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On a nonlinear sequential four-point fractional q-difference equation involving q-integral operators in boundary conditions along with stability criteria

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Abstract

In this paper, we consider a nonlinear sequential q-difference equation based on the Caputo fractional quantum derivatives with nonlocal boundary value conditions containing Riemann–Liouville fractional quantum integrals in four points. In this direction, we derive some criteria and conditions of the existence and uniqueness of solutions to a given Caputo fractional q-difference boundary value problem. Some pure techniques based on condensing operators and Sadovskii's measure and the eigenvalue of an operator are employed to prove the main results. Also, the Ulam–Hyers stability and generalized Ulam–Hyers stability are investigated. We examine our results by providing two illustrative examples.

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Keywords: q-operators; Fixed point; Fractional q-difference equation; Existence-uniqueness

1 Introduction

In several areas of sciences, such as biology, chemistry, economics, physics, and engineering, fractional calculus and its relevant differential equations and BVPs have been used extensively [1–3]. Indeed, fractional derivatives are not only a generalization of ordinary derivatives, but also they explain dynamical behavior of various physical processes specifically and effectively (real life phenomena) in contrast to integer order derivatives. References [4–18] are available for some improvements on the fractional differential equations theory.

By virtue of developments in fractional quantum calculus (q-FC), a number of scientists and researchers [19, 20] were attracted to a study of fractional q-difference equations, beginning in the nineteenth century, and wide interest lately [21–23].

In 2007, Atici et al. [24] studied some notions in relation to fractional q-calculus on time scales. Then in 2012, Annaby and Mansour presented their investigations by publishing a book on equations and BVPs in the context of fractional q-calculus [25]. Jarad et al. [26] turned to the stability notion on q-fractional non-autonomous systems and after that, Abdeljawad et al. [27] introduced Gronwall-type inequality in q-operator settings. By combining the two above notions, Butt et al. [28] investigated Ulam stability for a Caputo delay q-difference equation by means of q-Gronwall-type inequality. Also, some fascinating insights concerning IVPs and BVPs containing q-difference equations can be found in [29–35] and the references therein. Ahmad, Nieto, Alsaedi and Al-Hutami [36] turned to the q-difference FBVP with nonlocal integral conditions and implemented an existence analysis on the solutions of the proposed q-BVP which takes the format

$$\begin{cases} {}^{C}\mathcal{D}_q^{\rho}({}^{C}\mathcal{D}_q^{\beta} + b)u(t) = f(t, u(t)), \\ u(0) = d_1 I_q^{\gamma-1} u(\theta) = d_1 \int_0^{\theta} \frac{(\theta - qs)^{(\gamma-2)}}{\Gamma_q(\gamma-1)} u(s)\, d_q s, \\ u(1) = d_2 I_q^{\gamma-1} u(\pi) = d_2 \int_0^{\pi} \frac{(\pi - qs)^{(\gamma-2)}}{\Gamma_q(\gamma-1)} u(s)\, d_q s, \quad \gamma > 2, \theta > 0, \pi < 1, \end{cases} \tag{1}$$

where $f \in C([0,1] \times \mathbb{R}, \mathbb{R})$, $\rho, \beta \in (0,1]$, $q \in (0,1)$, $b, d_1, d_2 \in \mathbb{R}$ and ${}^{C}\mathcal{D}_q^{\rho}$, ${}^{C}\mathcal{D}_q^{\beta}$ denote the q-fractional derivatives in Caputo sense of orders ρ and β.

In 2014, Ahmad et al. [37] studied the existence criteria of the following q-difference equation involving two nonlinear terms and four-point nonlocal boundary conditions:

$$\begin{cases} {}^{C}\mathcal{D}_q^{\rho}({}^{C}\mathcal{D}_q^{\beta} + b)u(t) = mf(t, u(t)) + nI_q^{\gamma} g(t, u(t)), \\ w_1 u(0) - k_1 (t^{1-\beta}\, {}^{C}\mathcal{D}_q^{1} u(0))|_{t=0} = c_1 u(r_1), \quad 0 < r_1 < 1, \\ w_2 u(1) + k_2\, {}^{C}\mathcal{D}_q^{1} u(1) = c_2 u(r_2), \quad 0 < r_2 < 1, \end{cases} \tag{2}$$

in which $f, g \in C([0,1] \times \mathbb{R}, \mathbb{R})$, $\rho, \beta \in (0,1]$, $q \in (0,1)$, $b, m, n, w_1, w_2, k_1, k_2 \in \mathbb{R}$, $c_1, c_2 \in (0,1)$ and ${}^{C}\mathcal{D}_q^{\rho}$, ${}^{C}\mathcal{D}_q^{\beta}$ denotes the q-fractional derivatives in Caputo sense and I_q^{γ} denotes the fractional q-integral in Riemann–Liouville sense of order $\gamma \in (0,1)$.

In continuation to the investigation of the q-variant of fractional problems and inspired by the aforementioned work, we aim to examine this area from another angle. Several known methods of functional analysis are used to establish required results on the existence of solutions for a class of q-difference problem. More specifically, we consider the sequential four-point Caputo fractional q-difference boundary value problem (q-CFBVP) of the format

$$\begin{cases} \mathcal{D}_q^{\alpha}(\mathcal{D}_q^{\beta} u(t) - g(t, u(t))) = f(t, u(t)), \quad t \in \mathrm{J} := [0, T], \\ a_1 u(0) + b_1 \mathcal{D}_q^{\gamma} u(0) = \lambda_1 \int_0^{\eta_1} \frac{(\eta_1 - qs)^{(\sigma_1 - 1)}}{\Gamma_q(\sigma_1)} u(s)\, \mathrm{d}_q s, \quad \eta_1 \in (0, T), \sigma_1 > 0, \\ a_2 u(T) + b_2 \mathcal{D}_q^{\gamma} u(T) = \lambda_2 \int_0^{\eta_2} \frac{(\eta_2 - qs)^{(\sigma_2 - 1)}}{\Gamma_q(\sigma_2)} u(s)\, \mathrm{d}_q s, \quad \eta_2 \in (0, T), \sigma_2 > 0, \end{cases} \tag{3}$$

where $\mathcal{D}_q^{\mu}$ is the μth-q-difference derivative in the Caputo structure with $\mu \in \{\gamma, \beta, \alpha\}$ such that $0 < \alpha, \beta \leq 1, 0 < \gamma \leq 1$ and I_q^{θ} is the θth-q-difference integral in the Riemann–Liouville structure with $\theta > 0$ subject to $\theta \in \{\sigma_1, \sigma_2\}$ and also $f, g : \mathrm{J} \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions. a_1, a_2, b_1, b_2, λ_1, λ_2 are suitably chosen constants in $\mathbb{R}^+$.

Regarding to the novelty of the paper, in comparison to above q-problems, our supposed sequential q-CFBVP is more general. Under the given boundary value conditions, we have used both Caputo and Riemann–Liouville q-fractional operators in four different points of domain of the unknown solution function u simultaneously, in which the linear combinations of the unknown function and its fractional derivative is corresponding to a multiple of q-Riemann–Liouville integral in two mid-points. In this paper, we have designed an extended form of Langevin equations by providing a nonlinear function g in the left-side hand of the given boundary value problem (3). Also, to prove the existence of solutions for such an applied q-problem, we shall utilize some pure notions of functional analysis based on the measure of non-compactness, condensing operators and eigenvalue of the operator, which have been used in papers limited in this regard so far and this distinguishes our research from the work of others. Moreover, we here emphasize that this paper may have useful and effective applications in physics and quantum mechanics such as Langevin systems in the context of quantum operators.

The remaining part of this paper is organized as follows: Sect. 2 is devoted to the primitive notions of q-FC. At first, in Sect. 3, we give an auxiliary lemma which provides the solution of the supposed q-CFBVP (3) and then based on the obtained integral equation, by using fixed point theorems due to Sadovski, Krasnoselskii–Zabreiko and O'Regan, we establish the existence of solutions for the q-CFBVP (3) and also for its uniqueness, we utilize the famous Banach principle. In Sect. 4, the stability criteria of Ulam–Hyers type and its generalized type are checked. Additionally, in Sect. 5, we provide two examples which ensure the usability of the results presented in Sect. 3. The manuscript is ended by our conclusions in Sect. 6.

2 Preliminaries regarding q-operators

We collect some important basic notions of q-FC in this section. For details, we refer to [19, 21, 38, 39]. Let $q \in (0,1)$. A q-real number is denoted by $[m]_q$ and is defined as

$$[m]_q = \frac{1-q^m}{1-q}, \quad m \in \mathbb{R}.$$

The q-power function $(m-n)^k$ with $m, n \in \mathbb{R}$ is

$$(m-n)^{(0)} = 1, \qquad (m-n)^{(k)} = \prod_{j=0}^{k-1} \bigl(m - nq^j\bigr), \quad k \in \mathbb{N} \cup \{0\},$$

and, if $\beta \in \mathbb{R}$, then

$$(m-n)^{(\beta)} = m^\beta \prod_{i=0}^{\infty} \frac{m - nq^i}{m - nq^{\beta+i}}.$$

On the other side, $[c(m-n)]^{(\beta)} = c^\beta (m-n)^{(\beta)}$ holds for $c \in \mathbb{R}$ and also notice that $m^{(\beta)} = m^\beta$ if $n = 0$. The q-Gamma function is given by

$$\Gamma_q(\alpha) = \frac{(1-q)^{(\alpha-1)}}{(1-q)^{\alpha-1}}, \quad \alpha \in \mathbb{R} \backslash \{0, -1, \dots\},$$

and satisfies $\Gamma_q(\alpha+1) = [\alpha]_q \Gamma_q(\alpha)$.

The 1st-q-derivative of an arbitrary mapping ϕ is defined by the following rule:

$$(D_q \phi)(x) = \frac{\phi(qx) - \phi(x)}{(1-q)x}, \quad x \neq 0,$$

and for the higher orders, it becomes

$$D_q^0\phi = \phi, \qquad D_q^k\phi = D_q\bigl(D_q^{k-1}\phi\bigr), \quad k \in \mathbb{N} \cup \{0\}.$$

The 1st-q-integral of an arbitrary mapping ϕ given on the interval $[0,n]$ is given by

$$(I_q\phi)(x) = \int_0^x \phi(r)\,d_qr = x(1-q)\sum_{k=0}^{\infty}\phi\bigl(xq^k\bigr)q^k, \quad x \in [0,n].$$

If $m \in [0,n]$, then

$$\int_m^n \phi(r)\,d_qr = \int_0^n \phi(r)\,d_qr - \int_0^m \phi(r)\,d_qr.$$

Similarly, for the higher orders, it becomes

$$I_q^0\phi = \phi, \qquad I_q^k\phi = I_q\bigl(I_q^{k-1}\phi\bigr), \quad k \in \mathbb{N} \cup \{0\}.$$

For two first order q-operators D_q and I_q, we have

$$D_qI_q\phi(x) = \phi(x).$$

Here, we assemble some definitions about such q-operators from the fractional point of view.

Definition 1 ([39]) Let $\alpha \geq 0$. The αth-q-integral of the Riemann–Liouville type for ϕ defined on $[0,\infty)$ is given by $I_q^0\phi(t) = \phi(t)$ and

$$I_q^\alpha\phi(t) = \frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qr)^{(\alpha-1)}\phi(t)\,d_qr, \quad \alpha > 0.$$

Definition 2 ([19]) The Caputo αth-q-derivative for an absolutely continuous mapping ϕ is formulated by

$$D_q^\alpha\phi(t) = I_q^{[\alpha]-\alpha}D_q^{[\alpha]}\phi(t),$$

where $[\alpha]$ denotes the integer part of α.

For more information on the fractional q-operators, we refer the reader to [38].

Lemma 3 ([19]) *Let* $\alpha, \beta \in \mathbb{R}_+$. *Then we have the following formulas*:

(1) $I_q^\alpha I_q^\beta\phi(t) = I_q^{\alpha+\beta}\phi(t)$;
(2) $D_q^\alpha I_q^\alpha\phi(t) = \phi(t)$.

