_2\to\infty} p_2^{r_2} = 1.$$

Then

$$\lim_{m\to\infty} \delta_1 = 0 \quad \text{and} \quad \lim_{n\to\infty} \delta_2 = 0.$$

It directly follows from (3.25) that the sequence $\{U_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2} f\}_{m,n\in\mathbb{N}}$ converges uniformly to f for any $f \in C_b(I)$ on $[0,1] \times [0,1]$.

Let $\text{Lip}_M(\gamma_1,\gamma_2)$ ($\gamma_1,\gamma_2 \in (0,1]$) be a Lipschitz class defined by

$$\begin{aligned}
&\text{Lip}_M(\gamma_1,\gamma_2) \\
&\quad = \big\{f \in C_b(I) : \Delta_{u,v}[f:x,y] \leq M|u-x|^{\gamma_1}|v-y|^{\gamma_2}, (u,v),(x,y) \in [0,1] \times [0,1]\big\}.
\end{aligned}$$

Theorem 3.7 *Let* $f \in \text{Lip}_M(\gamma_1,\gamma_2)$. *Then, for any* $M > 0$ *and* $\gamma_1,\gamma_2 \in (0,1]$, *we have*

$$\left|f(x,y) - U_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}(f;p_1,p_2,q_1,q_2;x,y)\right| \leq M\delta_1^{\frac{\gamma_1}{2}}\delta_2^{\frac{\gamma_2}{2}}. \tag{3.31}$$

Proof By the definition of the operators $U_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}$ and linearity of the operators $S_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}$, we obtain

$$\begin{aligned} U_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}(f;p_1,p_2,q_1,q_2;x,y) &= S_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}\bigl(f(x,v)+f(u,y)-f(u,v);p_1,p_2,q_1,q_2;x,y\bigr) \\ &= S_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}\bigl(f(x,y)-\Delta_{u,v}[f:x,y];p_1,p_2,q_1,q_2;x,y\bigr) \\ &= f(x,y)S_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}(e_{00};p_1,p_2,q_1,q_2;x,y) \\ &\quad - S_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}\bigl(\Delta_{u,v}[f:x,y];p_1,p_2,q_1,q_2;x,y\bigr). \end{aligned}$$

By the hypothesis, we get

$$\begin{aligned} &\bigl|f(x,y)-U_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}(f;p_1,p_2,q_1,q_2;x,y)\bigr| \\ &\quad\le S_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}\bigl(\Delta_{u,v}[f:x,y];p_1,p_2,q_1,q_2;x,y\bigr) \\ &\quad\le MS_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}\bigl(|u-x|^{\gamma_1}|v-y|^{\gamma_2};p_1,p_2,q_1,q_2;x,y\bigr) \\ &\quad= MS_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}\bigl(|u-x|^{\gamma_1};p_1,q_1;x\bigr) \\ &\qquad\times S_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}\bigl(|v-y|^{\gamma_2};p_2,q_2;y\bigr). \end{aligned}$$

By using Hölder's inequality with $s_1=\frac{2}{\gamma_1}$, $t_1=\frac{2}{2-\gamma_1}$ and $s_2=\frac{2}{\gamma_2}$, $t_2=\frac{2}{2-\gamma_2}$, we have

$$\begin{aligned} \bigl|f(x,y)-U_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}(f;p_1,p_2,q_1,q_2;x,y)\bigr| &\le M\bigl(S_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}\bigl((u-x)^2;p_1,q_1;x\bigr)\bigr)^{\frac{\gamma_1}{2}} \\ &\quad\times\bigl(S_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}(e_0;p_1,q_1;x)\bigr)^{\frac{2-\gamma_1}{2}} \\ &\quad\times\bigl(S_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}\bigl((v-y)^2;p_1,q_1;y\bigr)\bigr)^{\frac{\gamma_2}{2}} \\ &\quad\times\bigl(S_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}(e_0;p_1,q_1;y)\bigr)^{\frac{2-\gamma_2}{2}}. \end{aligned}$$