Lemma 4 ([40]) *Let* $\alpha \in \mathbb{R}_+$ *and* $\beta \in (-1,\infty)$. *One has*

$$I_q^\alpha t^\beta = \frac{\Gamma_q(\beta+1)}{\Gamma_q(\alpha+\beta+1)}t^{\alpha+\beta}, \quad t > 0.$$

In particular, if $\phi \equiv 1$, *then*

$$I_q^{\alpha} 1(t) = \frac{1}{\Gamma_q(1+\alpha)} t^{(\alpha)}, \quad \textit{for all } t > 0.$$

Lemma 5 ([19]) *Let* $\alpha, \sigma > 0$. *Then*

$$I_q^{\alpha} D_q^{\sigma} \phi(t) = D_q^{\sigma} I_q^{\alpha} \phi(t) - \sum_{j=0}^{\sigma-1} \frac{t^{\alpha-\sigma+j}}{\Gamma_q(\alpha+j-\sigma+1)} D_q^j \phi(0).$$

Lemma 6 ([30, 40]) *Let* $k-1 < \alpha < k$. *Then*

$$I_q^{\alpha} D_q^{\alpha} \phi(t) = \phi(t) - \sum_{j=0}^{k-1} \frac{t^j}{\Gamma_q(j+1)} D_q^j \phi(0).$$

For the homogeneous q-difference equation $D_q^{\alpha}\phi(t) = 0$, the general series solution by Lemma 6 is given as $\phi(t) = \mu_0 + \mu_1 t + \mu_2 t^2 + \cdots + \mu_{k-1} t^{k-1}$ via $\mu_j \in \mathbb{R}$ and $k = [\alpha] + 1$ [19]. So, we have

$$\left(I_q^{\alpha} D_q^{\alpha} \phi\right)(t) = \phi(t) + \mu_0 + \mu_1 t + \mu_2 t^2 + \cdots + \mu_{k-1} t^{k-1}.$$

3 Results regarding existence property

In the present section, before moving to our fundamental results, we define $\|\cdot\|$ on $X = C(J, \mathbb{R})$ as $\|u\| = \sup_{t\in J} |u(t)|$, which in this phase, X transforms into a Banach space. Now, in the first place, we provide the next auxiliary lemma.

Lemma 7 *Let* $\psi \in C(\mathrm{J}, \mathbb{R})$, $\alpha, \beta, \gamma \in (0,1)$, $\sigma_1, \sigma_2 > 0$, $a_1, a_2, b_1, b_2, \lambda_1, \lambda_2 \in \mathbb{R}^+$ *and* $g_u(t) = g(t, u(t))$. *Then the solution of the linear sequential four-point q-CFBVP defined by*

$$\begin{cases} \mathcal{D}_q^{\alpha}(\mathcal{D}_q^{\beta} u(t) - g_u(t)) = \psi(t), & t \in \mathrm{J} := [0, T], \\ a_1 u(0) + b_1 \mathcal{D}_q^{\gamma} u(0) = \lambda_1 I_q^{\sigma_1} u(\eta_1), & 0 < \eta_1 < T, \sigma_1 > 0, \\ a_2 u(T) + b_2 \mathcal{D}_q^{\gamma} u(T) = \lambda_2 I_q^{\sigma_2} u(\eta_2), & 0 < \eta_2 < T, \sigma_2 > 0, \end{cases} \tag{4}$$

is given by

$$\begin{aligned} u(t) = {} & I_q^{\beta} g_u(t) + I_q^{\alpha+\beta} \psi(t) \\ & + \mu_1(t)\left[I_q^{\beta+\sigma_1} g_u(\eta_1) + I_q^{\alpha+\beta+\sigma_1} \psi(\eta_1)\right] \\ & + \mu_2(t)\left[\lambda_2\left(I_q^{\beta+\sigma_2} g_u(\eta_2) + I_q^{\alpha+\beta+\sigma_2} \psi(\eta_2)\right)\right. \\ & \left. - b_2\left(I_q^{\beta-\gamma} g_u(T) + I_q^{\alpha+\beta-\gamma} \psi(T)\right) - a_2\left(I_q^{\beta} g_u(T) + I_q^{\alpha+\beta} \psi(T)\right)\right], \end{aligned} \tag{5}$$

where

$$\begin{aligned}
&\mu_1(t) = \lambda_1\left(\frac{(\Lambda - \Lambda_2\Lambda_3)\Gamma_q(\beta+1) - t^\beta\Lambda_1\Lambda_3}{\Lambda\Lambda_1\Gamma_q(\beta+1)}\right),\\
&\mu_2(t) = \frac{\Lambda_1 t^\beta + \Lambda_2\Gamma_q(\beta+1)}{\Lambda\Gamma_q(\beta+1)},\\
&\Lambda_1 = a_1 - \frac{\lambda_1\eta_1^{\sigma_1}}{\Gamma_q(\sigma_1+1)}, \qquad \Lambda_2 = \frac{\lambda_1\eta_1^{\beta+\sigma_1}}{\Gamma_q(\beta+\sigma_1+1)},\\
&\Lambda_3 = a_2 - \frac{\lambda_2\eta_2^{\sigma_2}}{\Gamma_q(\sigma_2+1)},\\
&\Lambda_4 = \frac{a_2T^\beta}{\Gamma_q(\beta+1)} + \frac{b_2T^{\beta-\gamma}}{\Gamma_q(\beta-\gamma+1)} - \frac{\lambda_2\eta_2^{\beta+\sigma_2}}{\Gamma_q(\beta+\sigma_2+1)},
\end{aligned} \tag{6}$$

and Λ *is given by*

$$\begin{aligned}
\Lambda ={}& \left(\frac{a_2T^\beta}{\Gamma_q(\beta+1)} + \frac{b_2T^{\beta-\gamma}}{\Gamma_q(\beta-\gamma+1)} - \frac{\lambda_2\eta_2^{\beta+\sigma_2}}{\Gamma_q(\beta+\sigma_2+1)}\right)\left(a_1 - \frac{\lambda_1\eta_1^{\sigma_1}}{\Gamma_q(\sigma_1+1)}\right)\\
&+ \frac{\lambda_1\eta_1^{\beta+\sigma_1}}{\Gamma_q(\beta+\sigma_1+1)}\left(a_2 - \frac{\lambda_2\eta_2^{\sigma_2}}{\Gamma_q(\sigma_2+1)}\right) = \Lambda_4\Lambda_1 + \Lambda_2\Lambda_3 \neq 0.
\end{aligned} \tag{7}$$

Proof By using Lemma 6, we obtain the integral equation corresponding to (4):

$$u(t) = I_q^\beta g_u(t) + I_q^{\alpha+\beta}\psi(t) + \frac{t^\beta}{\Gamma_q(\beta+1)}k_0 + k_1, \quad k_0, k_1 \in \mathbb{R}. \tag{8}$$

Using the given boundary conditions in (4), we may obtain

$$I_q^{\sigma_i}u(t) = I_q^{\sigma_i+\beta}g_u(t) + I_q^{\sigma_i+\alpha+\beta}\psi(t) + k_0\frac{t^{\beta+\sigma_i}}{\Gamma_q(\beta+\sigma_i+1)} + k_1\frac{t^{\sigma_i}}{\Gamma_q(\sigma_i+1)}$$

for $i = 1, 2$ and

$$\mathcal{D}_q^\gamma x(t) = I_q^{\beta-\gamma}g_u(t) + I_q^{\alpha+\beta-\gamma}\psi(t) + k_0\frac{t^{\beta-\gamma}}{\Gamma_q(\beta-\gamma+1)}.$$

By categorizing similar terms, we obtain the expressions

$$\Lambda_1k_1 - \Lambda_2k_0 = \lambda_1\mathcal{I}_q^{\sigma_1+\beta}g_u(\eta_1) + \lambda_1 I_q^{\sigma_1+\beta+\alpha}\psi(\eta_1) \tag{9}$$

and

$$\begin{aligned}
\Lambda_3k_1 + \Lambda_4k_0 ={}& \lambda_2\big(I_q^{\sigma_2+\beta}g_u(\eta_2) + I_q^{\sigma_2+\beta+\alpha}\psi(\eta_2)\big)\\
&- a_2\big(I_q^\beta g_u(T) + I_q^{\beta+\alpha}\psi(T)\big) - b_2\big(I_q^{\beta-\gamma}g_u(T) + I_q^{\beta+\alpha-\gamma}\psi(T)\big).
\end{aligned} \tag{10}$$

Therefore, from (9) and (10), we get

$$k_0 = -\frac{\Lambda_3}{\Lambda}\big(\lambda_1 I_q^{\sigma_1+\beta}g_u(\eta_1) + \lambda_1 I_q^{\sigma_1+\beta+\alpha}\psi(\eta_1)\big)$$

$$
\begin{aligned}
&+ \frac{\Lambda_1}{\Lambda}\big[\lambda_2\big(I_q^{\sigma_2+\beta} g_u(\eta_2) + I_q^{\sigma_2+\beta+\alpha}\psi(\eta_2)\big) \\
&- a_2\big(I_q^{\beta} g_u(T) + I_q^{\beta+\alpha}\psi(T)\big) - b_2\big(I_q^{\beta-\gamma} g_u(T) + I_q^{\beta+\alpha-\gamma}\psi(T)\big)\big],
\end{aligned}
$$

and, by inserting k_0 into (9), we obtain

$$
\begin{aligned}
k_1 = {}& \frac{\Lambda - \Lambda_2\Lambda_3}{\Lambda_1\Lambda}\big(\lambda_1 I_q^{\sigma_1+\beta} g_u(\eta_1) + \lambda_1 I_q^{\sigma_1+\beta+\alpha}\psi(\eta_1)\big) \\
&+ \frac{\Lambda_2}{\Lambda}\big[\lambda_2\big(I_q^{\sigma_2+\beta} g_u(\eta_2) + I_q^{\sigma_2+\beta+\alpha}\psi(\eta_2)\big) - a_2\big(I_q^{\beta} g_u(T) + I_q^{\beta+\alpha}\psi(T)\big) \\
&- b_2\big(I_q^{\beta-\gamma} g_u(T) + I_q^{\beta+\alpha-\gamma}\psi(T)\big)\big].
\end{aligned}
$$

Substituting the value of k_0, k_1 in (8), we get (5), which completes the proof. □

Note that, for simplicity, we set $g(t,u(t)) = g_u(t)$ and $f(t,u(t)) = f_u(t)$ throughout the manuscript.

3.1 The first existence criterion

In this subsection, we prove an existence result for the sequential four-point q-CFBVP (3) by making use of Sadovskii's fixed-point theorem. Before moving towards it, we would like to recall several auxiliary facts which are our main tools. X is supposed as a Banach space.

Definition 8 Consider a bounded subset M of (X,d). The Kuratowski measure of noncompactness, denoted by $\alpha(M)$, is defined by

$$
\alpha(M) := \inf\left\{\epsilon > 0 : \exists \text{ finitely many sets } M_i \text{ s.t. } M = \bigcup_{i=1}^{n} M_i \text{ and } D(M_i) \le \epsilon\right\},
$$

where $D(M_i) = \sup\{|u - \tilde{u}| : u, \tilde{u} \in M_i\}$.

Definition 9 ([41]) Consider a bounded and continuous function $\Phi : \mathrm{Dom}(\Phi) \subseteq X \to X$ on X. For an arbitrary bounded set $M \subset \mathrm{Dom}(\Phi)$, the map Φ is condensing if

$$
\alpha\big(\Phi(M)\big) < \alpha(M),
$$

in which α is introduced above.

Lemma 10 ([42]) *Let $\mathcal{K}_1, \mathcal{K}_2 : E \subseteq X \to X$. The operator $\mathcal{K}_1 + \mathcal{K}_2$ is condensing if*

i. *$\mathcal{K}_1$ is k-contraction; that is, $\forall u, v \in E$ and $\exists k \in (0,1)$, so that*

$$
\|\mathcal{K}_1 u - \mathcal{K}_1 v\| \le k\|u - v\|;
$$

ii. *$\mathcal{K}_2$ is compact.*

Theorem 11 ([43]) *Consider the bounded, closed and convex subset B of X and the condensing mapping $\Phi : B \to B$. Then Φ has a fixed point.*

From now on, we put

$$\Theta_1 = \frac{T^{\alpha+\beta}}{\Gamma_q(\alpha+\beta+1)} + |\mu_1(T)| \frac{\eta_1^{\alpha+\beta+\sigma_1}}{\Gamma_q(\alpha+\beta+\sigma_1+1)} + |\mu_2(T)| \left(\lambda_2 \frac{\eta_2^{\alpha+\beta+\sigma_2}}{\Gamma_q(\alpha+\beta+\sigma_2+1)} + b_2 \frac{T^{\alpha+\beta-\gamma}}{\Gamma_q(\alpha+\beta-\gamma+1)} + a_2 \frac{T^{\alpha+\beta}}{\Gamma_q(\alpha+\beta+1)} \right) \tag{11}$$

and

$$\Theta_2 = \frac{T^{\beta}}{\Gamma_q(\beta+1)} + |\mu_1(T)| \frac{\eta_1^{\beta+\sigma_1}}{\Gamma_q(\beta+\sigma_1+1)} + |\mu_2(T)| \left(\lambda_2 \frac{\eta_2^{\beta+\sigma_2}}{\Gamma_q(\beta+\sigma_2+1)} + b_2 \frac{T^{\beta-\gamma}}{\Gamma_q(\beta-\gamma+1)} + a_2 \frac{T^{\beta}}{\Gamma_q(\beta+1)} \right). \tag{12}$$

Theorem 12 *Consider the following assertions*:

($\mathbf{B}_1$) $\exists L > 0$ *so that* $|f_u(t) - f_v(t)| \le L|u(t) - v(t)|$, $\forall t \in J$, $u, v \in \mathbb{R}$;

($\mathbf{B}_2$) $|f_u(t)| \le \sigma(t)$ *and* $|g_u(t)| \le \rho(t)$, *where* $\sigma, \rho \in C(J, \mathbb{R}^+)$.

Then the sequential four-point q-CFBVP (3) *has a solution on* J *if* $Q := L\Theta_1 < 1$, *by introducing* Θ_1 *as* (11).

Proof Consider a bounded, closed and convex subset $B_r = \{u \in X : \|u\| \le r\}$ of $X = C(J, \mathbb{R})$ for a fixed constant r. With regard to Lemma 7, define $\mathcal{K} : X \to X$ as follows:

$$\begin{aligned} \mathcal{K}u(t) = {} & I_q^{\beta} g_u(t) + I_q^{\alpha+\beta} f_u(t) \\ & + \mu_1(t)\big[I_q^{\beta+\sigma_1} g_u(\eta_1) + I_q^{\alpha+\beta+\sigma_1} f_u(\eta_1)\big] \\ & + \mu_2(t)\big[\lambda_2\big(I_q^{\beta+\sigma_2} g_u(\eta_2) + I_q^{\alpha+\beta+\sigma_2} f_u(\eta_2)\big) \\ & - b_2\big(I_q^{\beta-\gamma} g_u(T) + I_q^{\alpha+\beta-\gamma} f_u(T)\big) - a_2\big(I_q^{\beta} g_u(T) + I_q^{\alpha+\beta} f_u(T)\big)\big]. \end{aligned} \tag{13}$$

We split the operator $\mathcal{K}$ on the set B_r into $\mathcal{K} = \mathcal{K}_1 + \mathcal{K}_2$, where

$$\begin{aligned} \mathcal{K}_1 u(t) = {} & I_q^{\alpha+\beta} f_u(t) + \mu_1(t)\big(I_q^{\alpha+\beta+\sigma_1} f_u(\eta_1)\big) \\ & + \mu_2(t)\big[\lambda_2\big(I_q^{\alpha+\beta+\sigma_2} f_u(\eta_2)\big) - b_2\big(I_q^{\alpha+\beta-\gamma} f_u(T)\big) - a_2\big(I_q^{\alpha+\beta} f_u(T)\big)\big] \end{aligned} \tag{14}$$

and

$$\begin{aligned} \mathcal{K}_2 u(t) = {} & I_q^{\beta} g_u(t) + \mu_1(t)\big(I_q^{\beta+\sigma_1} g_u(\eta_1)\big) \\ & + \mu_2(t)\big[\lambda_2\big(I_q^{\beta+\sigma_2} g_u(\eta_2)\big) - b_2\big(I_q^{\beta-\gamma} g_u(T)\big) - a_2\big(I_q^{\beta} g_u(T)\big)\big]. \end{aligned} \tag{15}$$

We want to prove that the operators $\mathcal{K}_1$ and $\mathcal{K}_2$ follow all the assertions of Theorem 11. We proceed to implement the proof in four steps.

Step 1: $\mathcal{K}B_r \subset B_r$

Let us select r so that $r \geq \|\sigma\|\Theta_1 + \|\rho\|\Theta_2$, where Θ_2, Θ_1 are given by (11) and (12) and $\|\sigma\| = \sup_{t\in J}|\sigma(t)|$ and $\|\rho\| = \sup_{t\in J}|\rho(t)|$. For any $u \in B_r$, we have

$$\begin{aligned}\bigl|(\mathcal{K}u)(t)\bigr| \leq{}& \sup_{t\in J}\bigl\{I_q^{\beta}\bigl|g_u(t)\bigr| + I_q^{\alpha+\beta}\bigl|f_u(t)\bigr| \\ &+ \bigl|\mu_1(t)\bigr|\bigl[I_q^{\beta+\sigma_1}\bigl|g_u(\eta_1)\bigr| + I_q^{\alpha+\beta+\sigma_1}\bigl|f_u(\eta_1)\bigr|\bigr] + \bigl|\mu_2(t)\bigr|\bigl[\lambda_2\bigl(I_q^{\beta+\sigma_2}\bigl|g_u(\eta_2)\bigr| \\ &+ I_q^{\alpha+\beta+\sigma_2}\bigl|f_u(\eta_2)\bigr|\bigr) + b_2\bigl(I_q^{\beta-\gamma}\bigl|g_u(T)\bigr| + I_q^{\alpha+\beta-\gamma}\bigl|f_u(T)\bigr|\bigr) + a_2\bigl(I_q^{\beta}\bigl|g_u(T)\bigr| \\ &+ I_q^{\alpha+\beta}\bigl|f_u(T)\bigr|\bigr)\bigr]\bigr\} \leq \Theta_2\|\rho\| + \Theta_1\|\sigma\| < r,\end{aligned}$$

which implies that $\mathcal{K}B_r \subset B_r$.

Step 2: $\mathcal{K}_2$ is compact

In view of Step 1, we observe that the operator $\mathcal{K}_2$ is uniformly bounded; indeed for any $u \in B_r$:

$$\begin{aligned}\bigl|(\mathcal{K}_2u)(t)\bigr| \leq{}& I_q^{\beta}\bigl|g_u(t)\bigr| + \bigl|\mu_1(t)\bigr|\bigl(I_q^{\beta+\sigma_1}\bigl|g_u(\eta_1)\bigr|\bigr) \\ &+ \bigl|\mu_2(t)\bigr|\bigl[\lambda_2\bigl(I_q^{\beta+\sigma_2}\bigl|g_u(\eta_2)\bigr|\bigr) + b_2\bigl(I_q^{\beta-\gamma}\bigl|g_u(T)\bigr|\bigr) + a_2\bigl(I_q^{\beta}\bigl|g_u(T)\bigr|\bigr)\bigr] \\ \leq{}& \|\rho\|\biggl[\frac{T^{\beta}}{\Gamma_q(\beta+1)} + \bigl|\mu_1(T)\bigr|\frac{\eta_1^{\beta+\sigma_1}}{\Gamma_q(\beta+\sigma_1+1)} \\ &+ \bigl|\mu_2(T)\bigr|\biggl(\lambda_2\frac{\eta_2^{\beta+\sigma_2}}{\Gamma_q(\beta+\sigma_2+1)} + b_2\frac{T^{\beta-\gamma}}{\Gamma_q(\beta-\gamma+1)} + a_2\frac{T^{\beta}}{\Gamma_q(\beta+1)}\biggr)\biggr] \\ \leq{}& \Theta_2\|\rho\|.\end{aligned}$$

Now, take $t_1, t_2 \in J$ by assuming $t_1 < t_2$ and $u \in B_r$. Hence we have

$$\begin{aligned}&\bigl|\mathcal{K}_2u(t_2) - \mathcal{K}_2u(t_1)\bigr| \\ &\quad\leq I_q^{\beta}\bigl|g_u(t_2) - g_u(t_1)\bigr| + \bigl|\mu_1(t_2) - \mu_1(t_1)\bigr|\bigl(I_q^{\beta+\sigma_1}\bigl|g_u(\eta_1)\bigr|\bigr) \\ &\qquad+ \bigl|\mu_2(t_2) - \mu_2(t_1)\bigr|\bigl[\lambda_2\bigl(I_q^{\beta+\sigma_2}\bigl|g_u(\eta_2)\bigr|\bigr) + b_2\bigl(I_q^{\beta-\gamma}\bigl|g_u(T)\bigr|\bigr) + a_2\bigl(I_q^{\beta}\bigl|g_u(T)\bigr|\bigr)\bigr] \\ &\quad\leq \frac{\|\rho\|}{\Gamma_q(\beta+1)}\bigl[t_2^{\beta} - t_1^{\beta} + 2(t_2-t_1)^{\beta}\bigr] + \bigl|\mu_1(t_2) - \mu_1(t_1)\bigr|\bigl(I_q^{\beta+\sigma_1}\|\rho\|\bigr) \\ &\qquad+ \bigl|\mu_2(t_2) - \mu_2(t_1)\bigr|\bigl(\lambda_2\bigl(I_q^{\beta+\sigma_2}\|\rho\|\bigr) + b_2\bigl(I_q^{\beta-\gamma}\|\rho\|\bigr) + a_2\bigl(I_q^{\beta}\|\rho\|\bigr)\bigr).\end{aligned} \tag{16}$$

The right-hand side of (16) tends to zero (not depending upon u) as $t_2 \to t_1$. This shows that $\mathcal{K}_2$ is equicontinuous. From the above reasons, it is clear that $\mathcal{K}_2$ is relatively compact on B_r. Application of the Arzelà–Ascoli theorem proves the compactness of $\mathcal{K}_2$ on B_r.

Step 3: $\mathcal{K}_1$ is Q-contractive.

From ($\mathbf{B}_1$) and ($\mathbf{B}_2$) and for each $u, v \in B_r$, we have

$$\begin{aligned}\bigl|\mathcal{K}_1u(t) - \mathcal{K}_1v(t)\bigr| \leq{}& \sup_{t\in J}\bigl\{I_q^{\alpha+\beta}|f_u - f_v|(t) + \bigl|\mu_1(t)\bigr|\bigl(I_q^{\alpha+\beta+\sigma_1}|f_u - f_v|(\eta_1)\bigr) \\ &+ \bigl|\mu_2(t)\bigr|\bigl[\lambda_2\bigl(I_q^{\alpha+\beta+\sigma_2}|f_u - f_v|(\eta_2)\bigr) + b_2\bigl(I_q^{\alpha+\beta-\gamma}|f_u - f_v|(T)\bigr)\end{aligned}$$

$$+ a_2\big(I_q^{\alpha+\beta}|f_u - f_v|(T)\big)\big]\Big\}$$
$$\leq L\Theta_1\|u-v\|.$$

So, $\|\mathcal{K}_1u - \mathcal{K}_1v\| \leq L\Theta_1\|u-v\|$. Thus $\mathcal{K}_1$ is Q-contractive because of $Q := L\Theta_1 < 1$.

Step 4: $\mathcal{K}$ is condensing.