Considering Lemma 3.2, we get the degree of local approximation for B-continuous functions $f\in\mathrm{Lip}_M(\gamma_1,\gamma_2)$. □

4 Degree of approximation of *B*-differentiable functions

In this section, we consider the case of B-differentiable functions. Note that a function $f:\mathbb{R}\to\mathbb{R}$ is called B-differentiable if, for each $(x,y)\in\mathbb{R}$, the following equality holds:

$$\lim_{(u,v)\to(x,y)}\frac{\Delta_{u,v}[f:x,y]}{(u-x)(v-y)}=D_Bf<\infty, \tag{4.1}$$

where D_Bf is a B-derivative of f.

The mixed modulus of smoothness $\omega_{\mathrm{mixed}}:[0,\infty)\times[0,\infty)\to[0,\infty)$ is defined as follows.

Definition 4.1 For any $\delta_1,\delta_2\in[0,\infty)\times[0,\infty)$ and for all $(x,y),(u,v)\in I_1\times I_2$,

$$\omega_{\mathrm{mixed}}(\delta_1,\delta_2)=\sup\bigl\{\bigl|\Delta_{u,v}[f:x,y]\bigr|:|u-x|\le\delta_1,|v-y|\le\delta_2\bigr\}. \tag{4.2}$$

In [5], Badea et al. proved the following Shisha–Mond type theorem for B-differentiable functions by using GBS operators. In what follows, we try to prove this result by using (p,q)-GBS operators.

Theorem 4.2 *Let $L : C_b(I_1 \times I_2) \to C_b(I_1 \times I_2)$ be a positive linear operator and $Uf(x,y)$ be the associated GBS operator. Then the following inequality holds for any $f \in C_b(I_1 \times I_2)$, $(x,y) \in I_1 \times I_2$, and $\delta_1, \delta_2 \geq 0$:*

$$\begin{aligned}\bigl|f(x,y) - Uf(x,y)\bigr| \leq{}& \bigl|f(x,y)\bigr|\bigl|1 - L(e_{00};x,y)\bigr| \\ &+ 3\|D_B f\|_\infty \sqrt{L\bigl((e_{10}-x)^2;x,y\bigr)L\bigl((e_{01}-y)^2;x,y\bigr)} \\ &+ \Bigl\{\sqrt{L\bigl((e_{10}-x)^2;x,y\bigr)L\bigl((e_{01}-y)^2;x,y\bigr)} \\ &+ \delta_1^{-1}\sqrt{L\bigl((e_{10}-x)^4;x,y\bigr)L\bigl((e_{01}-y)^2;x,y\bigr)} \\ &+ \delta_2^{-1}\sqrt{L\bigl((e_{10}-x)^2;x,y\bigr)L\bigl((e_{01}-y)^4;x,y\bigr)} \\ &+ (\delta_1\delta_2)^{-1}L\bigl((e_{10}-x)^2;x,y\bigr)L\bigl((e_{01}-y)^2;x,y\bigr)\Bigr\}\omega_{\text{mixed}}(D_B f;\delta_1,\delta_2).\end{aligned}$$

Theorem 4.3 *Let $U_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}$ be a GBS operators associated with $S_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}$ and f has a bounded B-derivative $D_B f$. Then the following inequality holds*:

$$\begin{aligned}&\bigl|f(x,y) - U_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}(f;p_1,p_2,q_1,q_2;x,y)\bigr| \\ &\quad\leq \frac{M}{\sqrt{([m]_{p_1,q_1}+\beta_1)([n]_{p_2,q_2}+\beta_2)}} \\ &\qquad\times\Bigl\{\|D_B f\|_\infty + \omega_{\text{mixed}}\Bigl(D_B f;\bigl([m]_{p_1,q_1}+\beta_1\bigr)^{-1/2},\bigl([n]_{p_2,q_2}+\beta_2\bigr)^{-1/2}\Bigr)\Bigr\}.\end{aligned}$$