As $\mathcal{K}_1$ and $\mathcal{K}_2$ are continuous Q–contraction and compact, respectively, thus by Lemma 10, $\mathcal{K} : B_r \to B_r$ with $\mathcal{K} = \mathcal{K}_1 + \mathcal{K}_2$ is a condensing map on B_r. From the above arguments, by Theorem 11, we conclude that the map $\mathcal{K}$ has a fixed point, which leads to the existence of at least one solution for the sequential four-point q-CFBVP (3) in X. □

3.2 The second existence criterion

We now use another fixed point result due to Krasnoselskii–Zabreiko to demonstrate the following existence criterion for the sequential four-point q-CFBVP (3).

Theorem 13 ([44]) *Consider a completely continuous map $\mathcal{K}$ on a Banach space X. If a bounded linear map $\mathcal{L}$ exists on X so that 1 is not an eigenvalue of it and*

$$\lim_{\|u\|\to\infty} \frac{\|\mathcal{K}(u) - \mathcal{L}(u)\|}{\|u\|} = 0,$$

then $\mathcal{K}$ has a fixed point in X.

Theorem 14 *Consider the following assertions*:

(H1) *$f : J \times \mathbb{R} \to \mathbb{R}$ is continuous and for some $t \in J$, $f(t,0) \neq 0$ and*

$$\lim_{\|u\|\to\infty} \frac{f(t,u)}{u} = \lambda(t). \tag{17}$$

(H2) *The function $g : J \times \mathbb{R} \to \mathbb{R}$ is continuous and $\exists A \in \mathbb{R}_+$ so that*

$$\big|g\big(t,u(t)\big)\big| \leq A\big|u(t)\big|.$$

Then there exists at least one solution for the sequential four-point q-CFBVP (3) on J such that

$$\lambda_{\max} := \max_{t\in J}\big|\lambda(t)\big| < \frac{1 - A\Theta_2}{\Theta_1}, \tag{18}$$

where Θ_1 and Θ_2 are, respectively, given by (11) and (12).

Proof Consider a sequence $\{u_n\} \subset B_r$ which converges to u. We know that f and g are continuous, so, by letting $n \to \infty$, we get

$$|f_{u_n} - f_u|(t) \to 0, \qquad |g_{u_n} - g_u|(t) \to 0.$$

Thus, for $t \in J$, we write

$$\big|(\mathcal{K}u_n)(t) - (\mathcal{K}u)(t)\big| \leq I_q^{\beta}|g_{u_n} - g_u|(t) + I_q^{\alpha+\beta}|f_{u_n} - f_u|(t)$$

$$+ \mu_1(t)\big(I_q^{\beta+\sigma_1}|g_{u_n} - g_u|(\eta_1) + I_q^{\alpha+\beta+\sigma_1}|f_{u_n} - f_u|(\eta_1)\big)$$
$$+ \mu_2(t)\big(\lambda_2\big(I_q^{\beta+\sigma_2}|g_{u_n} - g_u|(\eta_2) + I_q^{\alpha+\beta+\sigma_1}|f_{u_n} - f_u|(\eta_2)\big)$$
$$+ b_2\big(I_q^{\beta-\gamma}|g_{u_n} - g_u|(T) + I_q^{\alpha+\beta-\gamma}|f_{u_n} - f_u|(T)\big)$$
$$+ a_2\big(I_q^{\beta}|g_{u_n} - g_u|(T) + I_q^{\alpha+\beta}|f_{u_n} - f_u|(T)\big)\big) \to 0. \tag{19}$$

Therefore the right-hand side of (19) tends to zero. Therefore, the continuity of $\mathcal{K}$ is established. Now, for $r > 0$, we set $N = \{u \in C(J,\mathbb{R}); \|u\| \leq r\}$ and $\|f^*\| = \sup_{(t,u)\in J\times N} |f_u(t)|$. Thus,

$$\begin{aligned}
\big|(\mathcal{K}u)(t)\big| &\leq I_q^{\beta}\big|g_u(t)\big| + I_q^{\alpha+\beta}\big|f_u(t)\big| \\
&\quad + \big|\mu_1(t)\big|\big[I_q^{\beta+\sigma_1}\big|g_u(\eta_1)\big| + I_q^{\alpha+\beta+\sigma_1}\big|f_u(\eta_1)\big|\big] \\
&\quad + \big|\mu_2(t)\big|\big[\lambda_2\big(I_q^{\beta+\sigma_2}\big|g_u(\eta_2)\big| + I_q^{\alpha+\beta+\sigma_2}\big|f_u(\eta_2)\big|\big) \\
&\quad + b_2\big(I_q^{\beta-\gamma}\big|g_u(T)\big| + I_q^{\alpha+\beta-\gamma}\big|f_u(T)\big|\big) + a_2\big(I_q^{\beta}\big|g_u(T)\big| + I_q^{\alpha+\beta}\big|f_u(T)\big|\big)\big] \\
&\leq \big\{I_q^{\alpha+\beta}1(t) + \big|\mu_1(t)\big|\big[I_q^{\alpha+\beta+\sigma_1}1(\eta_1)\big] + \big|\mu_2(t)\big|\big[\lambda_2 I_q^{\alpha+\beta+\sigma_2}1(\eta_2) \\
&\quad + b_2 I_q^{\alpha+\beta-\gamma}1(T) + a_2 I_q^{\alpha+\beta}1(T)\big]\big\}\big\|f^*\big\| \\
&\quad + \big\{I_q^{\beta}1(t) + \big|\mu_1(t)\big|\big[I_q^{\beta+\sigma_1}1(\eta_1)\big] + \big|\mu_2(t)\big|\big[\lambda_2 I_q^{\beta+\sigma_2}1(\eta_2) + b_2 I_q^{\beta-\gamma}1(T) \\
&\quad + a_2 I_q^{\beta}1(T)\big]\big\}Ar \leq \Theta_1\big\|f^*\big\| + \Theta_2 Ar,
\end{aligned}$$

which yields $\|\mathcal{K}u\| \leq \Theta_1\|f^*\| + A\Theta_2 r$. This shows the uniformly boundedness of $\mathcal{K}$. We now claim that $\mathcal{K}$ is equicontinuous.

Let $t_1, t_2 \in J$ via $t_1 < t_2$. Then, by setting $\|f^*\| = \sup_{(t,u)\in J\times N} |f_u(t)|$, we obtain

$$\begin{aligned}
&\big|\mathcal{K}u(t_2) - \mathcal{K}u(t_1)\big| \\
&\quad\leq I_q^{\beta}\big|g_u(t_2) - g_u(t_1)\big| + I_q^{\alpha+\beta}\big|f_u(t_2) - f_u(t_1)\big| \\
&\qquad + \big|\mu_1(t_2) - \mu_1(t_1)\big|\big[I_q^{\beta+\sigma_1}\big|g_u(\eta_1)\big| + I_q^{\alpha+\beta+\sigma_1}\big|f_u(\eta_1)\big|\big] \\
&\qquad + \big|\mu_2(t_2) - \mu_2(t_1)\big|\big[\lambda_2\big(I_q^{\beta+\sigma_2}\big|g_u(\eta_2)\big| + I_q^{\alpha+\beta+\sigma_2}\big|f_u(\eta_2)\big|\big) \\
&\qquad + b_2\big(I_q^{\beta-\gamma}\big|g_u(T)\big| + I_q^{\alpha+\beta-\gamma}\big|f_u(T)\big|\big) + a_2\big(I_q^{\beta}\big|g_u(T)\big| + I_q^{\alpha+\beta}\big|f_u(T)\big|\big)\big] \\
&\quad\leq \frac{Ar}{\Gamma_q(\beta+1)}\big[t_2^{\beta} - t_1^{\beta} + 2(t_2 - t_1)^{\beta}\big] \\
&\qquad + \frac{\|f^*\|}{\Gamma_q(\alpha+\beta+1)}\big[t_2^{\alpha+\beta} - t_1^{\alpha+\beta} + 2(t_2 - t_1)^{\alpha+\beta}\big] \\
&\qquad + \big|\mu_1(t_2) - \mu_1(t_1)\big|\big[I_q^{\beta+\sigma_1}1(\eta_1)Ar + I_q^{\alpha+\beta+\sigma_1}1(\eta_1)\big\|f^*\big\|\big] \\
&\qquad + \big|\mu_2(t_2) - \mu_2(t_1)\big|\big[\lambda_2\big(I_q^{\beta+\sigma_2}1(\eta_2)Ar + I_q^{\alpha+\beta+\sigma_2}1(\eta_2)\big\|f^*\big\|\big) \\
&\qquad + b_2\big(I_q^{\beta-\gamma}1(T)Ar + I_q^{\alpha+\beta-\gamma}1(T)\big\|f^*\big\|\big) + a_2\big(I_q^{\beta}1(T)Ar + I_q^{\alpha+\beta}1(T)\big\|f^*\big\|\big)\big].
\end{aligned}$$

It is clear that $|\mathcal{K}u(t_2) - \mathcal{K}u(t_1)| \to 0$ as $t_2 \to t_1$ independent of u. In consequence, from the above arguments, $\mathcal{K}$ is relatively compact on N. Application of the Arzelà–Ascoli theorem proves the compactness of $\mathcal{K}$ on N.

Now, by considering the sequential four-point q-CFBVP (3) to be linear by taking $f_u(t) = f(t,u(t)) = \lambda(t)u(t)$, the operator $\mathcal{L}$, by Lemma 7, is formulated by

$$\begin{aligned}\mathcal{L}u(t) = {} & I_q^{\beta} g_u(t) + I_q^{\alpha+\beta}\lambda(t)u(t) \\ & + \mu_1(t)\big[I_q^{\beta+\sigma_1} g_u(\eta_1) + I_q^{\alpha+\beta+\sigma_1}\lambda(\eta_1)u(\eta_1)\big] \\ & + \mu_2(t)\big[\lambda_2\big(I_q^{\beta+\sigma_2} g_u(\eta_2) + I_q^{\alpha+\beta+\sigma_2}\lambda(\eta_2)u(\eta_2)\big) \\ & - b_2\big(I_q^{\beta-\gamma} g_u(T) + I_q^{\alpha+\beta-\gamma}\lambda(T)u(T)\big) \\ & - a_2\big(I_q^{\beta} g_u(T) + I_q^{\alpha+\beta}\lambda(T)u(T)\big)\big].\end{aligned}$$

Our next claim is that 1 is not an eigenvalue of $\mathcal{L}$. If it is so, by (18), we estimate

$$\begin{aligned}\|u\| = {} & \sup_{t\in J}\big|(\mathcal{L}u)(t)\big| \\ \le {} & \sup_{t\in J}\big\{I_q^{\beta}\big|g_u(t)\big| + I_q^{\alpha+\beta}\big|\lambda(t)\big|\big|u(t)\big| \\ & + \big|\mu_1(t)\big|\big[I_q^{\beta+\sigma_1}\big|g_u(\eta_1)\big| + I_q^{\alpha+\beta+\sigma_1}\big|\lambda(\eta_1)\big|\big|u(\eta_1)\big|\big] \\ & + \big|\mu_2(t)\big|\big[\lambda_2\big(I_q^{\beta+\sigma_2}\big|g_u(\eta_2)\big| + I_q^{\alpha+\beta+\sigma_2}\big|\lambda(\eta_2)\big|\big|u(\eta_2)\big|\big) \\ & + b_2\big(I_q^{\beta-\gamma}\big|g_u(T)\big| + I_q^{\alpha+\beta-\gamma}\big|\lambda(T)\big|\big|u(T)\big|\big) \\ & + a_2\big(I_q^{\beta}\big|g_u(T)\big| + I_q^{\alpha+\beta}\big|\lambda(T)\big|\big|u(T)\big|\big)\big]\big\} \\ \le {} & (\lambda_{\max}\Theta_1 + A\Theta_2)\|u\| < \|u\|,\end{aligned}$$

which is not possible. Hence we established our claim.

Finally, we show that $\|\mathcal{K}(u) - \mathcal{L}(u)\|/\|u\|$ vanishes as $\|u\| \to \infty$. For $t \in J$, one may write

$$\begin{aligned}& \big|(\mathcal{K}u)(t) - (\mathcal{L}u)(t)\big| \\ & \quad \le I_q^{\alpha+\beta}\big|f_u(t) - \lambda(t)u(t)\big| \\ & \qquad + \big|\mu_1(t)\big|\big[I_q^{\alpha+\beta+\sigma_1}\big|f_u(\eta_1) - \lambda(\eta_1)u(\eta_1)\big|\big] \\ & \qquad + \big|\mu_2(t)\big|\big[\lambda_2 I_q^{\alpha+\beta+\sigma_2}\big|f_u(\eta_2) - \lambda(\eta_2)u(\eta_2)\big| \\ & \qquad + b_2 I_q^{\alpha+\beta-\gamma}\big|f_u(T) - \lambda(T)u(T)\big| + a_2 I_q^{\alpha+\beta}\big|f_u(T) - \lambda(T)u(T)\big|\big] \\ & \quad \le I_q^{\alpha+\beta}\left(\left|\frac{f_u(t)}{u(t)} - \lambda(t)\right|\big|u(t)\big|\right) \\ & \qquad + \big|\mu_1(t)\big|\left[I_q^{\alpha+\beta+\sigma_1}\left(\left|\frac{f_u(\eta_1)}{u(\eta_1)} - \lambda(\eta_1)\right|\big|u(\eta_1)\big|\right)\right] \\ & \qquad + \big|\mu_2(t)\big|\left[\lambda_2\left(I_q^{\alpha+\beta+\sigma_2}\left(\left|\frac{f_u(\eta_2)}{u(\eta_2)} - \lambda(\eta_2)\right|\big|u(\eta_2)\big|\right)\right)\right. \\ & \qquad \left. + b_2\left(I_q^{\alpha+\beta-\gamma}\left(\left|\frac{f_u(T)}{u(T)} - \lambda(T)\right|\big|u(T)\big|\right)\right) + a_2\left(I_q^{\alpha+\beta}\left(\left|\frac{f_u(T)}{u(T)} - \lambda(T)\right|\big|u(T)\big|\right)\right)\right].\end{aligned}$$

This means that

$$\frac{\|\mathcal{K}u - \mathcal{L}u\|}{\|u\|} \le I_q^{\alpha+\beta}\left(\left|\frac{f_u(t)}{u(t)} - \lambda(t)\right|\right)$$

$$+ \left|\mu_1(t)\right|\left[I_q^{\alpha+\beta+\sigma_1}\left(\left|\frac{f_u(\eta_1)}{u(\eta_1)} - \lambda(\eta_1)\right|\right)\right]$$
$$+ \left|\mu_2(t)\right|\left[\lambda_2\left(I_q^{\alpha+\beta+\sigma_2}\left(\left|\frac{f_u(\eta_2)}{u(\eta_2)} - \lambda(\eta_2)\right|\right)\right)\right.$$
$$\left. + b_2\left(I_q^{\alpha+\beta-\gamma}\left(\left|\frac{f_u(T)}{u(T)} - \lambda(T)\right|\right)\right) + a_2\left(I_q^{\alpha+\beta}\left(\left|\frac{f_u(T)}{u(T)} - \lambda(T)\right|\right)\right)\right].$$

By (17) and letting $\|u\| \to \infty$, it is concluded that $|\frac{f_u(\cdot)}{u} - \lambda(\cdot)| \to 0$. Thus we obtain

$$\lim_{\|u\|\to\infty} \frac{\|\mathcal{K}(u) - \mathcal{L}(u)\|}{\|u\|} = 0.$$

Consequently, by Theorem 13, the supposed sequential four-point q-CFBVP (3) admits a solution in X. The proof is ended. □

3.3 The third existence criterion

We now present our last existence criterion based on the O'Regan theorem [45].

Theorem 15 ([45]) *Consider a closed and convex set $E \neq \emptyset$ belonging to a Banach space X containing an open set O. Define $\mathcal{K} : \bar{O} \to E$ as $\mathcal{K} = \mathcal{K}_1 + \mathcal{K}_2$ subject to $\mathcal{K}(\bar{O})$ being bounded. Moreover, $\mathcal{K}_1 : \bar{O} \to E$ is continuous and completely continuous, $\mathcal{K}_2 : \overline{O} \to E$ is nonlinear contraction (i.e, a nonnegative nondecreasing function $\Upsilon : [0,\infty) \to [0,\infty)$ exists which satisfies $\Upsilon(t) < t$ for $t > 0$, and $\|\mathcal{K}_2 u - \mathcal{K}_2 u'\| \leq \Upsilon(\|u - u'\|)$, $\forall u, u' \in O$.) Then either*

(C1) *$\mathcal{K}$ has a fixed point $u \in \bar{O}$;*

or

(C2) *there exist $u \in \partial O$ and $\mu \in (0,1)$ such that $u = \mu\mathcal{K}(u)$.*

Theorem 16 *Let $f, g \in C(J \times \mathbb{R}, \mathbb{R})$ and assume that:*

(D1) *there exist a nonnegative mapping $b \in C(J, [0,\infty))$ and a nondecreasing function $\mathbb{T} : [0,\infty) \to (0,\infty)$ such that*

$$\left|f(t,u)\right| \leq b(t)\mathbb{T}\left(\|u\|\right), \quad \forall (t,u) \in J \times \mathbb{R};$$

(D2) *there exist a continuous function $\phi_1 : [0,\infty) \to [0,\infty)$ and $\kappa > 0$ such that*

$$\left|g(t,u) - g(t,v)\right| \leq \phi_1\left(\|u - v\|\right) \quad \text{and} \quad \phi_1\left(|u|\right) \leq \kappa|u|, \quad \forall t \in J, u, v \in \mathbb{R};$$

(D3) *there exists $\varepsilon > 0$ such that $\sup_{\varepsilon\in(0,\infty)}[\frac{\varepsilon}{\Theta_1 b^*\mathbb{T}(\varepsilon)+l\Theta_2}] > \frac{1}{1-\kappa\Theta_2}$, where $l = \sup_{t\in J}|g(t,0)|$ and $\kappa\Theta_2 < 1$.*

Then there exists a solution for the supposed sequential four-point q-CFBVP (3) on J.

Proof We consider $\mathcal{K} : X \to X$ defined by (13) as

$$\mathcal{K}u(t) = \mathcal{K}_1 u(t) + \mathcal{K}_2 u(t), \quad t \in J,$$

where the operators $\mathcal{K}_1$ and $\mathcal{K}_2$ are, respectively, given in (14) and (15). By (D3), $\exists\varepsilon > 0$ so that

$$\frac{\varepsilon}{\Theta_1 b^*\mathbb{T}(\varepsilon) + l\Theta_2} > \frac{1}{1 - k\Theta_2},$$

and take $B_\varepsilon = \{u \in X : \|u\| < \varepsilon\}$. We demonstrate the continuity and complete continuity of $\mathcal{K}_1$. Before this, we prove the uniform boundedness of $\mathcal{K}_1$. Taking any $u \in \bar{B}_\varepsilon$, we have

$$\begin{aligned}\left|(\mathcal{K}_1 u)(t)\right| &\leq I_q^{\alpha+\beta}\left|f_u(t)\right| + \left|\mu_1(t)\right|\left(I_q^{\alpha+\beta+\sigma_1}\left|f_u(\eta_1)\right|\right) \\ &\quad + \left|\mu_2(t)\right|\left[\lambda_2\left(I_q^{\alpha+\beta+\sigma_2}\left|f_u(\eta_2)\right|\right) + b_2\left(I_q^{\alpha+\beta-\gamma}\left|f_u(T)\right|\right) + a_2\left(I_q^{\alpha+\beta}\left|f_u(T)\right|\right)\right] \\ &\leq \Theta_1 b^* \mathbb{T}(\varepsilon),\end{aligned}$$

in which $b^* = \sup_{t\in J}|b(t)|$. Thus $\mathcal{K}_1$ is uniformly bounded. Let $t_1, t_2 \in J$ such that $t_1 < t_2$. Then

$$\begin{aligned}&\left|(\mathcal{K}_1 u)(t_2) - (\mathcal{K}_1 u)(t_1)\right| \\ &\quad\leq I_q^{\alpha+\beta}\left|f_u(t_2) - f_u(t_1)\right| + \left|\mu_1(t_2) - \mu_1(t_1)\right|\left(I_q^{\alpha+\beta+\sigma_1}\left|f_u(\eta_1)\right|\right) \\ &\qquad + \left|\mu_2(t_2) - \mu_2(t_1)\right|\left[\lambda_2\left(I_q^{\alpha+\beta+\sigma_2}\left|f_u(\eta_2)\right|\right)\right. \\ &\qquad \left. + b_2\left(I_q^{\alpha+\beta-\gamma}\left|f_u(T)\right|\right) + a_2\left(I_q^{\alpha+\beta}\left|f_u(T)\right|\right)\right] \\ &\quad\leq \frac{b^*\mathbb{T}(\varepsilon)}{\Gamma_q(\alpha+\beta+1)}\left[t_2^{\alpha+\beta} - t_1^{\alpha+\beta} + 2(t_2 - t_1)^{\alpha+\beta}\right] \\ &\qquad + \left|\mu_1(t_2) - \mu_1(t_1)\right|\left[I_q^{\alpha+\beta+\sigma_1}1(\eta_1)b^*\mathbb{T}(\varepsilon)\right] \\ &\qquad + \left|\mu_2(t_2) - \mu_2(t_1)\right|\left[\lambda_2 I_q^{\alpha+\beta+\sigma_2}1(\eta_2)b^*\mathbb{T}(\varepsilon)\right. \\ &\qquad \left. + b_2 I_q^{\alpha+\beta-\gamma}1(T)b^*\mathbb{T}(\varepsilon) + a_2 I_q^{\alpha+\beta}1(T)b^*\mathbb{T}(\varepsilon)\right],\end{aligned}$$

which tends to zero as $t_2 \to t_1$ free of u. This gives the equicontinuity of $\mathcal{K}_1$. Application of the Arzelà–Ascoli theorem proves the compactness of $\mathcal{K}_1$ and consequently its complete continuity. Furthermore, the continuity of $\mathcal{K}_1$ can be deduced from that of f by the hypothesis.

We now show that $\mathcal{K}_2$ is a nonlinear contraction. By (D2) and for $u, v \in B_\varepsilon$, we have

$$\begin{aligned}&\left|(\mathcal{K}_2 u)(t) - (\mathcal{K}_2 v)(t)\right| \\ &\quad\leq I_q^{\beta}|g_u - g_v|(t) + \left|\mu_1(t)\right|\left[I_q^{\beta+\sigma_1}|g_u - g_v|(\eta_1)\right] \\ &\qquad + \left|\mu_2(t)\right|\left[\lambda_2 I_q^{\beta+\sigma_2}|g_u - g_v|(\eta_2) + b_2 I_q^{\beta-\gamma}|g_u - g_v|(T) + a_2 I_q^{\beta}|g_u - g_v|(T)\right] \\ &\quad\leq \Theta_2\phi_1\left(\left\|u(t) - v(t)\right\|\right) \\ &\quad\leq \Theta_2\kappa\left\|u(t) - v(t)\right\|.\end{aligned}$$

By setting $\Upsilon(u) = \Theta_2\kappa u$, note that $\Upsilon(0) = 0$ and $\Upsilon(u) = \Theta_2\kappa u < u$ for $u > 0$ since $\kappa\Theta_2 < 1$. Thus

$$\|\mathcal{K}_2 u - \mathcal{K}_2 v\| \leq \Upsilon\left(\|u - v\|\right).$$

Hence $\mathcal{K}_2$ is a nonlinear contraction. Now again, by (D2), for arbitrary $u \in B_\varepsilon$, we estimate

$$\left|g_u(t)\right| = \left|g(t,u)\right| \leq \left|g(t,u) - g(t,0)\right| + \left|g(t,0)\right| \leq \phi_1\left(\|u\|\right) + \left|g(t,0)\right| \leq \kappa\varepsilon + l.$$

where $l = \sup_{t\in J} |g(t,0)|$. Hence, we get

$$\|\mathcal{K}_2 u\| \leq \Theta_2(\kappa\varepsilon + l),$$

which confirms the boundedness of $\mathcal{K}_2$. Thus, $\mathcal{K} = \mathcal{K}_1 + \mathcal{K}_2$ is bounded.