Proof Since $f \in C_b(I)$, we have

$$\Delta_{u,v}[f:x,y] = (u-x)(v-y)D_B f(\lambda,\mu), \quad \text{with } x < \lambda < u; y < \mu < v,$$

where

$$D_B f(\lambda,\mu) = \Delta_{u,v}D_B f(\lambda,\mu) + D_B f(\lambda,y) + D_B f(x,\mu) - D_B f(x,y).$$

Since $D_B f \in B(I)$, we can write

$$\begin{aligned}&\bigl|S_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}\bigl(\Delta_{u,v}[f:x,y];p_1,p_2,q_1,q_2,x,y\bigr)\bigr| \\ &\quad\leq S_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}\bigl(|u-x||v-y|\omega_{\text{mixed}}\bigl(D_B f;|\lambda-x|,|\mu-y|\bigr);p_1,p_2,q_1,q_2;x,y\bigr) \\ &\qquad+ 3\|D_B f\|_\infty S_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}\bigl(|u-x||v-y|;p_1,p_2,q_1,q_2;x,y\bigr).\end{aligned} \tag{4.3}$$

Since the mixed modulus of smoothness is nondecreasing, we have

$$\begin{aligned}\omega_{\text{mixed}}\bigl(D_B f;|\lambda-x|,|\mu-y|\bigr) &\leq \omega_{\text{mixed}}\bigl(D_B f;|u-x|,|v-y|\bigr) \\ &\leq \bigl(1-\delta_1^{-1}|u-x|\bigr)\bigl(1-\delta_2^{-1}|v-y|\bigr)\omega_{\text{mixed}}(D_B f;\delta_1,\delta_2).\end{aligned}$$

Now substituting in inequality (4.3), by linearity of the operators $S_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}$ and by the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
&\left|f(x,y) - U_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}(f;p_1,p_2,q_1,q_2;x,y)\right| \\
&\quad = \left|S_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}\left(\Delta_{u,v}(f;x,y)\right)\right| \\
&\quad \leq 3\|D_B f\|_\infty \sqrt{S_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}\left(\psi_x^2;p_1,p_2,q_1,q_2;x,y\right)S_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}\left(\psi_y^2;p_1,p_2,q_1,q_2;x,y\right)} \\
&\qquad + \Bigg[\sqrt{S_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}\left(\psi_x^2;p_1,p_2,q_1,q_2;x,y\right)S_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}\left(\psi_y^2;p_1,p_2,q_1,q_2;x,y\right)} \\
&\qquad + \delta_1^{-1}\sqrt{S_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}\left(\psi_x^4;p_1,p_2,q_1,q_2;x,y\right)S_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}\left(\psi_y^2;p_1,p_2,q_1,q_2;x,y\right)} \\
&\qquad + \delta_2^{-1}\sqrt{S_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}\left(\psi_x^2;p_1,p_2,q_1,q_2;x,y\right)S_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}\left(\psi_y^4;p_1,p_2,q_1,q_2;x,y\right)} \\
&\qquad + (\delta_1\delta_2)^{-1}S_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}\left(\psi_x^2\psi_y^2;p_1,p_2,q_1,q_2;x,y\right)\Bigg]\omega_{\text{mixed}}(D_B f;\delta_1,\delta_2).
\end{aligned}$$

By using the following equality for $(x,y),(u,v) \in I_1 \times I_2$:

$$\begin{aligned}
S_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}\left((u-x)^{2i}(v-y)^{2j};x,y\right) &= S_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}\left((u-x)^{2i};x,y\right) \\
&\quad \times S_{m,n,r_1,r_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}\left((v-y)^{2j};x,y\right),
\end{aligned}$$

$i,j = 1,2$, from (3.29), (3.30), and Lemma 3.3, for any $(x,y) \in [0,1] \times [0,1]$, we get the result. □

Remark 4.4 For $p = 1$, all the above results reduce to q-analogues and for $p = q = 1$ these results further reduce to the classical ones.

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