In the final step, we prove that the assumption (C2) of Theorem 15 does not hold. To prove this, consider the existence of $\mu \in (0,1)$ and $u \in \partial B_\varepsilon$ such that $u = \mu\mathcal{K}u$. So $\|u\| = \varepsilon$ and

$$\begin{aligned}
|u(t)| &= \mu|(\mathcal{K}u)(t)| \\
&= \mu|\mathcal{K}_1 u(t) + \mathcal{K}_2 u(t)| \\
&\leq |\mathcal{K}_1 u(t)| + |\mathcal{K}_2 u(t)| \\
&\leq \Theta_1 b^* \mathbb{T}(\varepsilon) + \Theta_2(\kappa\varepsilon + 1).
\end{aligned}$$

Taking the supremum for all $t \in J$ yields

$$\|u\| \leq \Theta_1 b^* \mathbb{T}(\varepsilon) + (\kappa\varepsilon + l)\Theta_2.$$

Hence, we get

$$\frac{\varepsilon}{\Theta_1 b^* \mathbb{T}(\varepsilon) + l\Theta_2} \leq \frac{1}{1 - \kappa\Theta_2},$$

which contradicts (D3). Thus $\mathcal{K}_1$ and $\mathcal{K}_2$ satisfy all the assertions of Theorem 15. Therefore, a fixed-point of $\mathcal{K}$ in B_ε exists, which is the same solution of the sequential four-point q-CFBVP (3). The proof is finished. □

3.4 The uniqueness property

Finally, we investigate the uniqueness property for the solutions of the sequential four-point q-CFBVP (3) by referring to the Banach principle.

Theorem 17 *Let*

(H_4) $\exists a > 0$ *satisfying*

$$|g_u(t) - g_v(t)| \leq a|u(t) - v(t)|, \quad \forall t \in J, u, v \in \mathbb{R};$$

(H_5) $\exists \ell > 0$ *satisfying*

$$|f_u(t) - f_v(t)| \leq \ell|u(t) - v(t)|, \quad \forall t \in J, u, v \in \mathbb{R}.$$

Then the sequential four-point q-CFBVP (3) *has a unique solution on J if*

$$\ell\Theta_1 + a\Theta_2 < 1, \tag{20}$$

where Θ_1, Θ_2 *are given in* (11) *and* (12), *respectively.*

Proof To prove the relevant result, define the ball $B_r = \{u \in X : \|u\| \leq r\}$ for some $r > 0$ satisfying

$$r \geq \frac{\Theta_1 f_0^* + \Theta_2 g_0^*}{1 - \ell\Theta_1 - a\Theta_2},$$

where $g_0^* = \sup_{t\in J} |g(t,0)|$ and $f_0^* = \sup_{t\in J} |f(t,0)|$ and Θ_1 and Θ_2 are, respectively, given by (11) and (12). Now, we prove $\mathcal{K}B_r \subset B_r$ in which the operator $\mathcal{K} : X \to X$ is illustrated as (13). Similar to Step 1 in Theorem 12, for $u \in B_r$, we get

$$\begin{aligned}
\big|(\mathcal{K}u)(t)\big| &\leq I_q^{\beta}\big|g_u(t)\big| + I_q^{\alpha+\beta}\big|f_u(t)\big| \\
&\quad + \big|\mu_1(t)\big|\big[I_q^{\beta+\sigma_1}\big|g_u(\eta_1)\big| \\
&\quad + I_q^{\alpha+\beta+\sigma_1}\big|f_u(\eta_1)\big|\big] + \big|\mu_2(t)\big|\big[\lambda_2\big(I_q^{\beta+\sigma_2}\big|g_u(\eta_2)\big| + I_q^{\alpha+\beta+\sigma_2}\big|f_u(\eta_2)\big|\big) \\
&\quad + b_2\big(I_q^{\beta-\gamma}\big|g_u(T)\big| + I_q^{\alpha+\beta-\gamma}\big|f_u(T)\big|\big) + a_2\big(I_q^{\beta}\big|g_u(T)\big| + I_q^{\alpha+\beta}\big|f_u(T)\big|\big)\big] \\
&\leq \big(\ell\|u\| + f_0^*\big)\sup_{t\in J}\big\{I_q^{\alpha+\beta}1(t) + \big|\mu_1(t)\big|\big[I_q^{\alpha+\beta+\sigma_1}1(\eta_1)\big] \\
&\quad + \big|\mu_2(t)\big|\big[\lambda_2 I_q^{\alpha+\beta+\sigma_2}1(\eta_2) + b_2 I_q^{\alpha+\beta-\gamma}1(T) + a_2 I_q^{\alpha+\beta}1(T)\big]\big\} \\
&\quad + \big(a\|u\| + g_0^*\big)\sup_{t\in J}\big\{I_q^{\beta}1(t) + \big|\mu_1(t)\big|\big[I_q^{\beta+\sigma_1}1(\eta_1)\big] + \big|\mu_2(t)\big|\big[\lambda_2 I_q^{\beta+\sigma_2}1(\eta_2) \\
&\quad + b_2 I_q^{\beta-\gamma}1(T) + a_2 I_q^{\beta}1(T)\big]\big\} \\
&\leq \Theta_1\big(\ell r + f_0^*\big) + \Theta_2\big(ar + g_0^*\big) < r,
\end{aligned}$$

which implies $\|\mathcal{K}(u)\| \leq r$. Thus, $\mathcal{K}$ maps B_r into itself. Next, we prove that $\mathcal{K}$ is a contraction. For $u, v \in X$, and applying (11) and (12), we have

$$\begin{aligned}
\big|(\mathcal{K}u)(t) - (\mathcal{K}v)(t)\big| &\leq I_q^{\beta}|g_u - g_v|(t) + I_q^{\alpha+\beta}|f_u - g_v|(t) \\
&\quad + \big|\mu_1(t)\big|\big[I_q^{\beta+\sigma_1}|g_u - g_v|(\eta_1) + I_q^{\alpha+\beta+\sigma_1}|f_u - f_v|(\eta_1)\big] \\
&\quad + \big|\mu_2(t)\big|\big[\lambda_2\big(I_q^{\beta+\sigma_2}|g_u - g_v|(\eta_2) + I_q^{\alpha+\beta+\sigma_2}|f_u - f_v|(\eta_2)\big) \\
&\quad + b_2\big(I_q^{\beta-\gamma}|g_u - g_v|(T) + I_q^{\alpha+\beta-\gamma}|f_u - f_v|(T)\big) \\
&\quad + a_2\big(I_q^{\beta}|g_u - g_v|(T) + I_q^{\alpha+\beta}|f_u - f_v|(T)\big)\big] \\
&\leq (\ell\Theta_1 + a\Theta_2)\|u - v\|.
\end{aligned}$$

Consequently, we get

$$\big\|\mathcal{K}(u)(t) - \mathcal{K}(v)(t)\big\| \leq (\ell\Theta_1 + a\Theta_2)\|u - v\|.$$

Since $\ell\Theta_1 + a\Theta_2 < 1$, the above inequality proves that $\mathcal{K}$ is a contraction. Thus application of the Banach principle shows that $\mathcal{K}$ has a unique fixed point, corresponding to unique solution of the sequential four-point q-CFBVP (3) on J. This ends the proof. □

4 The criterion of Ulam–Hyers stability

Due to the importance of the notion of stability for possible solutions of different dynamical systems, in this section, we review two Ulam–Hyers and generalized Ulam–Hyers stabilities for solutions of the sequential four-point q-CFBVP (3). For more information, see [46–48].

Definition 18 ([49]) The sequential four-point q-CFBVP (3) is Ulam–Hyers stable if $\exists c^* \in \mathbb{R}_+$ such that $\forall \varepsilon > 0$ and $\forall u^*(t) \in C(J,\mathbb{R})$ as a solution function satisfying

$$\left|\mathcal{D}_q^{\alpha}\left(\mathcal{D}_q^{\beta}u^*(t) - g\left(t,u^*(t)\right)\right) - f\left(t,u^*(t)\right)\right| < \varepsilon, \tag{21}$$

$\exists u(t) \in C(J,\mathbb{R})$ as the solution of the sequential four-point q-CFBVP (3) with

$$\left|u^*(t) - u(t)\right| \leq \varepsilon c^*, \quad t \in J.$$

Definition 19 ([49]) The sequential four-point q-CFBVP (3) is generalized Ulam–Hyers stable if $\exists H \in C(\mathbb{R}^+,\mathbb{R}^+)$ with $H(0) = 0$ such that $\forall \varepsilon > 0$ and $\forall u^*(t) \in C(J,\mathbb{R})$ as a solution of

$$\left|\mathcal{D}_q^{\alpha}\left(\mathcal{D}_q^{\beta}u^*(t) - g\left(t,u^*(t)\right)\right) - f\left(t,u^*(t)\right)\right| < \varepsilon,$$

$\exists u(t) \in C(J,\mathbb{R})$ as a solution of the sequential four-point q-CFBVP (3) with

$$\left|u^*(t) - u(t)\right| \leq H(\varepsilon), \quad t \in J.$$

Remark 1 ([49]) It is evident that Def. 18 $\Rightarrow$ Def. 19.

Remark 2 ([49]) It is notable that $u^*(t) \in C(J,\mathbb{R})$ is a solution for (21) iff $\exists G \in C(J,\mathbb{R})$ depending on u^* such that

(1) $|G(t)| < \varepsilon$, $t \in J$.
(2) $\mathcal{D}_q^{\alpha}(\mathcal{D}_q^{\beta}u^*(t) - g(t,u^*(t))) = f(t,u^*(t)) + G(t)$, $t \in J$.

Now, we can discuss the above stabilities for solutions to the sequential four-point q-CFBVP (3).

Theorem 20 *If* (H_4) *and* (H_5) *are fulfilled, then the sequential four-point q-CFBVP* (3) *is Ulam–Hyers stable on J and accordingly is generalized Ulam–Hyers stable whenever*

$$\ell\Theta_1 + a\Theta_2 < 1,$$

where Θ_1, Θ_2 *are in the same forms given in* (11) *and* (12), *respectively.*

Proof For each $\varepsilon > 0$ and each function $u^*(t) \in C(J,\mathbb{R})$ as a solution of the inequality

$$\left|\mathcal{D}_q^{\alpha}\left(\mathcal{D}_q^{\beta}u(t) - g\left(t,u(t)\right)\right) - f\left(t,u(t)\right)\right| < \varepsilon,$$

a function $G(t)$ exists which satisfies

$$\mathcal{D}_q^{\alpha}\left(\mathcal{D}_q^{\beta}u(t) - g\left(t,u(t)\right)\right) = f\left(t,u(t)\right) + G(t)$$

with $|G(t)| \leq \varepsilon$. It gives

$$\begin{aligned}u^*(t) &= I_q^{\beta} g_{u^*}(t) + I_q^{\alpha+\beta} f_{u^*}(t) + I_q^{\alpha+\beta} G(t)\\ &\quad + \mu_1(t)\big[I_q^{\beta+\sigma_1} g_{u^*}(\eta_1) + I_q^{\alpha+\beta+\sigma_1} f_{u^*}(\eta_1)\big]\\ &\quad + \mu_2(t)\big[\lambda_2\big(I_q^{\beta+\sigma_2} g_{u^*}(\eta_2) + I_q^{\alpha+\beta+\sigma_2} f_{u^*}(\eta_2)\big)\\ &\quad - b_2\big(I_q^{\beta-\gamma} g_{u^*}(T) + I_q^{\alpha+\beta-\gamma} f_{u^*}(T)\big) - a_2\big(I_q^{\beta} g_{u^*}(T) + I_q^{\alpha+\beta} f_{u^*}(T)\big)\big].\end{aligned}$$

On the other side, let a unique function $u(t) \in C(J,\mathbb{R})$ be the solution of (3). Then $u(t)$ is written by

$$\begin{aligned}u(t) &= I_q^{\beta} g_u(t) + I_q^{\alpha+\beta} f_u(t)\\ &\quad + \mu_1(t)\big[I_q^{\beta+\sigma_1} g_u(\eta_1) + I_q^{\alpha+\beta+\sigma_1} f_u(\eta_1)\big]\\ &\quad + \mu_2(t)\big[\lambda_2\big(I_q^{\beta+\sigma_2} g_u(\eta_2) + I_q^{\alpha+\beta+\sigma_2} f_u(\eta_2)\big)\\ &\quad - b_2\big(I_q^{\beta-\gamma} g_u(T) + I_q^{\alpha+\beta-\gamma} f_u(T)\big) - a_2\big(I_q^{\beta} g_u(T) + I_q^{\alpha+\beta} f_u(T)\big)\big].\end{aligned}$$

We estimate

$$\begin{aligned}\big|u^*(t) - u(t)\big| &\leq I_q^{\alpha+\beta}\big|G(t)\big| + I_q^{\beta}|g_{u^*} - g_u|(t) + I_q^{\alpha+\beta}|f_{u^*} - g_u|(t)\\ &\quad + \big|\mu_1(t)\big|\big[I_q^{\beta+\sigma_1}|g_{u^*} - g_u|(\eta_1) + I_q^{\alpha+\beta+\sigma_1}|f_{u^*} - f_u|(\eta_1)\big]\\ &\quad + \big|\mu_2(t)\big|\big[\lambda_2\big(I_q^{\beta+\sigma_2}|g_{u^*} - g_u|(\eta_2) + I_q^{\alpha+\beta+\sigma_2}|f_{u^*} - f_u|(\eta_2)\big)\\ &\quad + b_2\big(I_q^{\beta-\gamma}|g_{u^*} - g_u|(T) + I_q^{\alpha+\beta-\gamma}|f_{u^*} - f_u|(T)\big)\\ &\quad + a_2\big(I_q^{\beta}|g_{u^*} - g_u|(T) + I_q^{\alpha+\beta}|f_{u^*} - f_u|(T)\big)\big]\\ &\leq \frac{\varepsilon T^{\alpha+\beta}}{\Gamma_q(\alpha+\beta+1)} + (\ell\Theta_1 + a\Theta_2)\big\|u^* - u\big\|.\end{aligned}$$

Hence

$$\big\|u^* - u\big\| \leq \frac{\varepsilon T^{\alpha+\beta}}{\Gamma_q(\alpha+\beta+1)} + (\ell\Theta_1 + a\Theta_2)\big\|u^* - u\big\|,$$

where Θ_1, Θ_2 are the same constants as represented in (11) and (12), respectively. In consequence,

$$\big\|u^* - u\big\| \leq \frac{\varepsilon T^{\alpha+\beta}}{\Gamma_q(\alpha+\beta+1)[1 - (\ell\Theta_1 + a\Theta_2)]}.$$

By assuming $c^* = \frac{T^{\alpha+\beta}}{\Gamma_q(\alpha+\beta+1)[1-(\ell\Theta_1+a\Theta_2)]}$, the Ulam–Hyers stability for q-system (3) is satisfied. Also, for

$$H(\varepsilon) = \frac{\varepsilon T^{\alpha+\beta}}{\Gamma_q(\alpha+\beta+1)[1 - (\ell\Theta_1 + a\Theta_2)]}$$

with $H(0) = 0$, the condition of the generalized Ulam–Hyers stability is fulfilled for solutions of the q-system (3). This completes the proof. □

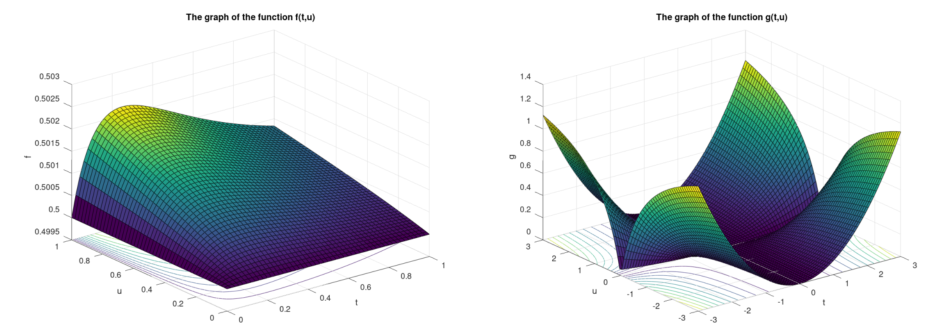

Figure 1 Graphs of the functions $f(t,u)$ and $g(t,u)$

5 Two examples

Here, we aim to present some examples to examine the obtained results.

Example 1 Let us consider the sequential four-point q-CFBVP with the following data:

$$\begin{cases} D_{1/4}^{1/3}[D_{1/4}^{2/3}u(t) - g(t,u(t))] = f(t,u(t)), \quad t \in J = [0,1], \\ u(0) + 2\mathcal{D}_{1/4}^{1/2}u(0) = 2/5 I_{1/4}^{3/4}u(1/2), \\ 2u(1) + \mathcal{D}_{1/4}^{1/2}u(1) = 3/7 I_{1/4}^{1/4}u(3/4), \end{cases} \tag{22}$$

where $\alpha = 1/3$, $\beta = 2/3$, $q = 1/4$, $T = 1$, $\gamma = 1/2$, $a_1 = b_2 = 1$, $a_2 = b_1 = 2$, $\sigma_1 = 3/4$, $\sigma_2 = 1/4$, $\lambda_1 = 2/5$, $\lambda_2 = 3/7$, $\eta_1 = 1/2$, $\eta_2 = 3/4$ and $g(t,u), f(t,u)$ are defined by

$$f(t,u) = \frac{tu}{56(1+t)^5}\left(\frac{|u|+2}{|u|+1}\right) + \frac{1}{2} \quad \text{and} \quad g(t,u) = \frac{3t^2}{6}\left(\frac{|u|}{3(|u|+1)}\right).$$

The continuity of f is obvious and we reach $f(t,0) = \frac{1}{2}$ (see Fig. 1). Now, we divide $f(t,u)$ by u and we get

$$\frac{f(t,u)}{u} = \frac{t}{56(1+t)^5}\left(1 + \frac{1}{|u|+1}\right) + \frac{1}{2u}.$$

Hence

$$\lim_{\|u\|\to\infty} \frac{f(t,u)}{u} = \frac{t}{56(1+t)^5}.$$

Setting $\lambda(t) = \frac{t}{56(1+t)^5}$, we get $\lambda_{\max} = 0.0179$. On the other side,

$$\bigl|g(t,u)\bigr| \le \frac{1}{6}|u|.$$

Letting $A = 1/6$, we obtain $\Theta_1 = 3.5597$ and $\Theta_2 = 4.8600$. since $(1 - A\Theta_2)/\Theta_1 = 0.0548 > \lambda_{\max}$, where Θ_1 and Θ_2 are, respectively, given by Eqs. (11) and (12). therefore, by Theorem 14, the sequential four-point q-CFBVP (22) has a solution on $[0,1]$.

Example 2 By considering $\alpha = 1/3$, $\beta = 2/3$, $q = 1/4$, $T = 1$, $\gamma = 1/2$, $a_1 = b_2 = 1$, $a_2 = b_1 = 2$, $\sigma_1 = 3/4$, $\sigma_2 = 1/4$, $\lambda_1 = 2/5$, $\lambda_2 = 3/7$, $\eta_1 = 1/2$, $\eta_2 = 3/4$ the sequential four-point q-CFBVP

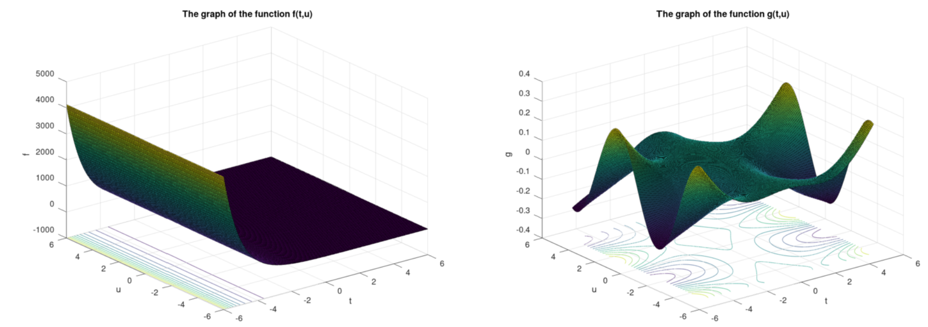

Figure 2 Graphs of the functions $f(t,u)$ and $g(t,u)$

is then given by

$$\begin{cases} D_{1/4}^{1/3}[D_{1/4}^{2/3}u(t) - g(t,u(t))] = f(t,u(t)), \quad t \in J = [0,1], \\ u(0) + 2\mathcal{D}_{1/4}^{1/2}u(0) = 2/5 I_{1/4}^{3/4}u(1/2), \\ 2u(1) + \mathcal{D}_{1/4}^{1/2}u(1) = 3/7 I_{1/4}^{1/4}u(3/4), \end{cases} \tag{23}$$

where $f(t,u)$ and $g(t,u)$ are given by (see Fig. 2)

$$f(t,u) = \frac{1}{3\sqrt{900+t^2}}\left(\arctan u + e^{-t}\right) \quad \text{and} \quad g(t,u) = \frac{1}{100(1+t^2)}\sin u + \frac{\cos t}{25}.$$

By usual computations, we obtain $\Theta_1 = 3.5597$ and $\Theta_2 = 4.8600$. Taking $a = 1/100$ and $\ell = 1/90$, it is clear that (H_4) and (H_5) are verified. Moreover, $\ell\Theta_1 + a\Theta_2 \approx 0.8775 < 1$. Thus, Theorem 17 is fulfilled and hence based on it, one can find that a unique solution exists for the sequential four-point q-CFBVP (23) on $[0,1]$. On the other side, as $\ell\Theta_1 + a\Theta_2 < 1$ is valid, so, by Theorem 20, the given sequential four-point q-CFBVP (23) is Ulam–Hyers and also generalized Ulam–Hyers stable on J.

6 Conclusions

In the present research, we considered a new boundary problem in the context of the quantum fractional operators. In other words, we defined a sequential q-fractional system of q-difference equation in which boundary conditions are designed as a linear combination of an unknown function and its q-derivative corresponding to a multiple of q-integrals in four points. The main focus of this research is on the solution's existence and its uniqueness with the help of some methods inspired by several pure concepts in functional analysis. We used three different fixed-point methods for this aim relying on the measure of non-compactness and condensing operators and compact operators. The existence of a unique solution is investigated based on the Banach criterion. The investigation of stability of the given q-CFBVP system in two formats based on Ulam–Hyers' conditions is implemented. Lastly, two examples are provided to ensure the findings. It is evident that this structure is more general and has many special applied cases. By assuming $g(t,u(t)) = -\mu \in \mathbb{R}$ and $a_1 = b_1 = a_2 = b_2 = 1$ and $\sigma_1 = \sigma_2 = 1$ and by letting $q \to 1$, our proposed sequential four-point q-CFBVP (3) is transformed into a fractional Langevin equation with integral conditions

$$\begin{cases} \mathcal{D}_q^{\alpha}(\mathcal{D}_q^{\beta}u(t) + \mu)) = f(t, u(t)), & t \in \mathrm{J} := [0, T], \\ u(0) + \mathcal{D}^{\gamma}u(0) = \lambda_1 \int_0^{\eta_1} u(s)\,\mathrm{d}s, & \eta_1 \in (0, T), \\ u(T) + \mathcal{D}^{\gamma}u(T) = \lambda_2 \int_0^{\eta_2} u(s)\,\mathrm{d}s, & \eta_2 \in (0, T), \end{cases}$$

which is considered as one of the most important equations in mathematical physics. Therefore, one can observe that the research study presented in the manuscript is not only new in the existing structure, but will also lead to other various quantum fractional problems as special cases. In future studies, we can generalize our boundary conditions to multi-point ones and investigate similar results in the context of newly-defined fractional (p, q)-operators in both cases of difference equations and inclusions.

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The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

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References

1. Hilfer, R.: Applications of Fractional Calculus in Physics. World Scientific, Singapore (2000)
2. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)
3. Miller, K.S., Ross, B.: An Introduction to the Fractional Calculus and Fractional Differential Equations. Wiley, New York (1993)
4. Baleanu, D., Etemad, S., Mohammadi, H., Rezapour, S.: A novel modeling of boundary value problems on the glucose graph. Commun. Nonlinear Sci. Numer. Simul. **100**, 105844 (2021). https://doi.org/10.1016/j.cnsns.2021.105844
5. Thabet, S.T.M., Etemad, S., Rezapour, S.: On a coupled Caputo conformable system of pantograph problems. Turk. J. Math. **45**(1), 496–519 (2021). https://doi.org/10.3906/mat-2010-70
6. Boutiara, A., Guerbati, K., Benbachir, M.: Caputo–Hadamard fractional differential equation with three-point boundary conditions in Banach spaces. AIMS Math. **5**(1), 259–272 (2019). https://doi.org/10.3934/math.2020017

7. Baleanu, D., Jajarmi, A., Mohammadi, H., Rezapour, S.: A new study on the mathematical modelling of human liver with Caputo–Fabrizio fractional derivative. Chaos Solitons Fractals **134**, 109705 (2020). https://doi.org/10.1016/j.chaos.2020.109705
8. Boutiara, A., Guerbati, K., Benbachir, M.: Caputo type fractional differential equation with nonlocal Erdelyi–Kober type integral boundary conditions in Banach spaces. Surv. Math. Appl. **15**, 399–418 (2020)
9. Aydogan, S.M., Baleanu, D., Mousalou, A., Rezapour, S.: On high order fractional integro-differential equations including the Caputo-Fabrizio derivative. Bound. Value Probl. **2018**, 90 (2018). https://doi.org/10.1186/s13661-018-1008-9
10. Mohammadi, H., Kumar, S., Rezapour, S., Etemad, S.: A theoretical study of the Caputo–Fabrizio fractional modeling for hearing loss due to Mumps virus with optimal control. Chaos Solitons Fractals **144**, 110668 (2021). https://doi.org/10.1016/j.chaos.2021.110668
11. Adiguzel, R.S., Aksoy, U., Karapinar, E., Erhan, I.M.: On the solutions of fractional differential equations via Geraghty type hybrid contractions. Appl. Comput. Math. **20**(2), 313–333 (2021)
12. Adiguzel, R.S., Aksoy, U., Karapinar, E., Erhan, I.M.: On the solutions of fractional differential equations via Geraghty type hybrid contraction. Math. Methods Appl. Sci. (2020). https://doi.org/10.1002/mma.6652
13. Adiguzel, R.S., Aksoy, U., Karapinar, E., Erhan, I.M.: Uniqueness of solution for higher-order nonlinear fractional differential equations with multi-point and integral boundary conditions. Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. **2021**, 155 (2021). https://doi.org/10.1007/s13398-021-01095-3
14. Baleanu, D., Rezapour, S., Saberpour, Z.: On fractional integro-differential inclusions via the extended fractional Caputo-Fabrizio derivation. Bound. Value Probl. **2019**, 79 (2019). https://doi.org/10.1186/s13661-019-1194-0
15. Baleanu, D., Etemad, S., Rezapour, S.: A hybrid Caputo fractional modeling for thermostat with hybrid boundary value conditions. Bound. Value Probl. **2020**, 64 (2020). https://doi.org/10.1186/s13661-020-01361-0
16. Abdeljawad, T., Agarwal, R.P., Karapinar, E., Kumari, P.S.: Solutions of the nonlinear integral equation and fractional differential equation using the technique of a fixed point with a numerical experiment in extended b-metric space. Symmetry **11**(5), 686 (2019). https://doi.org/10.3390/sym11050686
17. Abdo, M.S., Abdeljawad, T., Shah, K., Ali, S.M.: On nonlinear coupled evolution system with nonlocal subsidiary conditions under fractal-fractional order derivative. Math. Methods Appl. Sci. **44**(8), 6581–6600 (2021). https://doi.org/10.1002/mma.7210
18. Gul, R., Sarwar, M., Shah, K., Abdeljawad, T., Jarad, F.: Qualitative analysis of implicit Dirichlet boundary value problem for Caputo–Fabrizio fractional differential equations. J. Funct. Spaces **2020**, Article ID 4714032 (2020). https://doi.org/10.1155/2020/4714032
19. Ferreira, R.A.C.: Positive solutions for a class of boundary value problems with fractional q-differences. Comput. Math. Appl. **61**(2), 367–373 (2011). https://doi.org/10.1016/j.camwa.2010.11.012
20. Ernst, T.: A Comprehensive Treatment of q-Calculus. Birkhäuser, Basel (2012)
21. Jackson, F.H.: q-difference equations. Comput. Math. Appl. **32**(4), 305–314 (1910). https://doi.org/10.2307/2370183
22. Adams, C.R.: On the linear ordinary q-difference equation. Ann. Math. **30**(1), 195–205 (1910). https://doi.org/10.2307/1968274
23. Carmichael, R.D.: The general theory of linear q-difference equations. Am. J. Math. **34**(2), 147–168 (1912). https://doi.org/10.2307/2369887
24. Atici, F.M., Eloe, P.W.: Fractional q-calculus on a time scale. J. Nonlinear Math. Phys. **14**(3), 341–352 (2007). https://doi.org/10.2991/jnmp.2007.14.3.4
25. Annaby, M.H., Mansour, Z.S.: q-Fractional Calculus and Equations. Springer, Berlin (2012)
26. Jarad, F., Abdeljawad, T., Baleanu, D.: Stability of q-fractional non-autonomous systems. Nonlinear Anal., Real World Appl. **14**(1), 780–784 (2013). https://doi.org/10.1016/j.nonrwa.2012.08.001
27. Abdeljawad, T., Alzabut, J.: The q-fractional analogue for Gronwall-type inequality. J. Funct. Spaces **2013**, Article ID 543839 (2013). https://doi.org/10.1155/2013/543839
28. Butt, R.I., Abdeljawad, T., Alqudah, M.A., Ur Rehman, M.: Ulam stability of Caputo q-fractional delay difference equation: q-fractional Gronwall inequality approach. J. Inequal. Appl. **2019**, 305 (2019). https://doi.org/10.1186/s13660-019-2257-6
29. Ahmad, B., Etemad, S., Ettefagh, M., Rezapour, S.: On the existence of solutions for fractional q-difference inclusions with q-antiperiodic boundary conditions. Bull. Math. Soc. Sci. Math. Roum. **59(107)**(2), 119–134 (2016)
30. Abdeljawad, T., Baleanu, D.: Caputo q-fractional initial value problems and a q-analogue Mittag-Leffler function. Commun. Nonlinear Sci. Numer. Simul. **16**(12), 4682–4688 (2011). https://doi.org/10.1016/j.cnsns.2011.01.026
31. Etemad, S., Ntouyas, S.K., Ahmad, B.: Existence theory for a fractional q-integro-difference equation with q-integral boundary conditions of different orders. Mathematics **7**(8), 659 (2016). https://doi.org/10.3390/math7080659
32. Asawasamrit, S., Tariboon, J., Ntouyas, S.K.: Existence of solutions for fractional q-integro-difference equations with nonlocal fractional q-integral conditions. Abstr. Appl. Anal. **2014**, Article ID 474138 (2014). https://doi.org/10.1155/2014/474138
33. Etemad, S., Rezapour, S., Samei, M.E.: α-ψ-contractions and solutions of a q-fractional differential inclusion with three-point boundary value conditions via computational results. Adv. Differ. Equ. **2020**, 218 (2020). https://doi.org/10.1186/s13662-020-02679-w
34. Etemad, S., Ettefagh, M., Rezapour, S.: On the existence of solutions for nonlinear fractional q-difference equations with q-integral boundary conditions. J. Adv. Math. Stud. **8**(2), 265–285 (2015)
35. Rezapour, S., Imran, A., Hussain, A., Martínez, F., Etemad, S., Kaabar, M.K.A.: Condensing functions and approximate endpoint criterion for the existence analysis of quantum integro-difference FBVPs. Symmetry **13**(3), 469 (2021). https://doi.org/10.3390/sym13030469
36. Ahmad, B., Nieto, J.J., Alsaedi, A., Al-Hutami, H.: Boundary value problems of nonlinear fractional q-difference (integral) equations with two fractional orders and four-point nonlocal integral boundary conditions. Filomat **28**(8), 1719–1736 (2014). https://doi.org/10.2298/FIL1408719A
37. Ahmad, B., Nieto, J.J., Alsaedi, A., Al-Hutami, H.: Existence of solutions for nonlinear fractional q-difference integral equations with two fractional orders and nonlocal four-point boundary conditions. J. Franklin Inst. **351**(5), 2890–2909 (2014). https://doi.org/10.1016/j.jfranklin.2014.01.020

38. Rajkovic, P.M., Marinkovic, S.D., Stankovic, M.S.: On q-analogues of Caputo derivative and Mittag-Lefler function. Fract. Calc. Appl. Anal. **10**(4), 359–373 (2007)
39. Agarwal, R.P.: Certain fractional q-integrals and q-derivatives. Math. Proc. Camb. Philos. Soc. **66**(2), 365–370 (1969). https://doi.org/10.1017/S0305004100045060
40. Rajkovic, P.M., Marinkovic, S.D., Stankovic, M.S.: Fractional integrals and derivatives in q-calculus. Appl. Anal. Discrete Math. **1**(1), 311–323 (2007). https://doi.org/10.2298/AADM0701311R
41. Granas, A., Dugundji, J.: Fixed Point Theory. Springer, New York (2003)
42. Zeidler, E.: Nonlinear Functional Analysis and Its Application: Fixed Point Theorems. Springer, New York (1986)
43. Sadovskii, B.N.: A fixed point principle. Funct. Anal. Appl. **1**, 151–153 (1967). https://doi.org/10.1007/BF01076087
44. Krasnoselskii, M.A., Zabreiko, P.P.: Geometrical Methods of Nonlinear Analysis. Springer, New York (1984)
45. O'Regan, D.: Fixed-point theory for the sum of two operators. Appl. Math. Lett. **9**(1), 1–8 (1996). https://doi.org/10.1016/0893-9659(95)00093-3
46. Ameen, R., Jarad, F., Abdeljawad, T.: Boundary value problems of nonlinear fractional q-difference (integral) equations with two fractional orders and four-point nonlocal integral boundary conditions. Filomat **32**(15), 5265–5274 (2018). https://doi.org/10.2298/FIL1815265A
47. Ahmad, I., Nieto, J.J., Ur Rahman, G., Shah, K.: Fractional boundary value problems with multiply orders of fractional derivatives and integrals. Electron. J. Differ. Equ. **2020**(132), 1 (2020)
48. Boutiara, A., Etemad, S., Hussain, A., Rezapour, S.: The generalized U-H and U-H stability and existence analysis of a coupled hybrid system of integro-differential IVPs involving ϕ-Caputo fractional operators. Adv. Differ. Equ. **2021**, 95 (2021). https://doi.org/10.1186/s13662-021-03253-8
49. Rus, I.A.: Ulam stabilities of ordinary differential equations in a Banach space. Carpath. J. Math. **26**(1), 103–107 (2010)